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An iterated version of Lavrent'iev's method for ill-posed equations with approximately specified data

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Abstract — An iterated version of the Lavrentiev's method, in the setting of a Banach space, is suggested for obtaining stable approximate solutions for the ill-posed operator equation $Au = v$, when the data A and v are known only approximately. In the setting of a Hilbert space with appropriate a priori parameter choice, the suggested procedure yields order optimal error estimates. An iterated version of Tikhonov regularization yielding order optimal error estimate is a special case of the procedure. The assumption on the approximating operators show that the finite dimensional system arising out of it would be of smaller size for larger iterates. This aspect is compared with an assumption of [3] for a degenerate kernel method for integral equations of the first kind.

1. INTRODUCTION

Many inverse problems in science and engineering can be modelled as an ill-posed operator equation

$$Tx = y \tag{1.1}$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y with its range $R(T)$ not closed in Y (c. f. [1, 3]). A prototype of such an equation is the Fredholm integral equation of the first kind,

$$\int_a^b k(s,t)x(t)\,\mathrm{d}t = y(s), \qquad a \le s \le b \tag{1.2}$$

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with non-degenerate kernel $k(\cdot\,,\cdot)$. In this case the operator $K : L^2[a,b] \to L^2[a,b]$, defined by

$$(Kx)(s) = \int_a^b k(s,t)x(t)\,\mathrm{d}t, \qquad a \le s \le b \tag{1.3}$$

is a compact operator of infinite rank, and therefore the integral equation $Kx = y$ is ill-posed.

Regularization procedures are employed to obtain stable approximate solutions for an ill-posed equation (1.1). Tikhonov regularization is one of the widely used such procedure. In Tikhonov regularization, one solves the equation

$$(T^*T + \alpha I)x_\alpha = T^*y. \tag{1.4}$$

If the available data $\tilde{y}$ is an approximation to the actual data y, then one solves the equation

$$(T^*T + \alpha I)\tilde{x}_\alpha = T^*\tilde{y}. \tag{1.5}$$

in place of (1.4). Suppose $y \in D(T^\dagger) = R(T) + R(T)^\perp$, the domain of the Moore–Penrose generalized inverse $T^\dagger$ of T, and $\tilde{y} \in Y$ is such that $\|y - \tilde{y}\| \le \delta$ for a known error level $\delta > 0$. Then we have (see [2]):

- $x_\alpha \to \hat{x} = T^\dagger y$ as $\alpha \to 0$,
- $\|x_\alpha - \tilde{x}_\alpha\| \le \delta/\sqrt{\alpha}$,
- $\hat{x} = (T^*T)^\nu x_0$, $0 < \nu \le 1$, for some $x_0 \in X$, implies

  $$\|\hat{x} - x_\alpha\| \le \|x_0\|\alpha^\nu$$

  so that, with $\alpha = c_0\delta^{2/(2\nu+1)}$ for some $c_0 > 0$,

  $$\|\hat{x} - \tilde{x}_\alpha\| \le (c_0\|x_0\| + 1/\sqrt{c_0})\delta^{2\nu/(2\nu+1)}. \tag{1.6}$$

  Also, the above rate $O(\delta^{2\nu/(2\nu+1)})$ is optimal.

Recall that $\hat{x} = T^\dagger y$ is the unique solution in $N(T)^\perp$ of the operator equation

$$T^*Tx = T^*y,$$

and the operator $T^*T : X \to X$ is positive and self-adjoint. In case the given operator T itself is positive and self adjoint, then one may use the Lavrentiev's method in which one solves the equations

$$(T + \alpha I)u_\alpha = y \tag{1.7}$$

and

$$(T + \alpha I)\tilde{u}_\alpha = \tilde{y} \tag{1.8}$$

in place of (1.4) and (1.5) respectively. In this case it is known [10] that if $y \in R(T)$, then

- $u_\alpha \to \hat{x}$ as $\alpha \to 0$,
- $\|u_\alpha - \tilde{u}_\alpha\| \le \delta/\alpha$, and
- $\hat{x} = T^\nu u_0$, $0 < \nu \le 1$, for some $u_0 \in X$, implies

$$\|\hat{x} - u_\alpha\| \le \|u_0\|\alpha^\nu$$

so that, with $\alpha = c_0\delta^{1/(\nu+1)}$ for some $c_0 > 0$,

$$\|\hat{x} - \tilde{x}_\alpha\| \le (c_0\|x_0\| + 1/c_0)\delta^{\nu/(\nu+1)}. \tag{1.9}$$

It is also known that the above rate $O(\delta^{\nu/(\nu+1)})$ is optimal.

The purpose of this paper is to investigate an iterated version of the Lavrentiev's method in the setting of a Banach space to obtain stable approximations for the solution of an ill-posed operator equation when both the operator involved (which represents the modelling of the problem) and the data, the right hand side of the equation, are known only approximately. If the space is a Hilbert space, then we show, under apprορriate a priori choice of the parameter, that the method yields order optimal error estimate. An iterated verison of Tikhonov regularization yielding order optimal error estimate is a special case of the procedure.

The operator equation under discussion is

$$Au = v, \qquad v \in R(A) \tag{1.10}$$

where $A : X \to X$ is a bounded linear operator on a Banach space X with its range $R(A)$ not necessarily closed in X. It is a consequence of Bounded Inverse Theorem that if $R(A)$ is not closed, then A can not have a continuous inverse, so that in such case the above equation is ill-posed.

Regularization of the above equation has been studied by Schock [9] under the assumptions

(a) $(-\infty, 0) \subseteq \rho(A)$;

(b) $\exists M > 0$ such that $\|(A + \alpha I)^{-1}\| \le M/\alpha$ for all $\alpha > 0$;

(c) $R(A)$ dense in X.

Clearly the assumption (a) implies that the equation

$$(A + \alpha I)\tilde{u}_\alpha = \tilde{v}, \qquad \alpha > 0 \tag{1.11}$$

is uniquely solvable for every $\tilde{v} \in X$. Thus the above equation is well-posed. In applications $\tilde{v} \in X$ is the availbale data in place of v with some known error level $\delta > 0$, i.e., $\|v - \tilde{v}\| \le \delta$.

Note that the conditions (a), (b), (c) on A are satisfied if X is a Hilbert space and $A : X \to X$ is a positive operator. Thus in this case, the regularized equation (1.11) is the well-studied Lavrentiev's regularized equation. The above consideration includes the case of the normal equation $T^*Tx = T^*y$ associated with an ill-posed operator equation $Tx = y$, where $T : X \to Y$ is a bounded linear operator between (possibly different) Hilbert spaces with its range not necessarily closed in Y.

It is to be remarked that consideration of ill-posed equations in the setting of a Banach space is important in view of its applications to Abel integral equations [7, 8].

In the second section we show that the conditions (a)–(c) assumed on A imply that A is injective, and hence the equation (1.10) is uniquely solvable for every $v \in X$. Thus results of [10] are valid without explicit assumption of injectivity. For the sake of completion of exposition we include these results along with their proofs.

In applications, it is also the case that the operator A is either not available exactly, or to obtain numerical solutions it is often necessary to approximate the operator using a sequence of finite rank operators. In many such cases, for example, projection based methods and degenerate kernel methods for integral equations, one has a sequence (A_n) of bounded operators such that

$$\|A - A_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus in place of (1.11), the equation to be solved is

$$(A_n + \alpha I)\tilde{u}_{\alpha,n} = \tilde{v}, \qquad \alpha > 0 \tag{1.12}$$

for large enough n. In this context it is to be mentioned that Groetsch [3] considered a degenerate kernel method for the case of Tikhonov regularization of the integral equation (1.2) wherein the A_n is obtained by approximating the kernel of the integral operator K^*K by a convergent quadrature rule. The results of Groetsch have been generalized by Nair [5] and Nair and Schock [6]. In order to obtain the optimal result, a condition on (A_n) required by Groetsch [3] is

$$\|A - A_n\| = O(\alpha^2).$$

A natural question is whether we can modify (1.12) so that the new procedure requires small n than the former. The present paper is important in this context.

We introduce an iterated form of (1.12), namely

$$(A_n + \alpha I)u_{\alpha,n}^{(j)} = \tilde{v} + (A_n - A)u_{\alpha,n}^{(j-1)}, \qquad j = 1, \dots, k$$

with $u_{\alpha,n}^{(0)} = 0$. We show that the requirement on A_n for the existence, convergence and error estimates for the kth iterate is

$$\|A - A_n\| = O(\alpha^{1+1/k}).$$

Applications of the main theorems in second and third sections are considered in fourth and fifth sections for Levrentiev's regularization and Tikhonov regularization respectively, and optimal order estimates are derived under suitable a priori choice of the parameter.

2. REGULARIZATION IN BANACH SPACES

Let X be a complex Banach space and $A : X \to X$ be a bounded linear operator such that

(a) $(-\infty, 0) \subseteq \rho(A)$;

(b) $\exists M > 0$ such that $\|(A + \alpha I)^{-1}\| \le M/\alpha$ forall $\alpha > 0$;

(c) $R(A)$ is dense in X.

For $v \in X$, we consider the operator equation

$$Au = v.$$

First we prove a basic result.

Theorem 2.1. *For $\alpha > 0$, let $R_\alpha = \alpha(A + \alpha I)^{-1}$. Then we have the following.*

(i) If $v \in R(A)$ with $v = Au,\ u \in X$, then

$$\|R_\alpha v\| \le \alpha(1 + M)\|u\|.$$

(ii) For every $v \in X$,

$$R_\alpha v \to 0 \quad \textit{as} \quad \alpha \to 0.$$

(iii) A is injective.

Proof. (i) Let $v = Au$, $u \in X$. Then, for every $\alpha > 0$, we have

$$R_\alpha v = \alpha(A + \alpha)^{-1} Au = \alpha u - \alpha^2 (A + \alpha)^{-1} Au.$$

Therefore, using the assumption (b),

$$\|R_\alpha v\| \le \alpha(1 + M)\|u\|.$$

(ii) If $v \in R(A)$, then by the result in'(i), $R_\alpha v \to 0 \quad \text{as} \quad \alpha \to 0$. Also, by the assumption (b), $\|R_\alpha\| \le M$ for every $\alpha > 0$. Hence, using the assumption (c), it follows that

$$R_\alpha v \to 0 \quad \text{as} \quad \alpha \to 0$$

for every $v \in X$.

(iii) Suppose $u \in X$ is such that $Au = 0$. Then we have

$$(A + \alpha I)u = \alpha u,$$

so $u = R_\alpha u$, and by (ii), $u = R_\alpha u \to 0$ as $\alpha \to 0$. Hence $u = 0$. This proves that A is injective. $\square$

Now let $v \in R(A)$. Then by the above theorem, there exists a unique $\hat{u} \in X$ such that

$$A\hat{u} = v \tag{2.1}$$

and by the assumption (a), for every $\alpha > 0$, there exists a unique $u_\alpha \in X$ such that

$$(A + \alpha I)u_\alpha = v. \tag{2.2}$$

Theorem 2.2 (Schock). *For $v \in R(A)$, let $\hat{u}$, u_α for $\alpha > 0$ be as in* (2.1) *and* (2.2) *respectively. Then*

$$\|\hat{u} - u_\alpha\| \to 0 \quad \text{as} \quad \alpha \to 0.$$

If $\hat{u} \in R(A)$ with $\hat{u} = Aw$, $w \in X$, then

$$\|\hat{u} - u_\alpha\| \leq \alpha(1 + M)\|w\|.$$

Proof. Follows from Theorem 2.1 by observing that

$$\hat{u} - u_\alpha = \alpha(A + \alpha I)^{-1}\hat{u} = R_\alpha \hat{u}.$$

□

By the assumption (c) and Theorem 2.1(iii), it follows from Bounded Inverse Theorem, that

$$A^{-1} : R(A) \to X \quad \text{is continuous} \quad \Longleftrightarrow \quad R(A) = X.$$

Thus equation (2.1) is ill-posed whenever $R(A) \neq X$, whereas (2.2) is a well-posed equation.

Theorem 2.2 shows that u_α is an approximation of $\hat{u}$ for small enough α, and

$$\|\hat{u} - u_\alpha\| = O(\alpha)$$

whenever $\hat{u} \in R(A)$.

In applications the data $v \in R(A)$ may not be available exactly; one may have an approximation $\tilde{v}$ of v. In such situation what one considers in place of the equation (2.2) is

$$(A + \alpha I)\tilde{u}_\alpha = \tilde{v}. \tag{2.3}$$

Theorem 2.3 (Schock). *Let $\hat{u}$ and u_α and $\tilde{u}_\alpha$, $\alpha > 0$, be as in* (2.1), (2.2) *and* (2.3) *respectively. Then*

$$\|\hat{u} - \tilde{u}_\alpha\| \leq \|\hat{u} - u_\alpha\| + \frac{M}{\alpha}\|v - \tilde{v}\|.$$

If $\hat{u} \in R(A)$, $\|v - \tilde{v}\| \leq \delta$ for a given $\delta > 0$ and $\alpha = c_0\sqrt{\delta}$ for some constant $c_0 > 0$, then

$$\|\hat{u} - \tilde{u}_\alpha\| \leq [(1 + M)\|w\| + c_0 M]\sqrt{\delta},$$

where $\hat{u} = Aw$, $w \in X$.

Proof. From (1.4) and (2.3) we have

$$u_\alpha - \tilde{u}_\alpha = (A + \alpha I)^{-1}(v - \tilde{v})$$

so that

$$\|u_\alpha - \tilde{u}_\alpha\| \le \frac{M}{\alpha}\|v - \tilde{v}\|.$$

Hence

$$\|\hat{u} - \tilde{u}_\alpha\| \le \|\hat{u} - u_\alpha\| + \frac{M}{\alpha}\|v - \tilde{v}\|.$$

The rest of the results follow from Theorem 2.2. □

3. ITERATED REGULARIZED APPROXIMATION PROCEDURE

Let $A : X \to X$ be a bounded linear operator on a complex Banach space X satisfying the assumptions (a), (b), (c) of Section 2. Without loss of generality, we assume that $M = 1$. Let (A_n) be a sequence of bounded linear operators on X such that

$$\|A - A_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Lemma 3.1. *For $0 < \alpha \le \|A\|$ and $n \in \mathbf{N}$, let $D_{\alpha,n} = (A + \alpha I)^{-1}(A_n - A)$. If*

$$\|A - A_n\| \le \frac{\alpha}{2}$$

for some positive integer k, then we have the following

(i) $-1 \notin \sigma(D_{\alpha,n})$;

(ii) $-\alpha \notin \sigma(A_n)$;

(iii) $\|(I + D_{\alpha,n})^{-1} D_{\alpha,n}\| \le \frac{2}{\alpha}\|A - A_n\|$.

Proof. The assumption on α and A_n implies that $\|A - A_n\| \le \alpha/2$, so that

$$\|D_{\alpha,n}\| \le \frac{\|A - A_n\|}{\alpha} \le \frac{1}{2}.$$

Hence $-1 \notin \sigma(D_{\alpha,n})$, so that from the relation

$$A_n + \alpha I = (A + \alpha I)(I + D_{\alpha,n})$$

it follows that $-\alpha \notin \sigma(A_n)$. Also, we have

$$\|(I + D_{\alpha,n})^{-1} D_{\alpha,n}\| \le \frac{\|D_{\alpha,n}\|}{1 - \|D_{\alpha,n}\|} \le 2\|D_{\alpha,n}\| \le \frac{2}{\alpha}\|A - A_n\|.$$

□

Theorem 3.2. *For $v \in X$, let $\hat{u}$, u_α and $\tilde{u}_\alpha$ be as in* (2.1), (2.2), *and* (2.3) *respectively, and let $0 < \alpha \le \|A\|$. Let $n = n(\alpha)$ be a positive integer such that*

$$\|A - A_n\| \le \frac{\alpha}{2}. \tag{3.1}$$

Then the equation

$$(A_n + \alpha I)u_{\alpha,n}^{(j)} = \tilde{v} + (A_n - A)u_{\alpha,n}^{(j-1)}, \qquad j = 1, \ldots, k \tag{3.2}$$

with $u_{\alpha,n}^{(0)} = 0$ *is uniquely solvable for* $j = 1, 2, \ldots$. *If, in addition,*

$$\|A - A_n\| \leq \frac{\alpha}{2}\Big(\frac{\alpha}{\|A\|}\Big)^{1/k} \tag{3.3}$$

for some $k \in \mathbf{N}$, *then*

$$\|\hat{u} - u_{\alpha,n}^{(k)}\| \leq 2\big(\|\hat{u} - u_\alpha\| + \|u_\alpha - \tilde{u}_\alpha\|\big).$$

Proof. By Lemma 3.1 (ii), $-\alpha \notin \sigma(A_n)$ so that equation (3.2 is uniquely solvable for $j = 1, 2, \ldots$. From (3.2), with $j = k$, we have

$$u_{\alpha,n}^{(k)} = \tilde{u}_\alpha + D_{\alpha,n}u_{\alpha,n}^{(k-1)} - D_{\alpha,n}u_{\alpha,n}^{(k)}$$

so that

$$(I + D_{\alpha,n})u_{\alpha,n}^{(k)} = \tilde{u}_\alpha + D_{\alpha,n}u_{\alpha,n}^{(k-1)}.$$

Also,

$$(I + D_{\alpha,n})\tilde{u}_\alpha = \tilde{u}_\alpha + D_{\alpha,n}\tilde{u}_\alpha.$$

Hence

$$\tilde{u}_\alpha - u_{\alpha,n}^{(k)} = Q_{\alpha,n}(\tilde{u}_\alpha - u_{\alpha,n}^{(k-1)} = Q_{\alpha,n}^k \tilde{u}_\alpha, \tag{3.4}$$

where

$$Q_{\alpha,n} = (I + D_{\alpha,n})^{-1}D_{\alpha,n}.$$

Now using the relation

$$(A + \alpha I)u_\alpha = v = A\hat{u},$$

it follows that

$$u_\alpha = \frac{1}{\alpha}A(\hat{u} - u_\alpha)$$

so that

$$\tilde{u}_\alpha = \frac{1}{\alpha}A(\hat{u} - u_\alpha) - (u_\alpha - \tilde{u}_\alpha).$$

Thereore from (3.4), we have

$$\tilde{u}_\alpha - u_{\alpha,n}^{(k)} = \frac{1}{\alpha}Q_{\alpha,n}^k A(\hat{u} - u_\alpha) - Q_{\alpha,n}^k(u_\alpha - \tilde{u}_\alpha).$$

Thus,

$$\begin{aligned} \hat{u} - u_{\alpha,n}^{(k)} &= (\hat{u} - u_\alpha) + (u_\alpha - \tilde{u}_\alpha) + (\tilde{u}_\alpha - u_{\alpha,n}^{(k)}) \\ &= \Big(I + \frac{1}{\alpha}Q_{\alpha,n}^k A\Big)(\hat{u} - u_\alpha) + \big(I - Q_{\alpha,n}^k\big)(u_\alpha - \tilde{u}_\alpha) \end{aligned}$$

so that

$$\|\hat{u} - u_{\alpha,n}^{(k)}\| = \Big(1 + \frac{1}{\alpha}\|Q_{\alpha,n}^k A\|\Big)\|\hat{u} - u_\alpha\| + \big(1 + \|Q_{\alpha,n}\|^k\big)\|u_\alpha - \tilde{u}_\alpha\|. \tag{3.5}$$

Now, Lemma 3.1(iii) and the assumption $0 < \alpha \le \|A\|$ imply that

$$\|Q_{\alpha,n}\| \le \frac{2}{\alpha}\|A - A_n\| \le 1 \quad \text{and} \quad \|Q_{\alpha,n}\| \le \frac{2}{\alpha}\|A - A_n\| \le \Big(\frac{\alpha}{\|A\|}\Big)^{1/k}$$

so that

$$\frac{1}{\alpha}\|Q_{\alpha,n}^k A\| \le \frac{\|Q_{\alpha,n}\|^k\|A\|}{\alpha} \le 1.$$

Thus

$$\|\hat{u} - u_{\alpha,n}^{(k)}\| \le 2\big(\|\hat{u} - u_\alpha\| + \|u_\alpha - \tilde{u}_\alpha\|\big).$$

□

Remarks.

(a) We observe that

$$\begin{aligned}\|Q_{\alpha,n}A\| &= \|(I + D_{\alpha,n})^{-1}D_{\alpha,n}A\| \\ &\le 2\|D_{\alpha,n}A\| \\ &= 2\|(A + \alpha I)^{-1}(A - A_n)A\| \\ &\le \frac{2}{\alpha}\|(A - A_n)A\|,\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{\alpha}\|Q_{\alpha,n}^k A\| &\le \frac{1}{\alpha}\|Q_{\alpha,n}\|^{k-1}\|Q_{\alpha,n}A\| \\ &\le \frac{1}{\alpha}\Big(\frac{2}{\alpha}\|A - A_n\|\Big)^{k-1}\|Q_{\alpha,n}A\| \\ &\le \frac{2}{\alpha^2}\|(A - A_n)A\|\Big(\frac{2}{\alpha}\|A - A_n\|\Big)^{k-1}.\end{aligned}$$

Using this estimate in relation (3.5), we can replace the condition (3.3) in Theorem 3.2 by

$$\frac{2}{\alpha^2}\|(A - A_n)A\|\Big(\frac{2}{\alpha}\|A - A_n\|\Big)^{k-1} \le 1. \tag{3.6}$$

Note that

$$\frac{2}{\alpha^2}\|(A - A_n)A\|\Big(\frac{2}{\alpha}\|A - A_n\|\Big)^{k-1} \le \frac{\|A\|}{\alpha}\Big(\frac{2}{\alpha}\|A - A_n\|\Big)^{k}.$$

There are examples for which

$$\|(A - A_n)A\| << \|A\| \; \|A - A_n\|.$$

For instance, consider the case when X is a Hilbert space, $A : X \to X$ is a compact operator and (P_n) is a sequence of orthogonal projections such that

$\|x - P_n x\| \to 0$ as $n \to \infty$ for every $x \in X$. Taking $A_n = AP_n$, we have $\|A - A_n\| = \|A(I - P_n)\|$ and

$$\|(A - A_n)A\| = \|A(I - P_n)A\| \le \|A(I - P_n)\| \, \|(I - P_n)A\|.$$

Thus, in this case,

$$\|(A - A_n)A\| \le \|(I - P_n)A\| \, \|A - A_n\|.$$

Note that, by the assumptions on A and (P_n),

$$\|A - A_n\| \to 0 \quad \text{and} \quad \|(I - P_n)A\| \to 0$$

as $n \to \infty$. Hence, the condition (3.6) take the form

$$\frac{\|(I - P_n)A\|}{\alpha} \Big(\frac{2}{\alpha}\|A - A_n\|\Big)^k \le 1.$$

Clearly, this assumption is weaker that (3.3).

In case the operator A is self adjoint as well, then $\|(I - P_n)A\| = \|A - A_n\|$ so that the above condition is same as

$$\|A - A_n\| \le \frac{\alpha}{2^{k/(k+1)}}$$

which is guaranteed by the initial assumption $\|A - A_n\| \le \alpha/2$. Thus, in this particular example, the additional assumption (3.3) is redundant.

4. APPLICATION TO LAVRENTIEV'S REGULARIZATION

Let X be a Hilbert space, $A : X \to X$ be a positive self adjoint operator with its range $R(A)$ dense in X, and $v \in R(A)$.

Now the following result can be deduced from Theorem 3.1 and the results listed in Section 1.

Theorem 4.1. *Let (A_n) be a sequence of bounded operators on X such that*

$$\|A - A_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Assume that

- *$u \in R(A^\nu)$ with $u = A^\nu u_0$ for some $\nu \in (0, 1]$ and $u_0 \in Xm$*
- *$0 < \delta \le (\|A\|/c)^{1+\nu}$ for some $c \ge 1$, and*
- *$n \in \mathbb{N}$ is such that*

$$\|A - A_n\| \le \frac{c}{2}\Big(\frac{c}{\|A\|}\Big)^{1/k} \delta^{(k+1)/k(\nu+1)}$$

for some $k \in \mathbb{N}$. Then, with $\alpha = c\delta^{1/(\nu+1)}$,

$$\|u - u^{(k)}_{\alpha,n}\| \le 2\big(c\|u_0\| + 1/c\big)\delta^{\nu/(\nu+1)}.$$

5. APPLICATION TO TIKHONOV REGULARIZATION

Let X and Y be Hilbert spaces, $T : X \to Y$ be a bounded linear operator with its range $R(T)$ dense in Y and $y \in R(T)$. Let $\delta > 0$ and $\tilde{y} \in Y$ be such that $\|y - \tilde{y}\| \le \delta$. For $\alpha > 0$, let x_α and $\tilde{x}_\alpha$ be as in (1.4) and (1.5) respectively. Taking

$$A = T^*T, \quad v = T^*y \quad \text{and} \quad \tilde{v} = T^*\tilde{y}$$

the following result can be deduced from Theorem 3.1 and the results listed in Section 1.

Theorem 5.1. *Let (A_n) be a sequence of bounded linear operators on X such that*

$$\|A - A_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Assume that

- $\hat{x} \in R(A^\nu)$ *with* $\hat{x} = A^\nu x_0$ *for some* $\nu \in (0, 1]$ *and* $x_0 \in X$,
- $0 < \delta \le (\|A\|/c)^{1+2\nu}$ *for some constant* $c \ge 1$, *and*
- $n \in \mathbb{N}$ *is such that*

$$\|A - A_n\| \le \frac{c}{2}\left(\frac{c}{\|A\|}\right)^{1/k} \delta^{2(k+1)/k(2\nu+1)}$$

for some $k \in \mathbb{N}$. *Then, with* $\alpha = c\delta^{2/(2\nu+1)}$,

$$\|\hat{x} - u_{\alpha,n}^{(k)}\| \le 2(c\|x_0\| + 1/\sqrt{c})\delta^{2\nu/(2\nu+1)}.$$

6. CONCLUDING REMARKS

- We observe that, with appropriate a priori choice of the parameter α, the estimates in Theorems 4.1 and 5.1 are of the same order as in (1.9) and (1.6) respectively, which are obtained with no perturbation of the the operator.

- As mentioned in the introduction, Groetsch [3] considered a degenerate kernel method for the case of Tikhonov regularization of the integral equation (1.2) wherein the A_n is obtained by approximating the kernel of the integral operator K^*K by a convergent quadrature rule. He obtained the optimal order $O(\delta^{2/3})$ for $y \in R(K^*K)$ under the assumption

$$\alpha = c\delta^{2/3} \quad \text{and} \quad \|A - A_n\| = O(\delta^{4/3})$$

whereas the optimal result in Theorem 5.1 holds under weaker assumption on (A_n). For example, the order $O(\delta^{2/3})$ is obtained for $y \in R(K^*K)$ under the assumption

$$\alpha = c\delta^{2/3} \quad \text{and} \quad \|A - A_n\| = O(\delta^{2(k+1)/3k}).$$

This implies that the system required for solving the kth iterate is smaller for $k > 1$.

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