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A uniqueness result for linear complementarity problems over the Jordan spin algebra

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ABSTRACT

Given a Euclidean Jordan algebra $(V, \circ, \langle\cdot,\cdot\rangle)$ with the (corresponding) symmetric cone K, a linear transformation $L : V \to V$ and $q \in V$, the linear complementarity problem $\mathrm{LCP}(L, q)$ is to find a vector $x \in V$ such that

$$x \in K, \quad y := L(x) + q \in K \quad \text{and} \quad x \circ y = 0.$$

To investigate the global uniqueness of solutions in the setting of Euclidean Jordan algebras, the P-property and its variants of a linear transformation were introduced in Gowda et al. (2004) [3] and it is shown that if $\mathrm{LCP}(L, q)$ has a unique solution for all $q \in V$, then L has the P-property but the converse is not true in general. In the present paper, when $(V, \circ, \langle\cdot,\cdot\rangle)$ is the Jordan spin algebra, we show that $\mathrm{LCP}(L, q)$ has a unique solution for all $q \in V$ if and only if L has the P-property and L is positive semidefinite on the boundary of $\mathcal{K}$.

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1. Introduction

For $n > 1$, consider $\mathbb{R}^n$ with the usual inner-product $\langle x, y\rangle := x^T y$. We write a vector x in $\mathbb{R}^n$ by $[x_0, \bar{x}]^T$, where $x_0 \in \mathbb{R}$ and define the Jordan product of any two n-vectors x and y by

$$x \circ y := \begin{bmatrix} x^T y \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}.$$

The triple $(\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle)$ is called *Jordan-spin algebra*. The cone of squares

$$\mathcal{K}^n := \{x \in \mathbb{R}^n : x_0 \geq 0, \quad \bar{x}^T\bar{x} \leq x_0^2\} = \{x \circ x : x \ \in \mathbb{R}^n\}$$

is the well-known second-order cone or the Lorentz cone. The most interesting case happens when $\mathcal{K}^n$ is non-polyhedral which is true iff $n > 2$. Henceforth, we fix $n > 2$ and simply use $\mathcal{K}$ to denote $\mathcal{K}^n$. Given an $n \times n$ real matrix M and a vector q in $\mathbb{R}^n$, the second-order cone linear complementarity problem SOLCP(M, q) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad y := Mx + q \in \mathcal{K} \ \text{ and } \ x \circ y = 0 (\Leftrightarrow x^T y = 0).$$

Complementarity problems appear in various areas that include game theory, optimization and economics. SOLCP is a classical example of a linear complementarity problem defined on a non-polyhedral cone. The Jordan spin algebra associated with the second-order cone has rank two and has extra properties which allow us to go beyond the general study of complementarity problems. The text of Faraut and Koranyi [2] covers the foundations of Euclidean Jordan algebra.

Definition 1. We will say that an $n \times n$ real matrix M has the *Globally Uniquely Solvable* (GUS) property, if SOLCP(M, q) has a unique solution for all $q \in \mathbb{R}^n$.

In general it is very difficult to verify whether a linear transformation has the GUS-property. Investigations on GUS-property of a linear transformation in Euclidean Jordan algebras are found in Gowda and Sznajder [4]. One of the fundamental problems in SOLCP is to find conditions that characterize the GUS-property of an $n \times n$ matrix. The problem can be posed in a more general setting.

Given a finite dimensional real Hilbert space $\mathcal{H}$, and a closed convex cone $\mathcal{C}$ in $\mathcal{H}$, a linear transformation $T : \mathcal{H} \to \mathcal{H}$, and a vector $q \in \mathcal{H}$, the (cone) linear complementarity problem LCP$(T, \mathcal{H}, \mathcal{C})$ is to find a vector $x \in \mathcal{H}$ such that

$$x \in \mathcal{C}, \quad y := Tx + q \in \mathcal{C}^* \ \text{ and } \ \langle x, y\rangle = 0,$$

where $\mathcal{C}^*$, called the dual cone, is defined by

$$\mathcal{C}^* := \{u \in \mathcal{H} : \langle u, z\rangle \geq 0 \ \forall z \in \mathcal{C}\}.$$

By specializing $\mathcal{H} = \mathbb{R}^n$ and $\mathcal{C} = \mathcal{K}$ with the usual inner-product, we get SOLCP. When $H = \mathbb{R}^n$, $C = \mathbb{R}^n{}_+$ (the non-negative orthant) and $\langle x, y\rangle$ is the usual inner product, the above complementarity problem reduces, for an $n \times n$ matrix M, to the linear complementarity problem LCP(M, q): Find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad y := Mx + q \geq 0 \ \text{ and } \ x^T y = 0, \tag{1}$$

where the inequalities are defined in the component-wise sense. With numerous applications to many fields, see (Cottle, Pang and Stone [1]), the study of linear complementarity problem has received wide attention. In the LCP theory, the uniqueness of solution in LCP(M, q) for all q is addressed via the following equivalent conditions [Theorems 3.3.4 and 3.3.7 in Cottle, Pang and Stone [1]]:

(P1) Every principal minor of M is positive.
(P2) The implication

$$x \in \mathbb{R}^n, \quad x * Mx \leq 0 \implies x = 0$$

holds, where $x * Mx$ is the component-wise product of vectors x and Mx.
(P3) For every $q \in \mathbb{R}^n$, LCP(M, q) has a unique solution.

Generalizing (P2), Gowda et al. [3] introduced P-property for a linear transformation in a Euclidean Jordan algebra. However, in a non-polyhedral setting, it turns out that GUS and P-properties are not equivalent in general. Our findings in this paper are given below.

- The following are equivalent for an $n \times n$ matrix M.
  1. SOLCP(M, q) has a unique solution for all $q \in \mathbb{R}^n$.
  2. (a) M has the P-property on $\mathcal{K}$.
     (b) $x^T Mx \geq 0$ and $x^T M^{-1}x \geq 0 \ \forall x \in \partial\mathcal{K}$.
  3. (a) M^T has the P-property on $\mathcal{K}$.
     (b) $x^T Mx \geq 0$ and $x^T M^{-1}x \geq 0 \ \forall x \in \partial\mathcal{K}$.
- If M has P-property on $\mathcal{K}$, then M^T need not have P-property on $\mathcal{K}$.
- If A is a **Z**-matrix with respect to $\mathcal{K}$, then the following are equivalent:
  (i) SOLCP(A, q) has a unique solution for all $q \in \mathbb{R}^n$.
  (ii) A is positive stable and $x^T Ax \geq 0$ for all $x \in \partial\mathcal{K}$.

1.1. Notations

(i) To denote the topological interior and the boundary of $\mathcal{K}$, we use $\text{int}(\mathcal{K})$ and $\partial\mathcal{K}$ respectively.
(ii) Let J denote the $n \times n$ diagonal matrix $\text{diag}(1, -1, -1, \ldots, -1)$ and e denote the vector $(1, 0, \ldots, 0)^T$ in $\mathbb{R}^n$.
(iii) $\mathcal{C}(A)$ and $\mathcal{N}(A)$ will denote the column-space and null-space of a matrix A.
(iv) If M is an $n \times n$ matrix and $\tau \in \mathbb{R}$, we define $M_\tau := M - \tau J$.

1.2. Properties of Jordan spin algebra

Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ be written as

$$x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} \quad (x_0, y_0 \in \mathbb{R}).$$

(PR1) Recall that for any two vectors x and y, the Jordan spin algebra is defined by:

$$x \circ y := \begin{bmatrix} x^T y \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}.$$

The following are immediate.
(a) $\mathcal{K} = \{x \circ x : x \in \mathbb{R}^n\}$.
(b) Let $u, v, w \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. The following distributive law holds:

$$u \circ (\alpha v + \beta w) = \alpha u \circ v + \beta u \circ w$$
$$(\alpha v + \beta w) \circ u = \alpha v \circ u + \beta w \circ u.$$

(PR2) Any pair of non-zero vectors $\{c_1, c_2\}$ that satisfy the following conditions is called a *Jordan frame*.
(S1) $c_1 \circ c_2 = 0$.
(S2) $c_1 \circ c_1 = c_1$ and $c_2 \circ c_2 = c_2$.
(S3) $c_1 + c_2 = e$.
(PR3) *Spectral decomposition*: Let $x \in \mathbb{R}^n$. Then there exist real numbers λ_1 and λ_2 and a Jordan frame $\{c_1, c_2\}$ such that

$$x = \lambda_1 c_1 + \lambda_2 c_2.$$

The following can be verified easily:
(i) $x \in \partial\mathcal{K}$ if and only if $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_1\lambda_2 = 0$.
(ii) $x \in \text{int}(\mathcal{K})$ if and only if $\lambda_1 > 0$ and $\lambda_2 > 0$.

(PR4) Given a vector $x \in \mathbb{R}^n$, with the spectral decomposition $x = \lambda_1 c_1 + \lambda_2 c_2$, we define the determinant of x by $\det(x) := \lambda_1\lambda_2$. It is easy to verify the following:
   (a) $\det(x) = 0 \Longleftrightarrow x \in \partial\mathcal{K} \cup -\partial\mathcal{K} \Longleftrightarrow x_0^2 = \|\bar{x}\|^2$.
   (b) $\det(x) > 0 \Longleftrightarrow x \in \operatorname{int}(\mathcal{K}) \cup -\operatorname{int}(\mathcal{K}) \Longleftrightarrow x_0^2 > \|\bar{x}\|^2$.

(PR5) We say that the vectors x and y *operator commute* if they share a Jordan frame, that is,

$$x = \lambda_1 c_1 + \lambda_2 c_2, \quad y = \omega_1 c_1 + \omega_2 c_2,$$

for a Jordan frame $\{c_1, c_2\}$.

(PR6) The vectors x and y operator commute if and only if $\bar{x} = 0$ or $\bar{y} = 0$ or $\bar{x} = \alpha\bar{y}$ for some real number α. Thus if $x \in \mathbb{R}^n$, then x and Jx operator commute.

(PR7) If $\det(x) > 0$, $\det(y) \geq 0$ and $x \circ y = 0$, then $y = 0$.

(PR8) Any two non-zero vectors x and y in $\mathcal{K}$ are orthogonal if and only if $y = \mu Jx$ for some $\mu > 0$. Thus, $x \in \mathcal{K}$ and $y \in \mathcal{K}$ are orthogonal if and only if $x \circ y = 0$.

(PR9) Self-duality:

$$\mathcal{K} = \{x \in \mathbb{R}^n : x^T y \geq 0 \ \ \forall y \in \mathcal{K}\}.$$
$$\operatorname{int}(\mathcal{K}) = \{x \in \mathbb{R}^n : x^T y > 0 \ \ \forall 0 \neq y \in \mathcal{K}\}.$$

For more details, we refer to Tao [7]. The definition of P-property of a linear transformation with respect to the Jordan spin algebra is given below. For a discussion on P-property and its variants in a general symmetric cone, we refer to Gowda et al. [3].

Definition 2 *(P-property).* We will say that an $n \times n$ matrix M has the P-property on $\mathcal{K}$ if:

$$\left.\begin{array}{c} x \text{ and } Mx \text{ operator commute} \\ \text{and} \\ x \circ Mx \in -\mathcal{K} \end{array}\right\} \Longrightarrow x = 0.$$

Example 1. If S is an $n \times n$ matrix with $S(\mathcal{K}) \subseteq \mathcal{K}$, then $I - S$ has P-property on $\mathcal{K}$ if and only if $\rho(S) < 1$. See [5].

Example 2. Let a, b, λ and c be such that

$$a + |\lambda| > 0, \quad \lambda \neq 0, \quad a > \lambda c, \quad a > 0, \quad (c + \lambda)^2 < 4a \quad \text{and} \quad b \in \mathbb{R}.$$

Define $M := \begin{bmatrix} a & b & c \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{bmatrix}$. Then M has P-property on $\mathcal{K}$. See [6].

We record some basic results from Gowda et al. [3] which we will use without explicit mentioning.

Theorem 1. *Let M be an $n \times n$ matrix. Then the following are true:*

(i) *If M has the* GUS*-property, then M must have P-property on $\mathcal{K}$.*
(ii) GUS *and P-properties are not equivalent.*
(iii) *If M has the P-property, then* $\det(M) > 0$.

2. Results

To find new characterizations for the GUS-property, we first derive certain linear algebraic properties of a matrix that has P-property on $\mathcal{K}$.

Lemma 1. *If M is an $n \times n$ matrix, then the following are true:*

(i) *If* $\det(M) > 0$, *then MJ has a positive eigenvalue and* $\det(MJ) \neq 0$.
(ii) *If M has the P-property on $\mathcal{K}$, then the following are true.*
  (a) *MJ has a positive eigenvalue and non-singular. Further, if τ is any positive eigenvalue of MJ, then there exists $u \in \operatorname{int}(\mathcal{K})$ such that $\mathcal{N}(M_\tau) = \operatorname{span}\{u\}$.*
  (b) *If $\alpha \in \mathbb{R}^n$ is an eigenvalue of MJ, then $\mathcal{N}(M_\alpha) \cap \partial\mathcal{K} = \{0\}$.*
(iii) *Let $\tau > 0$ be such that $\mathcal{N}(M_\tau) = \operatorname{span}\{u\}$ for some $u \in \operatorname{int}(\mathcal{K})$. Then $\mathcal{C}(M_\tau) = \{M_\tau x : x \in \partial\mathcal{K}\}$.*
(iv) *Let $\tau > 0$ be a positive eigenvalue of MJ such that $\mathcal{N}(M_\tau) = \operatorname{span}\{u\}$ for some $u \in \operatorname{int}(\mathcal{K})$. Assume that M satisfy the following conditions*
  (a) *If $\alpha \in \mathbb{R}$ is an eigenvalue of MJ, then $\mathcal{N}(M_\alpha) \cap \partial\mathcal{K} = \{0\}$.*
  (b) $y^T My \geq 0 \ \forall y \in \partial\mathcal{K}$.
  Then there exists $v \in \operatorname{int}(\mathcal{K})$ such that $\mathcal{N}(M_\tau^T) = \operatorname{span}\{v\}$.

Proof. First we prove (i). As $\det(M) > 0$ and J is non-singular, $\det(MJ) \neq 0$. Define $f : [0, \infty) \to \mathbb{R}$ by $f(s) := \det(M - sJ)$. Since $\lim_{s\to\infty} \frac{f(s)}{s^n} = \det(-J) < 0$ and $f(0) > 0$, by intermediate value theorem, there exists $s' \in \mathbb{R}$ such that $f(s') = 0$. As $J^2 = I$, $f(s') = \det(MJ - s'I) = \det(M - s'J)\det(J) = 0$ and hence, $s' > 0$ is a positive eigenvalue of MJ. This proves (i).

Now we prove (ii). Assume that M has P-property on $\mathcal{K}$. Then $\det(M) > 0$ and therefore by (i), $\det(MJ) \neq 0$ and has a positive eigenvalue, say τ. Let $0 \neq v \in \mathbb{R}^n$ be an eigenvector of MJ and $w := Jv$. As $J^2 = I$, we get $Mw = \tau Jw$, i.e., $M_\tau w = 0$. We claim that $\det(w) > 0$. By assuming $\det(w) \leq 0$, we get a contradiction. If $\det(w) < 0$, then by spectral decomposition, we can write $w = \alpha e_1 - \beta e_2$, where $\alpha > 0$ and $\beta > 0$. The vectors w and Jw operator commute. So, w and Mw operator commute. Since $Je_1 = \mu e_2$ for some $\mu > 0$, we see that

$$Jw = \alpha Je_1 - \beta Je_2 = \alpha\mu e_2 - \beta(\frac{1}{\mu})e_1.$$

By distributive law,

$$w \circ Mw = -\tau\alpha\beta(\frac{e_1}{\mu} + \mu e_2).$$

Thus, $w \circ Mw \in -\operatorname{int}(\mathcal{K})$. Since M has the P-property, $w = 0$ and this is possible only when $v = 0$, which is a contradiction. Let $\det(w) = 0$. Then $Mw = \tau Jw$ and by using (PR4) and (PR8) we get $w \circ Mw = 0$. Since M has the P-property, $w = 0$ and thus $v = 0$, which is a contradiction. Therefore, $\det(w) > 0$. To this end, we have shown that if $u \in \mathcal{N}(M_\tau)$, then $\det(u) > 0$, i.e., $\pm u \in \operatorname{int}(\mathcal{K})$. We now show that $\operatorname{nullity}(M_\tau) = 1$. If $\operatorname{nullity}(M_\tau) > 1$, then there exist linearly independent vectors x and y in $\mathcal{N}(M_\tau)$ such that $x^T y = 0$. Since $\det(x) > 0$ and $\det(y) > 0$, by self-duality (PR9), either $x^T y > 0$ or $x^T y < 0$. So, x and y cannot be orthogonal. Thus, $\operatorname{nullity}(M_\tau) = 1$ and this proves (ii)(a).

If $x \in \mathcal{N}(M_\alpha) \cap \partial\mathcal{K}$, then $Mx = \alpha Jx$ and $x \in \partial\mathcal{K}$ and hence $x \circ Mx = 0$. Since M has the P-property, $x = 0$. This completes the proof of (ii)(b).

We now prove (iii). Let $v \in \operatorname{int}(\mathcal{K})$ be such that $\mathcal{N}(M_\tau) = \operatorname{span}\{v\}$. Suppose $w \in \mathcal{C}(M_\tau)$. Then, $w = M_\tau z$ for some $z \in v^\perp$. Since z is orthogonal to a vector in $\operatorname{int}(\mathcal{K})$, $\det(z) < 0$. As $\det(v) > 0$, by continuity, there exists $0 < \alpha < 1$ such that $\det((1 - \alpha)v + \alpha z) = 0$. Define $y := (1 - \alpha)v + \alpha z$. Since $y^T v > 0$ and $\det(y) = 0$, $y \in \partial\mathcal{K}$. If $x := \frac{y}{1-\alpha} = v + \frac{\alpha}{1-\alpha} z$, then we have

$$M_\tau x = \frac{\alpha}{1-\alpha} M_\tau z \quad \text{and} \quad x \in \partial\mathcal{K}.$$

Thus, $w = M_\tau(\frac{1-\alpha}{\alpha} x)$. The proof of (iii) is complete.

We now prove (iv). By our assumption $\operatorname{nullity}(M_\tau) = 1$ and hence $\operatorname{nullity}(M_\tau^T) = 1$. Let $\mathcal{N}(M_\tau^T) = \operatorname{span}\{x\}$. To complete the proof, by (PR4), it suffices to show that $\det(x) > 0$. Suppose $\det(x) < 0$. By (PR3), there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $x = \lambda_1 e_1 - \lambda_2 e_2$, where $\{e_1, e_2\}$ is a Jordan frame. Define $p := \frac{1}{\lambda_1} e_1 + \frac{1}{\lambda_2} e_2$. Then, $p \in \operatorname{int}(\mathcal{K})$ and $p^T x = 0$. Since $\mathcal{C}(M_\tau) = \{y : y^T x = 0\}$, it follows that $p \in \mathcal{C}(M_\tau)$. By (iii), there exists $0 \neq y \in \partial\mathcal{K}$ such that $M_\tau y = My - \tau Jy = -p$; hence $y^T My + p^T y = 0$. Since $p \in \operatorname{int}(\mathcal{K})$ and $y \in \partial\mathcal{K}$, $p^T y > 0$. As $y^T My \geq 0$, $y^T My + p^T y > 0$. This is a contradiction. Thus, $\det(x) \geq 0$. Suppose $\det(x) = 0$. Without loss of generality, assume $x \in \partial\mathcal{K}$. As Jx is orthogonal to x, $-Jx \in \mathcal{C}(M_\tau)$. By (iii), we can find $y \in \partial\mathcal{K}$ such that $M_\tau y = -Jx$, i.e., $My - \tau Jy = -Jx$. Since $y^T Jy = 0$, we get $y^T My + y^T Jx = 0$. As $y^T My \geq 0$ and $y^T Jx \geq 0$, it follows that $y^T Jx = 0$ and $y^T My = 0$. So, $y = \alpha x$ for some $\alpha > 0$. From $M_\tau y = -Jx$, we see that $Mx = \rho Jx$ for some $\rho \in \mathbb{R}$ and hence $x \in \mathcal{N}(M_\rho) \cap \partial\mathcal{K}$. By our assumption on M, we deduce that $x = 0$, which is a contradiction. Therefore $\det(x) > 0$. This proves (iv). □

2.1. *Necessary and sufficient conditions for* GUS-*property*

We now prove our main result regarding the global uniqueness of solutions in second-order cone linear complementarity problems by establishing a precise interconnection between the P and GUS-properties in SOLCP. We first note a useful result that follows easily from Theorem 2 in [8] and Lemma 1.

Theorem 2. *Let M be an $n \times n$ matrix. Then M has the* GUS*-property if and only if M satisfies the following conditions:*

(Z1) *MJ is non-singular and has a positive eigenvalue.*
(Z2) *There exist $\tau > 0$ and $v \in \operatorname{int}(\mathcal{K})$ such that $MJv = \tau v$. Further, all positive eigenvalues of MJ are equal to τ and* $\operatorname{rank}(MJ - \tau I) = n - 1$.
(Z3) *If $\alpha \in \mathbb{R}$, then $\mathcal{N}(M_\alpha) \cap \partial\mathcal{K} = \{0\}$.*
(Z4) *For all $y \in \partial\mathcal{K}$, $y^T My \geq 0$ and $y^T M^{-1} y \geq 0$.*

Proof. Assume that M has the GUS-property. Since P-property on $\mathcal{K}$ is a necessary condition for GUS, by Lemma 1, we get (Z1) and (Z3). (Z2) and (Z4) are immediate from Theorem 2 in [8].

We now prove the converse. By Theorem 2 in [8], it suffices to show that there exists $v \in \operatorname{int}(\mathcal{K})$ such that $\mathcal{N}(M_\tau^T) = \operatorname{span}\{v\}$. But this is immediate from item (iv) of Lemma 1. This completes the proof. □

Theorem 3. *Let M be an $n \times n$ matrix. Then the following are equivalent:*

(T1) *M has the* GUS*-property.*
(T2) *M satisfies the following:*
  (P1) *M has the P-property on $\mathcal{K}$.*
  (P2) *For all $x \in \partial\mathcal{K}$, $x^T Mx \geq 0$ and $x^T M^{-1} x \geq 0$.*
(T3) *M satisfies the following:*
  (R1) *M^T has the P-property on $\mathcal{K}$.*
  (R2) *For all $x \in \partial\mathcal{K}$, $x^T Mx \geq 0$ and $x^T M^{-1} x \geq 0$.*
(T4) *M^T has the* GUS*-property.*
(T5) *M satisfies the following:*
  (S1) *Both M and M^T have P-property.*
  (S2) *For all $x \in \partial\mathcal{K}$, $x^T Mx \geq 0$ and $x^T M^{-1} x \geq 0$.*

Proof. We first show that (T1) $\Longrightarrow$ (T2). Since GUS implies P-property, we get (P1). By Theorem 2(Z4), (P2) must be true. Thus, (T1) $\Longrightarrow$ (T2).

To prove (T2) $\Longrightarrow$ (T1), we verify all four conditions in Theorem 2. Item (Z4) in Theorem 2 follows immediately from (P2). By the P-property of M, $\det(M) > 0$ and hence by Lemma 1(i), MJ is non-singular and has a positive eigenvalue. Thus, M satisfies (Z1). Further, (Z3) is immediate from Lemma 1(ii)(b). We now verify (Z2). If $\tau > 0$ is a positive eigenvalue of MJ, then from Lemma 1(ii)(a), there exists $v \in \operatorname{int}(\mathcal{K})$ such that $\mathcal{N}(M_\tau) = \operatorname{span}\{v\}$. To complete the proof, we need to show that all the positive eigenvalues of MJ are equal and $\operatorname{rank}(MJ - \tau I) = n - 1$. Suppose there exist distinct positive eigenvalues of MJ (say, τ and τ'), such that $\tau' = \tau + c$, where $c > 0$. Let $x, y \in \text{int}(\mathcal{K})$ be such that

$$MJx = \tau x \text{ and } MJy = \tau' y.$$

Now we can find scalars α and β such that $0 \neq u := \alpha x + \beta y \in \partial\mathcal{K}$ and $\alpha\beta < 0$. Consider the following two possibilities:

(A) $\alpha > 0$ and $\beta < 0$.
(B) $\alpha < 0$ and $\beta > 0$.

Assume (A). Since

$$MJ(\alpha x) = \tau\alpha x \text{ and } MJ(\beta y) = (\tau + c)\beta y,$$

we get $MJu = \tau u + c\beta y$. By (P2), $(Ju)^T MJu \geq 0$. Further, $u^T Ju = 0$. Hence, $c\beta y^T Ju \geq 0$. Since $y \in \text{int}(\mathcal{K})$ and $u \in \partial\mathcal{K}$, we have $y^T Ju > 0$ and from $c > 0$ and $\beta < 0$, we have $c\beta y^T Ju < 0$ which is a contradiction. Thus (A) is not true. Now assume (B). Let $\theta' > 0$ be such that

$$\frac{1}{\tau'} + \frac{1}{\theta'} = \frac{1}{\tau}.$$

From the equations

$$M^{-1}y = \frac{1}{\tau'}Jy \text{ and } M^{-1}x = \frac{1}{\tau}Jx,$$

we find that

$$M^{-1}u = \frac{1}{\tau'}\alpha Ju + \frac{1}{\theta'}\alpha Jx.$$

Since $u^T M^{-1}u \geq 0$, we have

$$\frac{1}{\theta'}\alpha u^T Jx \geq 0.$$

Since $x \in \text{int}(\mathcal{K})$, $u \in \partial\mathcal{K}$, $\theta' > 0$ and $\alpha < 0$, we see that $\frac{1}{\theta'}\alpha u^T Jx < 0$. Thus (B) leads to a contradiction. Hence all the positive eigenvalues of MJ are equal. Suppose $\tau > 0$ is the positive eigenvalue of MJ. From Lemma 1(ii)(a), we see that $\text{nullity}(M_\tau) = 1$ and hence $\text{rank}(M_\tau) = n - 1$. Since $J^2 = I$, $\text{rank}(M_\tau) = \text{rank}(MJ - \tau I) = n - 1$. Thus M satisfies (Z2). Hence, M satisfies all sufficient conditions in Theorem 2. The proof of (T2) $\Longrightarrow$ (T1) is now complete.

By Corollary 2 in [8], M has the GUS-property if and only if M^T has GUS-property. Therefore (T1), (T2), (T3), (T4) and (T5) are equivalent. □

We have seen that M has the GUS-property if and only if M^T has GUS. Here is an example of a matrix that has P-property on $\mathcal{K}$ while its transpose does not posses the P-property.

Example 3. Let

$$\alpha := \sqrt{\sqrt{2}+2} \quad \text{and} \quad \beta := \sqrt{2(\sqrt{2}+1)}.$$

Define

$$M := \begin{bmatrix} 1 & \alpha & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad x := \begin{bmatrix} \sqrt{2}+1 \\ -\beta \\ -1 \end{bmatrix}.$$

Then

$$M^T x = \begin{bmatrix} 2+\sqrt{2} \\ 2\sqrt{\sqrt{2}+1} \\ \sqrt{2} \end{bmatrix}.$$

By (PR6) in Section 2, x and $M^T x$ operator commute. Further, $x \circ M^T x = 0$. So, M^T does not have the P-property. However, M has the P-property (see Example 2).

2.2. **Z**-*Transformations*

We now consider an interesting class of linear transformations namely **Z**-transformations over the second-order cone.

Definition 3. An $n \times n$ matrix M is called a **Z**-matrix on $\mathcal{K}$ if the following condition is satisfied:

$$x \in \mathcal{K}, \quad y \in \mathcal{K} \quad \text{and} \quad x^T y = 0 \implies y^T M x \leq 0.$$

It is well-known that if a matrix S satisfies $S(\mathcal{K}) \subseteq \mathcal{K}$, then $I - S$ is a **Z**-matrix. Thus the class of **Z**-matrices is very broad. If M is a **Z**-matrix on $\mathcal{K}$, then SOLCP(M, q) has a solution for all $q \in \mathbb{R}^n$ if and only if every eigenvalue of M (over the complex field) has a positive real part, i.e., M is *positive stable*. We refer to [5] for the proof of this result and more details on **Z**-matrices. For **Z**-matrices on $\mathcal{K}$, the following is conjectured in Tao [6]: For a **Z**-matrix M on $\mathcal{K}$, the following are equivalent: M has the GUS-property if and only if M has the P-property and M is positive semidefinite on $\partial\mathcal{K}$. We prove this result now.

Theorem 4. *Let M be a* **Z***-matrix on $\mathcal{K}$. Then the following are equivalent:*

(A) *M has the* GUS*-property.*
(B) *M is positive stable and $x^T M x \geq 0$ for all $x \in \partial\mathcal{K}$.*

Proof. We first prove (A) $\Longrightarrow$ (B). By our assumption SOLCP(M, q) has a unique solution for all $q \in \mathbb{R}^n$. From Theorem 7 in [5], we see that M is positive stable if and only if SOLCP(M, q) has a solution for all $q \in \mathbb{R}^n$. Hence M is positive stable. By Theorem 3, $x^T Mx \geq 0$ for all $x \in \partial\mathcal{K}$. This proves (A) $\Longrightarrow$ (B).

We now prove (B) $\Longrightarrow$ (A). By Theorem 13 in [5], M has the P-property on $\mathcal{K}$. Further, by Theorem 7 in [5], M has the P-property on $\mathcal{K}$ if and only if $M^{-1}(\mathcal{K}) \subseteq \mathcal{K}$. Therefore, $x^T M^{-1} x \geq 0$ for all $x \in \mathcal{K}$. By Theorem 3, we see that M has the GUS-property. This completes the proof. □

We conclude the paper with the following example where M and M^T have P-property, but M does not have the GUS-property.

Example 4. For $\alpha > 0$, let

$$M_\alpha := \begin{bmatrix} 1-\alpha & -\alpha & 0 \\ \alpha & 1+\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $M_\alpha = I - S_\alpha$, where

$$S_\alpha = \begin{bmatrix} \alpha & \alpha & 0 \\ -\alpha & -\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By an easy verification, $S_\alpha(\mathcal{K}) \subseteq \mathcal{K}$. Hence for any $\alpha > 0$, M_α is a **Z**-matrix on $\mathcal{K}$. Since ${S_\alpha}^2 = 0$, $\rho(S_\alpha) = 0$. Thus, M_α and M_α^T have P-property on $\mathcal{K}$ for any α. If $y \in \partial\mathcal{K}$, then it is easy to see that $y^T M_\alpha y = 2y_1^2 - \alpha(y_1^2 + y_2^2)$. If $\alpha > 2$ and $y = (1, 0, 1)$ then $y^T M_\alpha y < 0$. Hence for $\alpha > 2$, M_α does not have the GUS-property.

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