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AN OPTIMAL ORDER YIELDING DISCREPANCY PRINCIPLE FOR SIMPLIFIED REGULARIZATION OF ILL-POSED PROBLEMS IN HILBERT SCALES

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Recently, Tautenhahn and Hämarik (1999) have considered a monotone rule as a parameter choice strategy for choosing the regularization parameter while considering approximate solution of an ill-posed operator equation $Tx = y$, where T is a bounded linear operator between Hilbert spaces. Motivated by this, we propose a new discrepancy principle for the simplified regularization, in the setting of Hilbert scales, when T is a positive and selfadjoint operator. When the data y is known only approximately, our method provides optimal order under certain natural assumptions on the ill-posedness of the equation and smoothness of the solution. The result, in fact, improves an earlier work of the authors (1997).

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1. Introduction. Tikhonov regularization (cf. [2]) is one of the most widely used procedures to obtain stable approximate solution to an ill-posed operator equation

$$Tx = y, \tag{1.1}$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y. Suppose that the data y is not exactly known, but only an approximation of it, namely $\tilde{y}$, is available. Then, the regularized solution $\tilde{x}_\alpha$, by Tikhonov regularization, is obtained by minimizing the map

$$x \longmapsto \|Tx - \tilde{y}\|^2 + \alpha\|x\|^2 \tag{1.2}$$

for $\alpha > 0$. For $y \in R(T) + R(T)^\perp$, if $\hat{x}$ is the generalized solution of (1.1), that is, $\hat{x} = T^\dagger y$, where $T^\dagger$ is the *Moore-Penrose generalized inverse* of T, then estimates for the error $\|\hat{x} - \tilde{x}_\alpha\|$ are obtained by choosing the regularization parameter α appropriately. It is known that (see, e.g., [2]) if $\hat{x} \in R((T^*T)^\nu)$ for some $\nu > 0$ and if $\|y - \tilde{y}\| \le \delta$ for some noise level $\delta > 0$, then the optimal order for the above error is $O(\delta^\mu)$, where $\mu = \min\{2\nu/(2\nu+1), 2/3\}$.

In order to improve the error estimates available in Tikhonov regularization, Natterer [9] carried out error analysis in the framework of Hilbert scales. Subsequently, many authors extended, modified, and generalized Natterer's work to obtain error bounds under various contexts (see, e.g., Natterer [9], Hegland [3], Schröter and Tautenhahn [12], Mair [6], Nair et al. [8], and Nair [7]).

If T is a positive and selfadjoint operator on a Hilbert space, then the simplified regularization introduced by Lavrentiev is better suited than Tikhonov regularization in terms of speed of convergence and condition number in the case of finite-dimensional approximations (cf. Schock [11]).

In [1], simplified regularization in the framework of Hilbert scales was studied for the first time and obtained error estimates under a priori and a posteriori parameter choice strategies. The a posteriori choice of the parameter in that paper has a drawback that it can yield the optimal rate only under certain restricted smoothness assumption on the solution.

In this paper, we propose a new discrepancy principle, for choosing the regularization parameter α, for simplified regularization in the setting of Hilbert scales, which eliminates the drawback of the method in [1] yielding the optimal order for a range of values of smoothness. The discrepancy principle of this paper is motivated by a recent procedure adopted by Tautenhahn and Hämarik [13].

2. Preliminaries.

Let H be a Hilbert space and let $A : H \to H$ be a bounded, positive and selfadjoint operator on H. Recall that A is said to be a positive operator if $\langle Ax, x\rangle \geq 0$ for every $x \in H$. For $y \in R(A)$, the range of A, consider the operator equation

$$Ax = y. \tag{2.1}$$

Let $\hat{x}$ be the minimal norm solution of (2.1). It is well known that if $R(A)$ is not closed in H, then the problem of solving (2.1) for $\hat{x}$ is ill-posed in the sense that small perturbation in the data y can cause large deviations in the solution.

A prototype of (2.1) is an integral equation of the first kind,

$$\int_0^1 k(s,t)x(t)\,dt = y(s), \quad 0 \leq s \leq 1, \tag{2.2}$$

where $k(\cdot,\cdot)$ is a nondegenerate kernel which is square integrable, that is,

$$\int_0^1 \int_0^1 |k(s,t)|^2\,dt\,ds < \infty, \tag{2.3}$$

satisfying $k(s,t) = k(t,s)$ for all s, t in $[0,1]$, and such that the eigenvalues of the corresponding integral operator $A : L^2[0,1] \to L^2[0,1]$,

$$(Ax)(s) = \int_0^1 k(s,t)x(t)\,dt, \quad 0 \leq s \leq 1, \tag{2.4}$$

are all nonnegative. For example, consider the kernel $k(\cdot,\cdot)$ defined by

$$k(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le s \le t \le 1, \\ (1-t)s, & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

Clearly, $k(s,t) = k(t,s)$, so that $A : L^2[0,1] \to L^2[0,1]$, defined as in (2.4), is a selfadjoint operator. Moreover, the eigenvalues of this operator are $1/n^2\pi^2$ for $n = 1,2,\ldots$ (see Limaye [5, page 329]).

For considering the regularization of (2.1) in the setting of Hilbert scales, we consider a Hilbert scale $\{H_t\}_{t\in\mathbb{R}}$ generated by a strictly positive operator $L : D(L) \to H$ with its domain $D(L)$ dense in H satisfying

$$\|Lx\| \ge \|x\|, \quad x \in D(L). \tag{2.6}$$

Recall (cf. [4]) that the space H_t is the completion of $D := \bigcap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u,v\rangle_t = \langle L^t u, L^t v\rangle, \quad u,v \in D. \tag{2.7}$$

Moreover, if $\beta \le \gamma$, then the embedding $H_\gamma \hookrightarrow H_\beta$ is continuous, and therefore the norm $\|\cdot\|_\beta$ is also defined in H_γ and there is a constant $c_{0,1}$ such that

$$\|x\|_\beta \le c_{0,1}\|x\|_\gamma, \quad x \in H_\gamma. \tag{2.8}$$

We assume that the ill-posed nature of the operator A is related to the Hilbert scale $\{H_t\}_{t\in\mathbb{R}}$ according to the relation

$$c_1\|x\|_{-a} \le \|Ax\| \le c_2\|x\|_{-a}, \quad x \in H, \tag{2.9}$$

for some positive reals a, c_1, and c_2.

For the example of an integral operator given in the previous paragraph, one may take L to be defined by

$$Lx := \sum_{j=1}^{\infty} j^2\langle x,u_j\rangle u_j, \tag{2.10}$$

where $u_j(t) := \sqrt{2}\sin(j\pi t)$, $j \in \mathbb{N}$, and the domain of L is

$$D(L) := \left\{x \in L^2[0,1] : \sum_{j=1}^{\infty} j^4 |\langle x,u_j\rangle|^2 < \infty\right\}. \tag{2.11}$$

In this case, it can be seen that

$$H_t = \left\{x \in L^2[0,1] : \sum_{j=1}^{\infty} j^{4t} |\langle x,u_j\rangle|^2 < \infty\right\} \tag{2.12}$$

and the constants a, c_1, and c_2 in (2.9) are given by $a = 1$ and $c_1 = c_2 = 1/\pi^2$ (see Schröter and Tautenhahn [12, Section 4]).

As in [1], we consider the regularized solution of (1.1) as the solution of the well-posed equation

$$(A+\alpha L^s)x_\alpha = y, \quad \alpha > 0, \tag{2.13}$$

where s is a fixed nonnegative real number.

Suppose that the data $y \neq 0$ is known only approximately, say $\tilde{y} \neq 0$ with $\|y-\tilde{y}\| \le \delta$ for a known error level $\delta > 0$. Then, in place of (2.13), we consider

$$(A+\alpha L^s)\tilde{x}_\alpha = \tilde{y}. \tag{2.14}$$

It can be seen that the solution $\tilde{x}_\alpha$ of the above equation is the unique minimizer of the function

$$x \longmapsto \langle Ax,x\rangle - 2\langle \tilde{y},x\rangle + \alpha\langle L^s x,x\rangle, \quad x \in D(L). \tag{2.15}$$

We also observe that taking

$$A_s := L^{-s/2}AL^{-s/2}, \tag{2.16}$$

(2.13) and (2.14) take the forms

$$L^{s/2}(A_s+\alpha I)L^{s/2}x_\alpha = y, \qquad L^{s/2}(A_s+\alpha I)L^{s/2}\tilde{x}_\alpha = \tilde{y}, \tag{2.17}$$

respectively. Note that the operator A_s defined above is positive and selfadjoint bounded operator on H.

One of the crucial results for proving the results in [1] as well as the results in this paper is the following result, where functions f and g are defined by

$$f(t) = \min\{c_1^t, c_2^t\}, \quad g(t) = \max\{c_1^t, c_2^t\}, \quad t \in \mathbb{R}, \ |t| \le 1, \tag{2.18}$$

respectively.

Proposition 2.1 (see [1, Proposition 3.1]). *For $s \ge 0$ and $|\nu| \le 1$,*

$$f\left(\frac{\nu}{2}\right)\|x\|_{-\nu(s+a)/2} \le \|A_s^{\nu/2}x\| \le g\left(\frac{\nu}{2}\right)\|x\|_{-\nu(s+a)/2}, \quad x \in H. \tag{2.19}$$

Using the above proposition, the following result has been proved in [1].

Theorem 2.2 (see [1, Theorem 3.2]). *Suppose that $\hat{x} \in H_t$, $0 < t \le s+a$, and $\alpha > 0$. Then*

$$\|\hat{x} - \tilde{x}_\alpha\| \le \phi(s,t)\alpha^{t/(s+a)}\|x\|_t + \psi(s)\alpha^{-a/(s+a)}\delta, \tag{2.20}$$

where

$$\phi(s,t) = \frac{g\big((s-2t)/(2s+2a)\big)}{f\big(s/(2s+2a)\big)}, \qquad \psi(s) = \frac{g\big(-s/(2s+2a)\big)}{f\big(s/(2s+2a)\big)}. \tag{2.21}$$

In particular, if $\alpha = c_0\delta^{(s+a)/(t+a)}$ *for some constant* $c_0 > 0$, *then*

$$||\hat{x} - \tilde{x_\alpha}|| \le \eta(s,t)\delta^{t/(t+a)}, \tag{2.22}$$

where

$$\eta(s,t) = \max\Big\{\phi(s,t)\|\hat{x}\|_t c_0^{t/(t+a)}, \psi(s)c_0^{-a/(s+a)}\Big\}. \tag{2.23}$$

Let $R_\alpha = (A_s + \alpha I)^{-1}$. We will make use of the relation

$$||R_\alpha A_s^\tau|| \le \alpha^{\tau-1}, \quad \alpha > 0,\ 0 < \tau \le 1, \tag{2.24}$$

which follows from the spectral properties of the selfadjoint operator A_s, $s > 0$.

In [1], the authors considered parameter choice strategies, a priori and a posteriori, which yield the optimal rate $O(\delta^{t/(t+a)})$ if $\hat{x} \in H_t$ for certain specific values of t. The a posteriori parameter choice strategy in [1] is to choose α such that

$$\alpha^{p+1}||(A_s + \alpha I)^{-p-1}L^{-s/2}x|| = k\delta, \tag{2.25}$$

where $k > 1$ and $\tilde{y} \in X$ satisfy $0 < k\delta \le \|\tilde{y}\|_{-s/2}$. Under the above procedure, the optimal order $O(\delta^{t/(t+a)})$ is obtained for $t = s + p(s+a)$.

In the present paper, we propose a new discrepancy for choosing the regularization parameter α which yields the optimal rate

$$||\hat{x} - \tilde{x_\alpha}|| = O(\delta^{t/(t+a)}). \tag{2.26}$$

3. The discrepancy principle. Let s and a be fixed positive real numbers. For $\alpha > 0$ and nonzero $x \in H$, let

$$\Phi(\alpha,x) := \frac{\alpha||R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x||^2}{||R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x||}. \tag{3.1}$$

Note that, by assumption (2.9), $\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|$ is nonzero for every nonzero $x \in H$ so that the function $\Phi(\alpha,x)$ is well defined for every $\alpha > 0$ and for every nonzero $x \in H$.

We assume that the available data $\tilde{y}$ is nonzero and

$$\|y - \tilde{y}\| \le \delta \tag{3.2}$$

for some known error level $\delta > 0$. Our idea is to prove the existence of a unique α such that

$$\Phi(\alpha, \tilde{y}) = c\delta \tag{3.3}$$

for some known $c > 0$.

In due course we will make use of the relation

$$f\left(\frac{-s}{2s+2a}\right)\|x\| \leq \left\|A_s^{-s/(2s+2a)}L^{-s/2}x\right\| \leq g\left(\frac{-s}{2s+2a}\right)\|x\| \tag{3.4}$$

which can be easily derived from Proposition 2.1.

First we prove the monotonicity of the function $\Phi(\alpha, x)$ defined in (3.1).

Theorem 3.1. *For each nonzero $x \in H$, the function $\alpha \mapsto \Phi(\alpha, x)$ for $\alpha > 0$, defined in (3.1), is increasing and it is continuously differentiable with $\Phi'(\alpha, x) \geq 0$. In addition*

$$\lim_{\alpha \to 0} \Phi(\alpha, x) = 0, \qquad \lim_{\alpha \to \infty} \Phi(\alpha, x) = \left\|A_s^{-s/(2s+2a)}L^{-s/2}x\right\|. \tag{3.5}$$

Proof. Using (3.1), one can write

$$\begin{aligned}
\frac{d}{d\alpha}&\Phi(\alpha, x) \\
&= \frac{(d/d\alpha)\left(\Phi^2(\alpha, x)\right)}{2\Phi(\alpha, x)} \\
&= \frac{2\alpha\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2\left\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2}{2\alpha\left\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2} \\
&\quad\times\frac{(d/d\alpha)\left[\alpha\left\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2\right]}{\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^3} \\
&\quad-\frac{\alpha^2\left\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^4(d/d\alpha)\left[\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2\right]}{2\alpha\left\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^3}.
\end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}
\frac{d}{d\alpha}&\Phi(\alpha, x) \\
&= \frac{\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2(d/d\alpha)\left[\alpha\left\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2\right]}{\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^3} \\
&\quad-\frac{\alpha\left\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2(d/d\alpha)\left[\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^2\right]}{2\left\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\right\|^3}.
\end{aligned} \tag{3.7}$$

Let $\{E_\lambda : 0 \le \lambda \le a\}$ be the spectral family of A_s, where $a = \|A_s\|$. Then

$$\begin{aligned}
&\frac{d}{d\alpha}\Big(\alpha\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\Big)\\
&\quad= \frac{d}{d\alpha}\int_0^a \frac{\alpha}{\lambda^{s/(s+a)}(\lambda+\alpha)^3}\,d\langle E_\lambda L^{-s/2}x, L^{-s/2}x\rangle\\
&\quad= \int_0^a \Big[\frac{1}{\lambda^{s/(s+a)}(\lambda+\alpha)^3} - \frac{3\alpha}{\lambda^{s/(s+a)}(\lambda+\alpha)^4}\Big]d\langle E_\lambda L^{-s/2}x, L^{-s/2}x\rangle\\
&\quad= \|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 - 3\alpha\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2.
\end{aligned} \tag{3.8}$$

Similarly

$$\frac{d}{d\alpha}(\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|) = -4\|R_\alpha^{5/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2. \tag{3.9}$$

Therefore, from (3.7), using (3.8) and (3.9), we get

$$\begin{aligned}
&\frac{d}{d\alpha}\Phi(\alpha,x)\\
&\quad= \|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad\times\frac{\big[\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 - 3\alpha\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\big]}{\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^3}\\
&\qquad+\frac{2\alpha\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\|R_\alpha^{5/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2}{\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^3}.
\end{aligned} \tag{3.10}$$

The above equation can be rewritten as

$$\frac{d}{d\alpha}\Phi(\alpha,x) = \Psi_1(\alpha,x) + \Psi_2(\alpha,x), \tag{3.11}$$

where

$$\begin{aligned}
&\Psi_1(\alpha,x)\\
&\quad= \|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad\times\frac{\big[\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 - \alpha\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\big]}{\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^3},\\
&\Psi_2(\alpha,x)\\
&\quad= \Big(2\alpha\Big[\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad\times\|R_\alpha^{5/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2 - \|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^4\Big]\Big)\\
&\qquad\times\frac{1}{\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^3}.
\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}
&\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad=\langle (A_s+\alpha I)^{-3}A_s^{-s/(2s+2a)}L^{-s/2}x, A_s^{-s/(2s+2a)}L^{-s/2}x\rangle,\\
&\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad=\langle (A_s+\alpha I)^{-3}A_s^{-s/(2s+2a)}L^{-s/2}x, (A_s+\alpha I)^{-1}A_s^{-s/(2s+2a)}L^{-s/2}x\rangle,
\end{aligned}\tag{3.13}$$

we have

$$\begin{aligned}
&\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2-\alpha\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad=\|A_s^{a/(2s+2a)}R_\alpha^2L^{-s/2}x\|^2.
\end{aligned}\tag{3.14}$$

Also,

$$\begin{aligned}
&\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^4\\
&\qquad=[\langle R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x, R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\rangle]^2\\
&\qquad=[\langle R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x, R_\alpha^{5/2}A_s^{-s/(2s+2a)}L^{-s/2}x\rangle]^2\\
&\qquad\le\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\|R_\alpha^{5/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2.
\end{aligned}\tag{3.15}$$

Hence

$$\Psi_1(\alpha,x)\ge 0,\qquad \Psi_2(\alpha,x)\ge 0,\tag{3.16}$$

so that

$$\frac{d}{d\alpha}(\Phi(\alpha,x))=\Psi_1(\alpha,x)+\Psi_2(\alpha,x)\ge 0.\tag{3.17}$$

To prove the last part of the theorem we observe that

$$\begin{aligned}
&\alpha^2\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|-\Phi(\alpha,x)\\
&\qquad=\frac{\alpha^2\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2-\alpha\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2}{\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|}.
\end{aligned}\tag{3.18}$$

Since

$$\begin{aligned}
&\alpha^2\|R_\alpha^{2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad=\alpha\langle R_\alpha^{3}A_s^{-s/(2s+2a)}L^{-s/2}x, \alpha R_\alpha A_s^{-s/(2s+2a)}L^{-s/2}x\rangle,\\
&\alpha\|R_\alpha^{3/2}A_s^{-s/(2s+2a)}L^{-s/2}x\|^2\\
&\qquad=\alpha\langle R_\alpha^{3}A_s^{-s/(2s+2a)}L^{-s/2}x, A_s^{-s/(2s+2a)}L^{-s/2}x\rangle,
\end{aligned}\tag{3.19}$$

and since $\alpha R_\alpha - I = A_s R_\alpha = R_\alpha A_s$, we have

$$\begin{aligned}\alpha^2\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\| - \Phi(\alpha,x) &= \frac{-\alpha\langle R_\alpha^3 A_s^{-s/(2s+2a)}L^{-s/2}x, A_s R_\alpha A_s^{-s/(2s+2a)}L^{-s/2}x\rangle}{\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|} \\ &= \frac{-\alpha\|A_s^{a/(2s+2a)}R_\alpha^2 L^{-s/2}x\|^2}{\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|} \le 0.\end{aligned} \tag{3.20}$$

Hence

$$\Phi(\alpha,x) \ge \alpha^2\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\| \ge \alpha^2\frac{\|A_s^{-s/(2s+2a)}L^{-s/2}x\|}{(\|A_s\|+\alpha)^2}. \tag{3.21}$$

Also, we have

$$\begin{aligned}\Phi(\alpha,x) &= \frac{\alpha\langle R_\alpha A_s^{-s/(2s+2a)}L^{-s/2}x, R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\rangle}{\|R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}x\|} \\ &\le \alpha\|R_\alpha A_s^{-s/(2s+2a)}L^{-s/2}x\|.\end{aligned} \tag{3.22}$$

Hence

$$\begin{aligned}&\left(\frac{\alpha}{\|A_s\|+\alpha}\right)^2\|A_s^{-s/(2s+2a)}L^{-s/2}x\| \\ &\qquad\le \Phi(\alpha,x) \le \alpha\|R_\alpha A_s^{-s/(2s+2a)}L^{-s/2}x\|.\end{aligned} \tag{3.23}$$

From this, it follows that

$$\lim_{\alpha\to 0}\Phi(\alpha,x) = 0, \qquad \lim_{\alpha\to\infty}\Phi(\alpha,x) = \|A_s^{-s/(2s+2a)}L^{-s/2}x\|. \tag{3.24}$$

This completes the proof. □

For the next theorem, in addition to (3.2), we assume that

$$\|A_s^{-s/(2s+2a)}L^{-s/2}\tilde{y}\| \ge c\delta \tag{3.25}$$

for some $c > 0$. This assumption will be satisfied if, for example,

$$\delta \le \frac{\tilde{f}(s)}{c+\tilde{f}(s)}\|y\|, \qquad \tilde{f}(s) := f\left(\frac{-s}{2s+2a}\right) \tag{3.26}$$

since, by (3.2), we have $\|\tilde{y}\| \ge \|y\| - \delta$, and by (3.4),

$$\|A_s^{-s/(2s+2a)}L^{-s/2}\tilde{y}\| \ge f\left(\frac{-s}{2s+2a}\right)\|\tilde{y}\|, \tag{3.27}$$

where f is as in (2.18).

Now, the following theorem is a consequence of Theorem 3.1.

Theorem 3.2. *Assume that (3.2) and (3.25) are satisfied. Then there exists a unique* $\alpha := \alpha(\delta)$ *satisfying*

$$\Phi(\alpha, \tilde{y}) = c\delta. \tag{3.28}$$

4. Error estimates. In order to obtain Hölder-type error bounds, that is, error
bounds of the form

$$\|\tilde{x_\alpha} - \hat{x}\| = O(\delta^\tau) \tag{4.1}$$

for some τ, we assume that the solution $\hat{x}$ of (2.1) satisfies the source condition (as in [1, 10]):

$$\hat{x} \in M_{\rho,t} := \{x \in H_t : \|x\|_t \le \rho\} \tag{4.2}$$

for some $t > 0$.

Lemma 4.1. *Suppose that* $\hat{x}$ *belongs to* $M_{\rho,t}$ *for some* $t \le s$*, and* $\alpha := \alpha(\delta) > 0$ *is the unique solution of (3.28), where* $c > g(-s/(2s+2a))$*. Then*

$$\alpha \ge c_0 \delta^{(s+a)/(t+a)}, \qquad c_0 = \frac{c - g(-s/(2s+2a))}{g((s-2t)/(2s+2a))\rho}. \tag{4.3}$$

Proof. Note that by (3.22), Proposition 2.1, and (2.24), we have

$$\begin{aligned}
\Phi(\alpha, \tilde{y}) &\le \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} \tilde{y}\| \\
&\le \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| + \alpha \|R_\alpha A_s^{-s/(2s+2a)} A_s L^{s/2} \hat{x}\| \\
&\le \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| + \alpha \|R_\alpha A_s^{(s+2a)/(2s+2a)} L^{s/2} \hat{x}\| \\
&\le \alpha \|R_\alpha A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| \\
&\quad + \alpha \|R_\alpha A_s^{(t+a)/(s+a)} A_s^{(s-2t)/(2s+2a)} L^{s/2} \hat{x}\| \\
&\le \|\alpha R_\alpha\| \|A_s^{-s/(2s+2a)} L^{-s/2} (\tilde{y} - y)\| \\
&\quad + \|\alpha R_\alpha A_s^{(t+a)/(s+a)}\| \|A_s^{(s-2t)/(2s+2a)} L^{s/2} \hat{x}\| \\
&\le g\left(\frac{-s}{2s+2a}\right)\delta + g\left(\frac{s-2t}{2s+2a}\right)\rho\alpha^{(t+a)/(s+a)}.
\end{aligned} \tag{4.4}$$

Thus

$$\left[c - g\left(\frac{-s}{2s+2a}\right)\right]\delta \le g\left(\frac{s-2t}{2s+2a}\right)\rho\alpha^{(t+a)/(s+a)}, \tag{4.5}$$

which implies

$$\alpha \ge c_0 \delta^{(s+a)/(t+a)}, \qquad c_0 = \frac{c - g(-s/(2s+2a))}{g((s-2t)/(2s+2a))\rho}. \tag{4.6}$$

This completes the proof. □

Theorem 4.2. *Under the assumptions in Lemma 4.1,*

$$\|\hat{x}-x_\alpha\| = O(\delta^\kappa), \quad \kappa := \frac{t}{t+a}. \tag{4.7}$$

Proof. Since x_α is the solution of (2.13), we have

$$\begin{aligned}\hat{x}-x_\alpha &= \hat{x}-(A+\alpha L^s)^{-1}y \\ &= \alpha L^{-s/2}(A_s+\alpha I)^{-1}L^{s/2}\hat{x} = \alpha L^{-s/2}R_\alpha L^{s/2}\hat{x}.\end{aligned} \tag{4.8}$$

Therefore, by (3.4), we have

$$f\left(\frac{s}{2s+2a}\right)\|\hat{x}-x_\alpha\| \le \|\alpha A_s^{s/(2s+2a)}R_\alpha L^{s/2}\hat{x}\|. \tag{4.9}$$

To obtain an estimate for $\|\alpha A_s^{s/(2s+2a)}R_\alpha L^{s/2}\hat{x}\|$, first we will make use of the following moment inequality

$$\|B^u x\| \le \|B^v x\|^{u/v}\|x\|^{1-u/v}, \quad 0 \le u \le v, \tag{4.10}$$

where B is a positive selfadjoint operator. Precisely, we use (4.10) with

$$\begin{aligned} u = \frac{t}{a}, \qquad v = 1+\frac{t}{a}, \qquad B = \alpha R_\alpha A_s^{a/(s+a)}, \\ x = \alpha^{1-t/a}R_\alpha^{1-t/a}A_s^{(s-2t)/(2s+2a)}L^{s/2}\hat{x}.\end{aligned} \tag{4.11}$$

Then since

$$\begin{aligned}\|x\| &\le \|A_s^{(s-2t)/(2s+2a)}L^{s/2}\hat{x}\| \\ &\le g\left(\frac{s-2t}{2s+2a}\right)\|L^{s/2}\hat{x}\|_{t-s/2} \le g\left(\frac{s-2t}{2s+2a}\right)\rho,\end{aligned} \tag{4.12}$$

we have

$$\begin{aligned}&\|\alpha A_s^{s/(2s+2a)}R_\alpha L^{s/2}\hat{x}\| \\ &\quad = \|B^{t/a}x\| \le \|B^{1+t/a}x\|^{t/(t+a)}\|x\|^{a/(t+a)} \\ &\quad \le \|\alpha^2 R_\alpha^2 A_s^{(2a+s)/(2s+2a)}L^{s/2}\hat{x}\|^{t/(t+a)}\|x\|^{a/(t+a)} \\ &\quad \le \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}y\|^{t/(t+a)}\|x\|^{a/(t+a)} \\ &\quad \le g\left(\frac{s-2t}{2s+2a}\right)^{a/(t+a)}\rho^{a/(t+a)}\|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}y\|^{t/(t+a)}.\end{aligned} \tag{4.13}$$

Further, by (2.24) and (3.20),

$$\begin{aligned}\|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}y\| &\le \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}(y-\tilde{y})\| \\ &\quad + \|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)}L^{-s/2}\tilde{y}\| \\ &\le \delta + \Phi(\alpha,\tilde{y}).\end{aligned} \tag{4.14}$$

Therefore, if $\alpha := \alpha(\delta)$ is the unique solution of (3.28), then we have

$$\|\alpha^2 R_\alpha^2 A_s^{-s/(2s+2a)} L^{-s/2} y\| \le (1+c)\delta. \tag{4.15}$$

Now the result follows from (4.9), (4.13), (4.14), and (4.15). □

Theorem 4.3. *Under the assumptions in Lemma 4.1,*

$$\|\hat{x} - \tilde{x}_\alpha\| = O(\delta^\kappa), \quad \kappa := \frac{t}{t+a}. \tag{4.16}$$

Proof. Let x_α and $\tilde{x}_\alpha$ be the solutions of (2.13) and (2.14), respectively. Then by triangle inequality, (2.24), and Proposition 2.1,

$$\begin{aligned}
\|\hat{x} - \tilde{x}_\alpha\| &\le \|\hat{x} - x_\alpha\| + \|x_\alpha - \tilde{x}_\alpha\| \\
&= \|\hat{x} - x_\alpha\| + \|L^{-s/2} R_\alpha L^{-s/2}(y - \tilde{y})\| \\
&\le \|\hat{x} - x_\alpha\| + \frac{1}{f(s/(2s+2a))}\|A_s^{s/(2s+2a)} R_\alpha L^{-s/2}(y - \tilde{y})\| \\
&\le \|\hat{x} - x_\alpha\| + \frac{1}{f(s/(2s+2a))}\|A_s^{s/(s+a)} R_\alpha A_s^{-s/(2s+2a)} L^{-s/2}(y - \tilde{y})\| \\
&\le \|\hat{x} - x_\alpha\| + \frac{1}{f(s/(2s+2a))}\|A_s^{s/(s+a)} R_\alpha\| \|A_s^{-s/(2s+2a)} L^{-s/2}(y - \tilde{y})\| \\
&\le \|\hat{x} - x_\alpha\| + \frac{g(-s/(2s+2a))}{f(s/(2s+2a))}\delta\alpha^{-a/(s+a)}.
\end{aligned} \tag{4.17}$$

The proof now follows from Lemma 4.1 and Theorem 4.2. □

Remark 4.4. We observe that unlike the discrepancy principle in [1], the discrepancy principle (3.3) gives the optimal order $O(\delta^{t/(t+a)})$ for all $0 < t \le s$.

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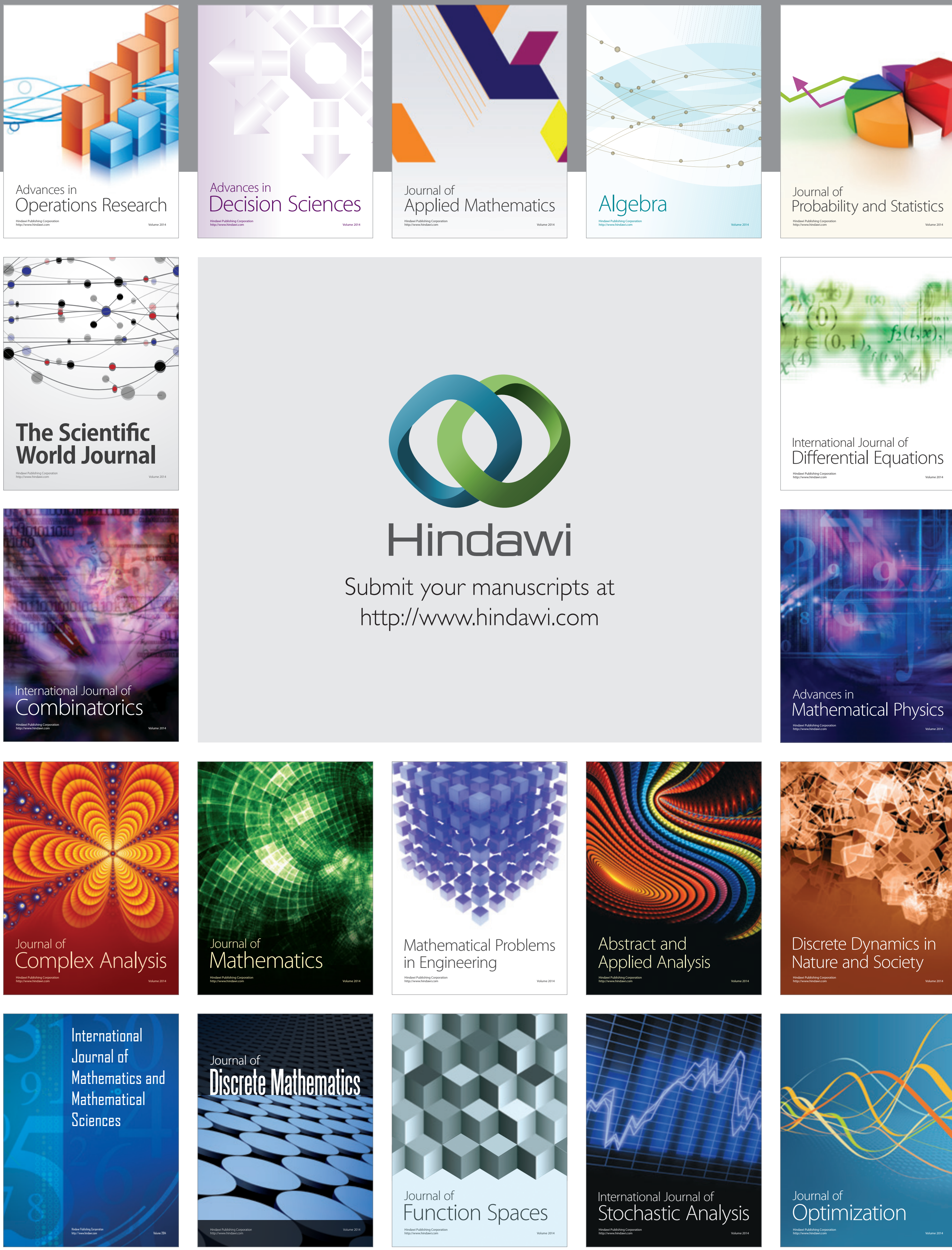
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