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IFAC-PapersOnLine 48-13 (2015) 001–006

Alternative Passive Maps for Infinite-Dimensional Systems Using Mixed-Potential Functions

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Abstract: This paper aims at developing a Brayton-Moser analogue of an infinite-dimensional system in the port-Hamiltonian framework, defined with respect to a Stokes-Dirac structure. It is shown that such a formulation leads to defining alternative passive maps, which differ from those in the port-Hamiltonian framework via a "power-like" function called the mixed-potential function. This mixed-potential function can also be used for stability analysis. We present our results for a general port-Hamiltonian system, with Maxwell's equations and the transmission line, with nonzero boundary conditions, as examples.

Keywords: Distributed-parameter systems, infinite-dimensional systems, Brayton-Moser equations, gradient systems, passivity, port-Hamiltonian systems.

1. INTRODUCTION

Energy-based methods for modeling and control of complex physical systems has been an active area of research for the past two decades. In particular the port-Hamiltonian-based formulation has proven to be effective in modeling and control of complex physical systems from several domains, both finite- and infinite-dimensional. (Duindam et al. (2009)). Port-Hamiltonian systems are inherently passive with the Hamiltonian, which is assumed to be bounded from below, serving as the storage function and the port variables are power-conjugate. This resulted in the development of so-called "Energy Shaping" methods for control of physical systems. In some cases the standard power-conjugate port variables do not necessarily help in achieving the control objectives due to the dissipation obstacle, motivating the search for alternative passive maps. One possible alternative that has been explored extensively in the finite-dimensional case is the Brayton-Moser (BM) framework for modeling electrical RLC networks (Brayton and Moser. (1964a)) and Brayton and Moser. (1964b)). The BM framework been successfully adapted towards towards analysing passivity of RLC circuits (Jeltsema et al. (2003)) and for control of physical systems via "Power Shaping" . For further details on the various energy-based and power-based modeling techniques, the reader is refered to (Jeltsema and Scherpen (2009)).

The majority of the literature on the BM framework restricts to the finite-dimensional case only. One of the first results in the context of infinite-dimensional systems was presented by Brayton and Miranker. In (Brayton and Miranker. (1964)), a stability theory using the BM framework is developed for a transmission line that is connected to a nonlinear load. However the proposed Lyapunov functional does not preserve the pseudo-gradient structure of the system, which is essential for boundary control, and the generation of alternative passive maps along the lines of the finite-dimensional case is not straightforward. In (Jeltsema and van der Schaft (2007)), the authors present a BM formulation of the Maxwell equations. However, only zero boundary conditions are considered. Recently, in (Pasumarthy et al. (2014)), we have presented some results on the control by interconnection of a transmission line by "Power Shaping" in the BM framework.

In this paper, we present a BM analogue of an infinite-dimensional port-Hamiltonian systems, defined with respect to a constant Stokes-Dirac structure (van der Schaft and Maschke (2002)). The main results include the derivation of a new passivity property for infinite-dimensional systems by "differentiating" one of the port variables (possibly the boundary port) and using a storage function that is directly related to the power of the system, while preserving the BM structure. This new storage function is also instrumental in analysing the stability of the system. The results are presented for a general port-Hamiltonian system, with Maxwell's equations, under zero boundary conditions and for transmission line, with non zero boundary conditions, as examples.

The remainder of the paper is organized as follows. In Section 2, we defined the Stokes-Dirac structure, and its Brayton Moser formulation. In Sections 3 and 4, we use the BM framework to model the Maxwell's equations and the telegrapher's equations of a transmission line respectively. In both cases, we derive admissible pairs and analyze stability with zero boundary conditions. In case of Transmission line we consider nonzero boundary

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.10.205

conditions and present a family of "admissible pairs" and derive new passivity properties.

Notations and Math Preliminaries

Let Z be an n dimensional Riemannian manifold with a smooth $(n-1)$ dimensional boundary ∂Z. $\Omega^k(Z)$, $k = 0, 1, \ldots, n$ denotes the space of all exterior k-forms on Z. The dual space $\left(\Omega^k(Z)\right)^*$ of $\Omega^k(Z)$ can be identified with $\Omega^{n-k}(Z)$ with a pairing between $\alpha \in \Omega^k(Z)$ and $\beta \in \left(\Omega^k(Z)\right)^*$ given by $\langle \beta | \alpha \rangle = \int_Z \beta \wedge \alpha$. Here $\wedge$ is the usual wedge product of differential forms, resulting in the n-form $\beta \wedge \alpha$. Similar pairings can be established between the boundary variables.

The operator d denotes the exterior derivative and maps k forms on Z to $k+1$ forms on Z. The Hodge star operator $*$ (corresponding to Riemannian metric on Z) converts p forms to $(n-p)$ forms. Given $\alpha, \beta \in \Omega^k(Z)$ and $\gamma \in \Omega^l(Z)$, the wedge product $\alpha \wedge \gamma \in \Omega^{k+l}(Z)$. We additionally have the following properties: [1]

$$\alpha \wedge \gamma = (-1)^{kl} \gamma \wedge \alpha \ , \quad **\alpha = (-1)^{k(n-k)} \alpha, \tag{1}$$

$$\int_z \alpha \wedge *\beta = \int_z \beta \wedge *\alpha, \tag{2}$$

$$\mathrm{d}\left(\alpha \wedge \gamma\right) = \mathrm{d}\alpha \wedge \gamma + (-1)^k \alpha \wedge \mathrm{d}\gamma. \tag{3}$$

Given a functional $H(\alpha_p, \alpha_q)$, we compute its variation as

$$\begin{aligned} \partial H &= H(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q) - H(\alpha_p, \alpha_q) \\ &= \int_z \left[\delta_p H \wedge \partial\alpha_p + \delta_q H \wedge \partial\alpha_q\right], \end{aligned} \tag{4}$$

where $\alpha_p,\ \partial\alpha_p \in \Omega^p(Z)$ and $\alpha_q,\ \partial\alpha_q \in \Omega^q(Z)$ and $\delta_p H \in \Omega^{n-p}(Z)$ and $\delta_p H \in \Omega^{n-q}(Z)$ are variational derivative of $H(\alpha_p, \alpha_q)$ with respective to α_p and α_q. Further, the time derivative of $H(\alpha_p, \alpha_q)$ is

$$\frac{dH}{dt} = \int_z \left(\delta_p H \wedge \frac{\partial \alpha_p}{\partial t} + \delta_q H \wedge \frac{\partial \alpha_q}{\partial t} \right).$$

Let $G : \Omega^{n-p}(Z) \to \Omega^{n-p}(Z)$ and $R : \Omega^{n-q}(Z) \to \Omega^{n-q}(Z)$, we call $G \geq 0$, if and only if $\forall \alpha_p \in \Omega^p(Z)$

$$\int_Z \left(\alpha_p \wedge *G\alpha_p\right) \geq 0$$

G is said to be symmetric if $\langle \alpha_p | G\alpha_p \rangle = \langle G\alpha_p | \alpha_p \rangle$.

Lastly, given $f(z,t) : Z \times \mathbb{R} \to \mathbb{R}$, we denote $\frac{\partial f}{\partial t}(z,t)$ as f_t, similarly $\frac{\partial f}{\partial z}(z,t)$ as f_z.

2. FROM INFINITE-DIMENSIONAL PORT HAMILTONIAN SYSTEMS TO BRAYTON MOSER EQUATIONS

2.1 Infinite-Dimensional Port-Hamiltonian Systems

Define the linear space $\mathcal{F}_{p,q} = \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z)$ called the space of flows and $\mathcal{E}_{p,q} = \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z)$, the space of efforts, with integers p, q satisfying $p + q = n + 1$. Then, the linear subspace $\mathcal{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$

$$\mathcal{D} = \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \;\middle|\; \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} *G & (-1)^r \mathrm{d} \\ \mathrm{d} & *R \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ \mathrm{e}_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p|_{\partial Z} \\ e_q|_{\partial Z} \end{bmatrix} \right\},$$

with $r = pq + 1$, is a Stokes-Dirac structure with dissipation, (van der Schaft and Maschke (2002)) with respect to the bilinear form

$$\begin{aligned} &\langle\langle \left(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1\right), \left(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2\right) \rangle\rangle = \\ &\quad \langle e_p^2 | f_p^1 \rangle + \langle e_p^1 | f_p^2 \rangle + \langle e_q^2 | f_q^1 \rangle + \langle e_q^1 | f_q^2 \rangle + \langle e_b^2 | f_b^1 \rangle + \langle e_b^1 | f_b^2 \rangle. \end{aligned}$$

Consider a distributed-parameter port-Hamiltonian system on $\Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z)$, with energy variables $(\alpha_p, \alpha_q) \in \Omega^p(Z) \times \Omega^q(Z)$ representing two different physical energy domains interacting with each other. The total stored energy is defined as

$$H := \int_Z \mathcal{H} \in \mathbb{R},$$

where $\mathcal{H}$ is the Hamiltonian density (energy per volume element). Let $G \geq 0$ and $R \geq 0$ represent the dissipative terms in the system. Then, setting $f_p = -(\alpha_p)_t$ and $f_q = -(\alpha_q)_t$, and $e_p = \delta_p H$ and $e_q = \delta_q H$, the system

$$\begin{aligned} -\frac{\partial}{\partial t} \begin{bmatrix} \alpha_p \\ \alpha_q \end{bmatrix} &= \begin{bmatrix} *G & (-1)^r \mathrm{d} \\ \mathrm{d} & *R \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix}, \\ \begin{bmatrix} f_b \\ \mathrm{e}_b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix}, \end{aligned} \tag{5}$$

with $r = pq + 1$, represents an infinite-dimensional port-Hamiltonian system.

The time-derivative of the Hamiltonian is computed as

$$\frac{dH}{dt} \leq \int_{\partial Z} e_b \wedge f_b.$$

This means that the increase in energy in the spatial domain is less than or equal to power supplied to the system through its boundary. This implies that the system is passive, with respect to the boundary variables, with storage function H, which is assumed to be bounded from below.

2.2 The Brayton-Moser Mixed-Potential

In the Brayton-Moser (BM) framework, the dynamics of a (finite-dimensional) RLC network can be written as

$$A(u)\dot{u} = P_u(u) \tag{6}$$

where u is the vector consisting of inductor currents and capacitor voltages, P_u is the gradient of some scalar function P called the mixed-potential function, and $A(u)$ is a non-singular matrix. See (Brayton and Moser. (1964a)) and Brayton and Moser. (1964b)) for more details.

We aim to write (5) in an equivalent BM form for infinite-dimensional systems. Let us (for now) assume that the relation between the energy and co-energy variables is linear and given as

$$\alpha_p = *\epsilon\ e_p \text{ and } \alpha_q = *\mu\ e_q. \tag{7}$$

[1] For details on the theory of differential forms we refer to (Abraham et al. (1988)))

where $\mu, \epsilon \in \mathbb{R}$. Applying the Hodge star operator to both sides of (5) and arranging terms using (7), we get

$$\begin{aligned} -\epsilon \dot{e}_p &= *\left((-1)^r \mathrm{d}e_q + G * e_p\right)(-1)^{(n-p)\times p}, \\ -\mu \dot{e}_q &= *\left(\mathrm{d}e_p + R * e_q\right)(-1)^{(n-q)\times q}. \end{aligned} \tag{8}$$

Let us first consider the case of a system that is lossless, i.e., when R and G are identically equal to zero in (5).

Define P to be a functional of the form

$$P := \int_Z e_q \wedge de_p.$$

Its variation is given as

$$\begin{aligned} \delta P &= P(e_p + \partial e_p, e_q + \partial e_q) - P \\ &= e_q \wedge \mathrm{d}\partial e_p + \partial e_q \wedge \mathrm{d}e_p + \cdots \end{aligned}$$

Using the relation $e_q \wedge \mathrm{d}\partial e_p = (-1)^{pq} \partial e_p \wedge \mathrm{d}e_q + (-1)^{n-q}\mathrm{d}\left(e_q \wedge \partial e_p\right)$, and the identity (4), we have

$$\delta_{e_q} P = \mathrm{d}e_q (-1)^{(n-q)\times q}, \quad \delta_{e_p} P = (-1)^{pq} \mathrm{d}e_p (-1)^{(n-p)\times p},$$

we can write (8) in the following BM-type of fashion:

$$\begin{bmatrix} -\mu & 0 \\ 0 & \epsilon \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} e_q \\ e_p \end{bmatrix} = \begin{bmatrix} *\delta_{e_q} P \\ *\delta_{e_p} P \end{bmatrix}. \tag{9}$$

To incorporate dissipation we proceed as follows. Consider instead a functional P defined as

$$P(e_p, e_q) = \int_Z e_q \wedge \mathrm{d}e_p + \frac{1}{2} R e_q \wedge * e_q - \frac{1}{2} G e_p \wedge * e_p \tag{10}$$

The variation in P is computed as

$$\begin{aligned} P &= e_q \wedge \mathrm{d}\partial e_p + \partial e_q \wedge \mathrm{d}e_p + \frac{1}{2}(e_q \wedge R * \partial e_q \\ &\quad + \partial e_q \wedge * e_q) - \frac{1}{2}(e_p \wedge G * \partial e_p + \partial e_p \wedge * e_p)) \\ &= \int_Z \partial e_q \wedge \mathrm{d}e_p + \partial e_p \wedge (-1)^{pq} \mathrm{d}e_q + \frac{1}{2}(e_q \wedge R * \partial e_q \\ &\quad + \partial e_q \wedge * e_q) - \frac{1}{2}(e_p \wedge G * \partial e_p + \partial e_p \wedge * e_p)) \\ &= \int_Z \partial e_q \wedge (\mathrm{d}e_p + R * e_q) + \partial e_p \wedge ((-1)^{pq} \mathrm{d}e_q - G * e_p) \end{aligned}$$

where we have used the relation $e_q \wedge \mathrm{d}\partial e_p = (-1)^{pq} \partial e_p \wedge \mathrm{d}e_q + (-1)^{n-q}\mathrm{d}\left(e_q \wedge \partial e_p\right)$, together with properties of the wedge form and the Hodge star operator defined in (2) and (3). Lastly, by making use of (4) we can write

$$\begin{bmatrix} \delta_{e_q} P \\ \delta_{e_p} P \end{bmatrix} = \begin{bmatrix} (\mathrm{d}e_p + R * e_q)(-1)^{(n-q)\times q} \\ ((-1)^{pq} \mathrm{d}e_q - G * e_p)(-1)^{(n-p)\times p} \end{bmatrix}, \tag{11}$$

whereas the form of the BM equations remain as in (9).

The system of equations (9) can be written in a concise way, similar to (6) as

$$A u_t = \delta_u P. \tag{12}$$

where $u = [e_p, e_q]^\top$ and $A = \mathrm{diag}(-\mu, \epsilon)$.

2.3 Boundary dynamics

The systems (12) can be interconnected to other systems via the boundary of the infinite-dimensional system, which can either be finite or infinite-dimensional in nature. To include dynamics we need to append the dynamics (12) to incorporate the boundary dynamics, i.e.,

$$\begin{bmatrix} A & 0 \\ 0 & A_b \end{bmatrix} \begin{bmatrix} u_t \\ u_t^b \end{bmatrix} = \begin{bmatrix} \delta_u P_d \\ \delta_{u^b} P_d^b \end{bmatrix}, \tag{13}$$

where u^b represents the states of the systems that are interconnected at the boundary and with a new mixed-potential function

$$P_d(e_p, e_q) = \int_Z P(e_p, e_q) + \int_{\partial Z} P^b(e_p, e_q),$$

with P^b representing the mixed-potential function associated with the boundary dynamics.

The variation in P_d is given by

$$\begin{aligned} \delta P_d = \int_Z \delta_{e_q} P \wedge \partial e_q + \delta_{e_p} P \wedge \partial e_p + \int_{\partial Z} \Big(\delta_{e_q} P^b \wedge \partial e_p \\ + \left(\delta_{e_p} P^b + (-1)^{(n-p)\times p} e_q \right) \wedge \partial e_p \Big) \end{aligned}$$

Now, with $U = [u, u^b]^\top$ and

$$\delta_U P_d = \begin{bmatrix} \delta_{e_q} P \\ \delta_{e_p} P \\ \delta_{e_q} P^b|_{\partial Z} \\ \left(\delta_{e_p} P^b + (-1)^{(n-p)\times p} e_q \right)|_{\partial Z} \end{bmatrix}, \tag{14}$$

the BM equations incorporating boundary dynamics can be written as

$$\mathcal{A} U_t = \delta_U P_d,$$

where $\mathcal{A} = \mathrm{diag}(A, A_b)$.

Once we have written down the dynamics in the BM framework, we can impose the question: does the mixed-potential function serve as a storage function (or a Lyaunov function) to infer (new) passivity (or equivalently, stability) properties of the system? In the remainder of the paper, we aim to answer this question with the aid of two examples.

3. EXAMPLE: MAXWELL'S EQUATIONS

Consider an electromagnetic medium with spatial domain $Z \subset \mathbb{R}^3$ with a smooth two-dimensional boundary ∂Z. The energy variables are the electric field induction $\mathcal{D} = \frac{1}{2}\mathcal{D}_{ij} z_i \wedge z_j$ and the magnetic field induction $\mathcal{B} = \frac{1}{2}\mathcal{B}_{ij} z_i \wedge z_j$ on Z. The associated co-energy variables are electric field intensity $\mathcal{E}$ and magnetic field intensity $\mathcal{H}$. These 1-forms are related to the energy variables (2-forms) through the constitutive relationships of the medium as

$$*\mathcal{D} = \epsilon \mathcal{E}, \;\; *\mathcal{B} = \mu \mathcal{H}, \tag{15}$$

where $\epsilon(z,t)$ and $\mu(z,t)$ denote the electric permittivity and the magnetic permeability, respectively. The Hamiltonian H can be written as

$$H(\mathcal{D}, \mathcal{B}) = \int_Z \frac{1}{2}\left(\mathcal{E} \wedge \mathcal{D} + \mathcal{H} \wedge \mathcal{B}\right). \tag{16}$$

Therefore, $\delta_{\mathcal{D}} H = \mathcal{E}$ and $\delta_{\mathcal{B}} H = \mathcal{H}$. Taking into account dissipation in the system, the dynamics can be written in the port-Hamiltonian form as

$$-\frac{\partial}{\partial t}\begin{bmatrix}\mathcal{D}\\ \mathcal{B}\end{bmatrix} = \begin{bmatrix}0 & -\mathrm{d}\\ \mathrm{d} & 0\end{bmatrix}\begin{bmatrix}\delta_{\mathcal{D}}H\\ \delta_{\mathcal{B}}H\end{bmatrix} + \begin{bmatrix}J_d\\ 0\end{bmatrix} = \begin{bmatrix}*\sigma & -\mathrm{d}\\ \mathrm{d} & 0\end{bmatrix}\begin{bmatrix}\delta_{\mathcal{D}}H\\ \delta_{\mathcal{B}}H\end{bmatrix} \quad (17)$$

where $*J_d = \sigma\mathcal{E}$, J_d denotes the current density and $\sigma(z,t)$ is the specific conductivity of the material. In addition, we define the boundary variables as $f_b = \delta_D H|_{\partial Z}$ and $e_b = \delta_B H|_{\partial Z}$. Hence, we obtain

$$\frac{d}{dt}H \leq \int_{\partial Z} \mathcal{H} \wedge \mathcal{E}.$$

For $n = 3$, $p = q = 2$, and $\alpha_p = \mathcal{D}$, $\alpha_q = \mathcal{B}$ with H given in (16), Maxwell's equations given in (17) forms a Stokes-Dirac structure.

3.1 The Brayton-Moser form of Maxwell's equations

In order to write the Maxwell's equations in BM form, we proceed in terms of the co-energy variables, i.e., $\mathcal{H}$ and $\mathcal{E}$. Define the corresponding mixed-potential functional as

$$P = \int_Z \mathcal{H} \wedge d\mathcal{E} - \frac{1}{2}\sigma\mathcal{E} \wedge *\mathcal{E}, \quad (18)$$

which yields the following form of Maxwell's equations in terms of the mixed potental

$$\begin{bmatrix}-\mu I_3 & 0\\ 0 & \epsilon I_3\end{bmatrix}\begin{bmatrix}\mathcal{H}_t\\ \mathcal{E}_t\end{bmatrix} = \begin{bmatrix}*d\mathcal{E}\\ -\sigma\mathcal{E} + *d\mathcal{H}\end{bmatrix} = \begin{bmatrix}*\delta_{\mathcal{H}}P\\ *\delta_{\mathcal{E}}P\end{bmatrix}. \quad (19)$$

A simple boundary condition, $(\mathcal{H} + *\sigma_d\mathcal{E})|_{\partial Z} = J^s$, can be incorporated with a mixed-potential function

$$P^b = \frac{1}{2}\int_{\partial Z} \mathcal{E} \wedge \sigma_d * \mathcal{E},$$

where σ_d is specific conductance at boundary. The dynamics (19), together with the boundary condition can be written as

$$\mathcal{A}U_t = \delta_U P_d + BJ^s,$$

with $U = \left[u, u^b\right]^\top$, $P_d = P + P^b$, $\mathcal{A} = \mathrm{diag}(A, A^b)$, $A = \mathrm{diag}(-\mu I_3, \epsilon I_3)$, $A^b = 0$, and $B = [0,0,0,-J^s]^\top$.

3.2 Admissible pairs and stability

To infer stability properties of the system (19), let us begin with the case of zero energy flow through the boundary of the system. The mixed-potential function (18) obtained using (10) is not positive definite. Hence, we cannot use it as Lyapunov or storage functional. Moreover, the rate of change of this function is computed as

$$\dot{P} = \int_Z \left(-\mu\dot{\mathcal{H}} \wedge *\dot{\mathcal{H}} + \epsilon\dot{\mathcal{E}} \wedge *\dot{\mathcal{E}}\right),$$

which clearly is not sign-definite. We thus need to look for other *admissible pairs* ($\tilde{\mathcal{A}}$, $\tilde{P}$) like in the case of finite-dimensional systems (Jeltsema et al. (2003)) that can be used prove stability of the system while preserving the dynamics of (19), or $\tilde{A}u_t = *\delta_u\tilde{P}$ for the more general case, of course with zero boundary energy flow. Moreover, the admissible pair should be such that the symmetric part of $\tilde{A}$ is negative semi-definite. This can be achieved in the following way (Brayton and Miranker. (1964)), (Jeltsema and van der Schaft (2007)). Let

$$\tilde{P} = \lambda P + \frac{1}{2}\int_Z \left(\delta_{\mathcal{H}}P \wedge M_1 * \delta_{\mathcal{H}}P + \delta_{\mathcal{E}}P \wedge M_2 * \delta_{\mathcal{E}}P\right),$$

with λ an arbitrary constant and symmetric mappings M_1 and M_2 from $\Omega^2(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$. Here the aim is to find λ, M_1 and M_2 such that

$$\dot{\tilde{P}} = u_t^\top \tilde{\mathcal{A}} u_t \leq -K||u_t||^2 \leq 0, \quad (20)$$

where $K \geq 0$ is a constant determined by $\tilde{\mathcal{A}}$. If we can find such a pair $(\tilde{\mathcal{P}}, \tilde{\mathcal{A}})$, which satisfies the above condition, then we can conclude stability of the system.

A constructive process to obtain such pairs is as follows. The variation in $\tilde{P}$ is computed as

$$\begin{bmatrix}\delta_{\mathcal{H}}\tilde{P}\\ \delta_{\mathcal{E}}\tilde{P}\end{bmatrix} = \begin{bmatrix}\lambda I & M_2\mathrm{d}*\\ M_1\mathrm{d}* & (\lambda I - \sigma M_2)\end{bmatrix}\begin{bmatrix}\delta_{\mathcal{H}}P\\ \delta_{\mathcal{E}}P\end{bmatrix}.$$

Applying the Hodge star operator on both sides and using (19), we get

$$*\begin{bmatrix}\delta_{\mathcal{H}}\tilde{P}\\ \delta_{\mathcal{E}}\tilde{P}\end{bmatrix} = \begin{bmatrix}-\mu\lambda I & \epsilon M_2 * \mathrm{d}\\ -\mu M_1 * \mathrm{d} & \epsilon\,(\lambda I - \sigma M_2)\end{bmatrix}\begin{bmatrix}\mathcal{H}_t\\ \mathcal{E}_t\end{bmatrix}.$$

Considering

$$\tilde{A} = \begin{bmatrix}-\mu\lambda I & \epsilon M_2 * \mathrm{d}\\ -\mu M_1 * \mathrm{d} & \epsilon\,(\lambda I - \sigma M_2)\end{bmatrix}$$

we have the following alternative BM representation

$$\tilde{A}u_t = *\delta_u\tilde{P}. \quad (21)$$

Furthermore, if λ, M_1, and M_2 are selected such that $\epsilon M_2 = \mu M_1$ and $0 \leq \lambda \leq \sigma\|M_2\|_s$, where $\|\cdot\|_s$ denotes the spectral norm, then the symmetric part of $\tilde{A} = \mathrm{diag}(-\mu\lambda I, -\epsilon\,(\sigma M_2 - \lambda I))$ is negative definite. Noting that P can be simplified to

$$\begin{aligned}P &= \int_z \mathcal{H} \wedge d\mathcal{E} - \frac{1}{2}\sigma\mathcal{E} \wedge *\mathcal{E}\\ &= \int_z -\frac{1}{2\sigma}\left[\delta_{\mathcal{E}}P \wedge *\delta_{\mathcal{E}}P\right] + \frac{1}{2\sigma}d\mathcal{H} \wedge *d\mathcal{H}\end{aligned}$$

this leads to

$$\begin{aligned}\tilde{P} = &\int_z \delta_{\mathcal{E}}P \wedge \frac{\sigma M_2 - \lambda I}{2\sigma} * \delta_{\mathcal{E}}P\\ &+ \frac{1}{2\sigma}d\mathcal{H} \wedge *d\mathcal{H} + \frac{1}{2}\left(\delta_{\mathcal{H}}P \wedge M_1 * \delta_{\mathcal{H}}P\right) \geq 0.\end{aligned}$$

The time-derivative of $\tilde{P}$ is

$$\dot{\tilde{P}} = -\int_Z \left(\mu\lambda\mathcal{H}_t \wedge *\mathcal{H}_t + \mathcal{E}_t \wedge *(\lambda I - \sigma M_2)\mathcal{E}_t\right) \leq 0,$$

thus implying stability.

Remark 1. To eliminate inequalities like

$$\sigma^{-1}\sqrt{\epsilon\mu^{-1}}\|*\mathrm{d}\| < 1$$

as given in (Jeltsema and van der Schaft (2007)), we choose $M_1 > 0$ and $M_2 > 0$ such that $\epsilon M_2 = \mu M_1$.

4. BRAYTON-MOSER FORM AND ADMISSIBLE PAIRS FOR THE TRANSMISSION LINE

In this Section, we first derive the BM equivalent of the dynamics of a transmission line modeled by the telegrapher's equations. Similar to the case of Maxwell's equations we find the admissible pairs under zero boundary energy flow conditions and infer stability of the system.

The spatial domain of the transmission line is set to $Z = [0,1] \subset \mathbb{R}$, with boundary $\partial Z = \{0,1\}$. The charge density $q(z,t) \in \Omega^1(Z)$ and the flux density $\phi(z,t) \in \Omega^1(Z)$ constitute the energy variables, whereas the associated co-energy variables are the voltage $v(z,t) \in \Omega^0(Z)$ and the current $i(z,t) \in \Omega^0(Z)$, respectively. For simplicity, the relation between the energy and co-energy variables is assumed to be linear, and is given by

$$*q = Cv, \ *\phi = Li, \tag{22}$$

where C, and L are the spatial capacitance and inductance per unit length, respectively, which are assumed to be independent of z. The Hamiltonian H, which is the total energy of the system, is given by

$$H = \frac{1}{2}\int_Z (v \wedge q + i \wedge \phi)\,. \tag{23}$$

Taking the dissipation terms into account, the telegrapher's equations written in port-Hamiltonian form as van der Schaft and Maschke (2002)

$$-\frac{\partial}{\partial t}\begin{bmatrix} q \\ \phi \end{bmatrix} = \begin{bmatrix} *G & \mathrm{d} \\ \mathrm{d} & *R \end{bmatrix}\begin{bmatrix} \delta_q H \\ \delta_\phi H \end{bmatrix}, \tag{24}$$

where $\delta_q H = v$ and $\delta_\phi H = i$, and R and G denote the distributed resistance and conductance per unit length, respectively. Furthermore, we define the boundary variables as $f_b = \delta_q H|_{\partial Z}$ and $e_b = \delta_\phi H|_{\partial Z}$. The rate of change of the Hamiltonian is given by

$$\frac{d}{dt}H = i(0,t)v(0,t) - i(1,t)v(1,t).$$

4.1 The Brayton-Moser form

The dynamics of the transmission line (23) can be written in an equivalent BM form as follows. Define the mixed-potential functional as

$$P = \int_Z \left(-v \wedge \mathrm{d}i + \frac{1}{2}Ri \wedge *i - \frac{1}{2}Gv \wedge *v\right). \tag{25}$$

Then, using the voltage and current as the state variables, we can rewrite the dynamics as follows

$$\begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} i_t \\ v_t \end{bmatrix} = \begin{bmatrix} *\delta_i P \\ *\delta_v P \end{bmatrix} = \begin{bmatrix} Ri + *\mathrm{d}v \\ -Gv - *\mathrm{d}i \end{bmatrix}. \tag{26}$$

4.2 Admissible pairs and stability

Similar to the case of the Maxwell's equations, we cannot use P and A directly to infer stability. We therefore we need to generate new admissible pairs $\tilde{P}$ and $\tilde{A}$ satisfying (20) and (21) such that $\tilde{P} \geq 0$ and $\tilde{A}+\tilde{A}^\top \leq 0$, resulting in stability. As in the case of Maxwell's equations, we propose a $\tilde{P}$ of the form

$$\tilde{P} = \lambda P + \frac{1}{2}\int_Z \delta_u P \wedge M * \delta_u P, \tag{27}$$

where $u = [i, v]^\top$. Selecting

$$M = \begin{bmatrix} \dfrac{\alpha}{R} & m_2 \\ m_2 & \dfrac{\beta}{G} \end{bmatrix},$$

where α, β, m_2 are positive constants satisfying $\alpha\frac{L}{R} = \beta\frac{C}{G}$ and λ is a unit-less constant. Such a choice will be clear in the following discussions, which will eventually lead to a stability criterion. Furthermore, it is easy to check that $\tilde{P}$ has units of power. To simplify the calculations, we define new positive constants θ, γ, and ζ as follows:

$$\begin{aligned} \theta &:= \ \alpha\frac{L}{R} = \beta\frac{C}{G}, \ m_2 := \frac{2\gamma}{CR+LG}, \\ \zeta &:= \frac{2\gamma}{\sqrt{LC}(\alpha+\beta)} \implies m_2 = \frac{\zeta\theta}{\sqrt{LC}}. \end{aligned} \tag{28}$$

To show that $\tilde{P} \geq 0$ we start with simplifying the right hand side of (27) in the following way. Define

$$\begin{aligned} \Delta_1 &:= \left(\zeta\sqrt{\frac{L}{2}}(Gv + i_z) - \sqrt{\frac{C}{2}}(Ri + v_z)\right), \\ \Delta_2 &:= \left(\zeta\sqrt{\frac{C}{2}}(Ri + v_z) - \sqrt{\frac{L}{2}}(Gv + i_z)\right). \end{aligned} \tag{29}$$

Using (28)–(29), and after some calculations, we have that $\tilde{P}$ as defined in (27) takes the form

$$\tilde{P} = \frac{\alpha(1-\zeta^2)+\lambda}{2R}(Ri + v_z)^2 + \Delta_2^2 - \frac{\lambda}{2R}v_z^2 - \frac{\lambda}{2}Gv^2 \tag{30}$$

$$= \frac{\beta(1-\zeta^2)-\lambda}{2G}(Gv + i_z)^2 + \Delta_1^2 + \frac{\lambda}{2G}i_z^2 + \frac{\lambda}{2}Ri^2 \tag{31}$$

which implies that $\tilde{P} \geq 0$ as long as the following conditions are satisfied

$$-\alpha(1-\zeta^2) \leq \lambda \leq \beta(1-\zeta^2), \quad 0 \leq \zeta \leq 1 \tag{32}$$

Further the variational derivative of $\tilde{P}$ with respective to u is calculated as

$$\delta_u\tilde{P} = \begin{bmatrix} -L(\lambda + \alpha - m_2\dfrac{\partial}{\partial z}) & -C(Rm_2 + \dfrac{\beta}{G}\dfrac{\partial}{\partial z}) \\ -L(Gm_2 + \dfrac{\alpha}{R}\dfrac{\partial}{\partial z}) & C(\lambda - \beta + m_2\dfrac{\partial}{\partial z}) \end{bmatrix}\begin{bmatrix} i_t \\ v_t \end{bmatrix}.$$

Therefore

$$\tilde{A} = \begin{bmatrix} -L(\lambda + \alpha - m_2\dfrac{\partial}{\partial z}) & -C(Rm_2 + \dfrac{\beta}{G}\dfrac{\partial}{\partial z}) \\ -L(Gm_2 + \dfrac{\alpha}{R}\dfrac{\partial}{\partial z}) & C(\lambda - \beta + m_2\dfrac{\partial}{\partial z}) \end{bmatrix} \tag{33}$$

satisfy the gradient form (21).

Noting that conjugate of $\dfrac{\partial}{\partial z}$ is $-\dfrac{\partial}{\partial z}$ and using $\alpha\dfrac{L}{R} = \beta\dfrac{C}{G}$ from (28), the symmetric part of $\tilde{A}$ (33) is simplified to be

$$\frac{\tilde{A}+\tilde{A}^*}{2} = \begin{bmatrix} -L(\lambda+\alpha) & -\gamma \\ -\gamma & C(\lambda-\beta) \end{bmatrix}$$

The symmetric part of $\tilde{A}$ is negative semi definite as long as the following conditions are satisfied,

$$-\alpha \leq \lambda \leq \beta, \text{and } (\lambda+\alpha)(\lambda-\beta) + \frac{(\alpha+\beta)^2}{4}\zeta^2 \leq 0. \tag{34}$$

Hence, we have the following proposition (the proof can be found in (Pasumarthy and Kosaraju (2015))).

Proposition 2. If there exist non zero α, β, λ and ζ satisfying (28), (32), and (34) then $\tilde{P}$ defined in (27) and $\tilde{A}$ defined in (33) with M are admissible pairs for the transmission line. Additionally if the symmetric part of $\tilde{A}$ is negative semi definite, i.e., if (34) holds true, then the system of equations (26) is stable.

4.3 Transmission line with nonzero boundary conditions

Consider a transmission line whose boundary is interconnected to certain circuit elements as shown in Figure (1). At $z = 0$ is a resistor R_0 in series with inductor L_0 connected to a voltage source E_0. At $z = 1$ the transmission line is terminated with a resistor R_1 .

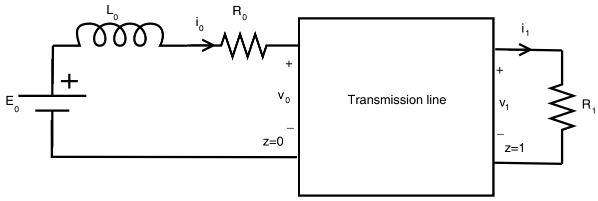

Fig. 1. Transmission line with nonzero boundary.

Hence, the boundary conditions are given by

$$v_0 + R_0 i_0 + L_0 \frac{di_0}{dt} = E_0, \qquad v_1 = R_1 i_1, \tag{35}$$

where $v_0 = v(0,t)$, $i_0 = i(0,t)$, $v_1 = v(1,t)$, and $i_1 = i(1,t)$. Furthermore, let $u = [i, v]^\top$, $u_0 = [i_0, v_0]^\top$, $u_1 = [i_1, v_1]^\top$, and $U = [u, u_0, u_1]^\top$. The total mixed-potential function $P_d = P + P^0 + P^1$ and $\mathcal{A}$ are as follows

$$P = \int_0^1 \left(\frac{1}{2} R i^2 - \frac{1}{2} G v^2 - v i_z \right) dz,$$

$$P^0 = \frac{1}{2} R_0 i_0^2, \quad P^1 = \frac{1}{2} R_1 i_1^2,$$

$$\mathcal{A} = \text{diag} \left\{ \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} -L_0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\},$$

where P and A are defined in (25) and (26), respectively. The input matrix $B = [0\ 0\ -1\ 0\ 0\ 0]^\top$.

The transmission line dynamics governed by (24) together with the boundary dynamics (35) can be written as

$$\mathcal{A} U_t = \delta_U P_d + B E_0.$$

It can be easily checked that using P as a storage function does not result in any kind of passivity properties of the system. Therefore, we need to find new $\tilde{P}_d \geq 0$ and $\tilde{\mathcal{A}} \leq 0$. To include the boundary conditions (35), we do the following. Let $\tilde{P}_d = \tilde{P} + \tilde{P}^1 + \tilde{P}^0$ and $\tilde{\mathcal{A}} = \text{diag}(\tilde{A}, \tilde{A}_0, \tilde{A}_1)$ such that

$$\tilde{\mathcal{A}} U_t = \delta_U \tilde{P}_d + BS, \tag{36}$$

where $\tilde{P}$ and $\tilde{A}$ are defined in (27) and (33) with $\lambda = 1$, and $\tilde{P}^0 = \frac{1}{2} R_0 i_0^2$, $\tilde{P}^1 = \frac{1}{2} R_1 i_1^2$,

$$\tilde{A}^0 = \begin{bmatrix} -\dfrac{\theta^2}{m_2 C} & \theta \\ \theta & -m_2 C \end{bmatrix}, \ \tilde{A}^1 = \begin{bmatrix} \dfrac{\theta^2}{m_2 C} & -\theta \\ -\theta & m_2 C \end{bmatrix},$$

and $S = E_0$. Under the assumption that ζ and θ are chosen such that, $L_0 = \frac{1}{m_2 C}(1 - \zeta^2)\theta^2$ and $\frac{\theta}{m_2} = R_1$, we can show that the time derivative $\tilde{P}_d = \tilde{P} + \tilde{P}^0 + \tilde{P}^1$ is computed as

$$\frac{d}{dt} \tilde{P}_d \leq E_0 \frac{di_0}{dt},$$

which implies that the system is passive with respect to input E_0 and output $\frac{di_0}{dt}$. Details of all computational steps are worked out in (Pasumarthy and Kosaraju (2015)).

5. CONCLUSION

This paper provides some means to generate new passive maps for infinite-dimensional systems, while preserving the pseudo-gradient (Brayton-Moser) structure. Preserving the structure is the key for boundary control by interconnection of infinite-dimensional systems, which will be considered in a future work.

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