Research Article

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A sampling theorem for the twisted shift-invariant space

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Abstract: Recently, a characterization of frames in twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ has been obtained in [16]. Using this result, we prove a sampling theorem on a subspace of a twisted shift-invariant space in this paper.

Keywords: Canonical dual frames, frames, sampling theorem, shift-invariant space, twisted translation

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1 Introduction

The fundamental Shannon's sampling theorem states that any function f belonging to the Paley–Wiener space

$$B_\pi = \{f \in L^2(\mathbb{R}) : \operatorname{supp}\widehat{f} \subset [-\pi, \pi]\}$$

can be reconstructed from its samples $\{f(k) : k \in \mathbb{Z}\}$ by the formula

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}(x-k),$$

where $\operatorname{sinc} y = \frac{\sin \pi y}{\pi y}$ and $\widehat{f}$ denotes the Fourier transform of f, given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx, \quad \xi \in \mathbb{R}.$$

Paley and Wiener extended Shannon's sampling theorem to a non-uniform sampling set in [15]. They showed that if $X = \{x_k \in \mathbb{R} : k \in \mathbb{Z}\}$ is such that $|x_k - k| < 1/\pi^2$, then any function f belonging to the class $\{f \in L^2(\mathbb{R}) : \operatorname{supp}\widehat{f} \subset [-\pi, \pi]\}$ can be recovered from its samples $\{f(x_k) : k \in \mathbb{Z}\}$. Duffin and Eachus [7] showed that the result is true if $|x_k - k| < 0.22$. Later, Kadee [13] showed that the maximum bound for $|x_k - k|$ has to be less than 0.25. For a more general sampling set, the sampling condition is stated in terms of Beurling density.

Sampling theorems have been studied on wavelet subspaces in [22, 23]. In particular, in [23] for any closed shift-invariant subspace V_0 of $L^2(\mathbb{R})$, a necessary and sufficient condition under which there is a sampling expansion for every $f \in V_0$ was shown. In sampling theory, non-uniform sampling in shift-invariant spaces is given importance to for the past fifteen years. We refer to a few papers [1–3, 8–11, 18–20] in this connection.

Characterizations of shift-invariant spaces in $L^2(\mathbb{R}^n)$ in terms of range functions were studied by Bownik in [4]. The study of shift-invariant spaces and frames has been extended to locally compact abelian groups

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in [5, 14] and non-abelian compact groups in [17]. Radha and Adhikari [16] introduced twisted shift-invariant spaces in $L^2(\mathbb{R}^{2n})$ and studied characterizations of orthonormal systems, Bessel sequences, frames and Riesz bases of twisted translates in terms of the kernel of the Weyl transform. The twisted translation and twisted shift-invariant space are defined as follows.

Definition 1.1. Let $\varphi \in L^2(\mathbb{R}^{2n})$. For $(k, l) \in \mathbb{Z}^{2n}$, we define the twisted translation of φ, denoted by $T^t_{(k,l)}\varphi$, as

$$T^t_{(k,l)}\varphi(x, y) = e^{\pi i(\langle x,l\rangle - \langle y,k\rangle)}\varphi(x - k, y - l), \quad (x, y) \in \mathbb{R}^{2n}.$$

Definition 1.2. For $\varphi \in L^2(\mathbb{R}^{2n})$, we define the twisted shift-invariant space of φ, denoted by $V^t(\varphi)$, as $\overline{\operatorname{span}}\{T^t_{(k,l)}\varphi : (k, l) \in \mathbb{Z}^{2n}\}$ in $L^2(\mathbb{R}^{2n})$.

The aim of our paper is to obtain a sampling theorem in a twisted shift-invariant space $V^t(\varphi)$ on $L^2(\mathbb{R}^{2n})$. However, we are able to get a reconstruction formula for a function f belonging to a subspace of $V^t(\varphi)$. We organize the paper as follows: In Section 2, we provide basic definitions and state some results which are available in the literature. In Section 3, we study canonical dual frames in a twisted shift-invariant space. In fact, for a certain function $\varphi \in L^2(\mathbb{R}^{2n})$, we explicitly show the existence of $\tilde{\varphi} \in L^2(\mathbb{R}^n)$ such that twisted translates of $\tilde{\varphi}$ is the canonical dual of twisted translates of φ. In Section 4, we prove our main result, namely a sampling theorem on a subspace of a twisted shift-invariant space. In fact, we give a necessary and sufficient condition for obtaining a reconstruction formula for functions belonging to a subspace of $V^t(\varphi)$ from their samples $\{f(k, j) : k \in \mathbb{Z}^n\}$ for each fixed $j \in \mathbb{Z}^n$. We also provide a necessary condition for obtaining a reconstruction formula for functions belonging to a subspace of $V^t(\varphi)$ from their samples $\{f(k, j) : k, j \in \mathbb{Z}^n\}$. However, we are not able to get the sufficient condition of this theorem.

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space.

Definition 2.1. A sequence $\{f_k : k \in \mathbb{Z}\}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.1}$$

If only the inequality on the right-hand side holds in (2.1), then $\{f_k : k \in \mathbb{Z}\}$ is called a Bessel sequence for $\mathcal{H}$.

The operator $S : \mathcal{H} \to \mathcal{H}$ defined by $Sf := \sum_{k\in\mathbb{Z}}\langle f, f_k\rangle f_k$ is called the frame operator associated with the frame $\{f_k\}$. Then S is bounded, invertible, self-adjoint, and positive. Further, $\{S^{-1}f_k : k \in \mathbb{Z}\}$ is also a frame for $\mathcal{H}$ and is called the canonical dual frame of $\{f_k : k \in \mathbb{Z}\}$. Using $\{S^{-1}f_k\}$, one can write

$$f = \sum_{k\in\mathbb{Z}} \langle f, S^{-1}f_k\rangle f_k \quad \text{for all } f \in \mathcal{H}. \tag{2.2}$$

For further details on frames we refer to [6, 12].

Definition 2.2. For $f \in L^1(\mathbb{C}^n)$, the Weyl transform of f is defined as

$$W(f) = \int_{\mathbb{C}^n} f(z)\pi_1(z, 0)\, dz,$$

where $\pi_\lambda(z, t)$, for $\lambda \neq 0$, denotes the Schrödinger representation on the Heisenberg group $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ given by

$$\pi_\lambda(z, t)\varphi(\xi) = e^{2\pi i\lambda t}e^{2\pi i\lambda(\langle x,\xi\rangle + \frac{1}{2}\langle x,y\rangle)}\varphi(\xi + y), \quad z = x + iy,\ \varphi \in L^2(\mathbb{R}^n).$$

The Weyl transform $W(f)$ is an integral operator with kernel $K_f(\xi, \eta)$ given by

$$\int_{\mathbb{R}^n} f(x, \eta - \xi)e^{\pi i\langle x,\xi+\eta\rangle}\, dx.$$

This map W can be uniquely extended to a bijection from the class of tempered distributions $S'(\mathbb{C}^n)$ onto the space of continuous linear maps from $S(\mathbb{R}^n)$ into $S'(\mathbb{R}^n)$. For a further study of the Weyl transform, we refer to [21].

We shall now state some definitions and results which were given in [16].

Lemma 2.3. *Let $\varphi \in L^2(\mathbb{R}^{2n})$. Then the kernel of the Weyl transform of $T^t_{(k,l)}\varphi$ satisfies the relation*

$$K_{T^t_{(k,l)}\varphi}(\xi,\eta) = e^{\pi i\langle 2\xi+l,k\rangle}K_\varphi(\xi+l,\eta). \tag{2.3}$$

Definition 2.4. For $\varphi \in L^2(\mathbb{R}^{2n})$, the function w_φ is defined as

$$w_\varphi(\xi) = \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2\,d\eta, \quad \xi\in\mathbb{R}^n.$$

Definition 2.5. A function $\varphi \in L^2(\mathbb{R}^{2n})$ is said to satisfy "condition C" if

$$\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}K_\varphi(\xi+m,\eta)\overline{K_\varphi(\xi+m+l,\eta)}\,d\eta = 0 \quad \text{a.e. } \xi\in\mathbb{T}^n,\ \text{for all } l\in\mathbb{Z}^n\setminus\{0\}.$$

Theorem 2.6 ([16]). *If $\{T^t_{(k,l)}\varphi : (k,l)\in\mathbb{Z}^{2n}\}$ is a Bessel sequence in $L^2(\mathbb{R}^{2n})$ with bound B, then $w_\varphi(\xi)\le B$ a.e. $\xi\in\mathbb{T}^n$.*

Theorem 2.7 ([16]). *Let $\varphi\in L^2(\mathbb{R}^{2n})$ and satisfying condition C. Then $\{T^t_{(k,l)}\varphi : (k,l)\in\mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$ with frame bounds A, B if and only if $A\le w_\varphi(\xi)\le B$ a.e. $\xi\in\Omega_\varphi$, where $\Omega_\varphi=\{\xi\in\mathbb{T}^n : w_\varphi(\xi)\neq 0\}$.*

Theorem 2.8 ([16]). *Let $\varphi\in L^2(\mathbb{R}^{2n})$. Suppose $\{T^t_{(k,l)}\varphi : (k,l)\in\mathbb{Z}^{2n}\}$ is a frame for $L^2(\mathbb{R}^{2n})$ with frame operator S. Then*

$$S^{-1}T^t_{(k,l)}\varphi = T^t_{(k,l)}S^{-1}\varphi.$$

Let $\varphi\in L^2(\mathbb{R}^{2n})$ and satisfying condition C. Suppose $A^t(\varphi)=\operatorname{span}\{T^t_{(k,l)}\varphi : (k,l)\in\mathbb{Z}^{2n}\}$ and $V^t(\varphi)=\overline{A^t(\varphi)}$. Consider $f\in A^t(\varphi)$, i.e.,

$$f=\sum_{(k',l')\in\mathcal{F}}c_{k',l'}T^t_{(k',l')}\varphi,$$

where $\mathcal{F}$ is a finite set. Define $\rho(\xi)=\{\rho_{l'}(\xi)\}_{l'\in\mathbb{Z}^n}$ for $\xi\in\mathbb{T}^n$, where

$$\rho_{l'}(\xi)=\sum_{k'}c_{k',l'}e^{\pi i\langle 2\xi+l',k'\rangle}.$$

Define $J_\varphi(f)=\rho$. The map J_φ initially defined on $A^t(\varphi)$ can be extended to an isometric isomorphism between $V^t(\varphi)$ and $L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n),w_\varphi)$. Moreover, it was proved that $f\in V^t(\varphi)$ if and only if

$$K_f(\xi,\eta)=\sum_{l'\in\mathbb{Z}^n}\rho_{l'}(\xi)K_\varphi(\xi+l',\eta), \tag{2.4}$$

where $\rho(\xi)=\{\rho_{l'}(\xi)\}_{l'\in\mathbb{Z}^n}$ and $\rho\in L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n),w_\varphi)$.

We shall make use of the following lemma in [23], which is a simple application of Parseval's identity.

Lemma 2.9. *Let $\{x_k\}$, $\{y_k\}$ be the Fourier coefficients of $f, g\in L^2(\mathbb{T}^n)$, respectively. Then*

$$\int_{\mathbb{T}^n}|f(\xi)g(\xi)|^2\,d\xi=\sum_{n\in\mathbb{Z}^n}\Big|\sum_{k\in\mathbb{Z}^n}x_k y_{n-k}\Big|^2.$$

3 Canonical dual frames in twisted shift-invariant space

Lemma 3.1. *Let $\varphi\in L^2(\mathbb{R}^{2n})$. For $\{c_k\}\in\ell^2(\mathbb{Z}^n)$, define a function L_φ on $\mathbb{R}^{2n}$ by*

$$L_\varphi(\xi,\eta)=\Big(\sum_{k\in\mathbb{Z}^n}c_k e^{2\pi i\langle k,\xi\rangle}\Big)K_\varphi(\xi,\eta).$$

Assume that $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ *is a Bessel sequence with bound B. Then* $L_\varphi \in L^2(\mathbb{R}^{2n})$ *and*

$$\sum_{k\in\mathbb{Z}^n} c_k K_{T^t_{(k,0)}\varphi}$$

converges to L_φ *in* $L^2(\mathbb{R}^{2n})$.

Proof. Since $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a Bessel sequence with bound B, by Theorem 2.6, $w_\varphi(\xi) \le B$ a.e. $\xi \in \mathbb{T}^n$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |L_\varphi(\xi,\eta)|^2\,d\xi\,d\eta &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\sum_{k\in\mathbb{Z}^n} c_k e^{2\pi i\langle k,\xi\rangle} K_\varphi(\xi,\eta)\Big|^2\,d\xi\,d\eta \\
&= \int_{\mathbb{T}^n}\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\Big|\sum_{k\in\mathbb{Z}^n} c_k e^{2\pi i\langle k,\xi\rangle} K_\varphi(\xi+m,\eta)\Big|^2\,d\eta\,d\xi \\
&= \int_{\mathbb{T}^n}\Big|\sum_{k\in\mathbb{Z}^n} c_k e^{2\pi i\langle k,\xi\rangle}\Big|^2 \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2\,d\eta\,d\xi \\
&= \int_{\mathbb{T}^n}\Big|\sum_{k\in\mathbb{Z}^n} c_k e^{2\pi i\langle k,\xi\rangle}\Big|^2 w_\varphi(\xi)\,d\xi \\
&\le B\int_{\mathbb{T}^n}\Big|\sum_{k\in\mathbb{Z}^n} c_k e^{2\pi i\langle k,\xi\rangle}\Big|^2\,d\xi \\
&= B\sum_{k\in\mathbb{Z}^n}|c_k|^2 < \infty.
\end{aligned}$$

Thus $L_\varphi \in L^2(\mathbb{R}^{2n})$. Now

$$\begin{aligned}
\Big\|\sum_{|k|\le n} c_k K_{T^t_{(k,0)}\varphi} - L_\varphi\Big\|^2_{L^2(\mathbb{R}^{2n})} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\sum_{|k|\le n} c_k K_{T^t_{(k,0)}\varphi}(\xi,\eta) - \Big(\sum_{k\in\mathbb{Z}^n} c_k e^{2\pi i\langle k,\xi\rangle}\Big)K_\varphi(\xi,\eta)\Big|^2\,d\xi\,d\eta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\sum_{|k|\le n} c_k e^{2\pi i\langle k,\xi\rangle} K_\varphi(\xi,\eta) - \Big(\sum_{k\in\mathbb{Z}^n} c_k e^{2\pi i\langle k,\xi\rangle}\Big)K_\varphi(\xi,\eta)\Big|^2\,d\xi\,d\eta,
\end{aligned}$$

using (2.3). Then

$$\begin{aligned}
\Big\|\sum_{|k|\le n} c_k K_{T^t_{(k,0)}\varphi} - L_\varphi\Big\|^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\sum_{|k|\le n} c_k e^{2\pi i\langle k,\xi\rangle} - \sum_{k\in\mathbb{Z}} c_k e^{2\pi i\langle k,\xi\rangle}\Big|^2 |K_\varphi(\xi,\eta)|^2\,d\xi\,d\eta \\
&= \int_{\mathbb{T}^n}\Big|\sum_{|k|\le n} c_k e^{2\pi i\langle k,\xi\rangle} - \sum_{k\in\mathbb{Z}} c_k e^{2\pi i\langle k,\xi\rangle}\Big|^2 \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2\,d\eta\,d\xi \\
&= \int_{\mathbb{T}^n}\Big|\sum_{|k|\le n} c_k e^{2\pi i\langle k,\xi\rangle} - \sum_{k\in\mathbb{Z}} c_k e^{2\pi i\langle k,\xi\rangle}\Big|^2 w_\varphi(\xi)\,d\xi \\
&\le B\Big\|\sum_{|k|\le n} c_k e_k - \sum_{k\in\mathbb{Z}} c_k e_k\Big\|^2_{L^2(\mathbb{T}^n)} \\
&= B\sum_{|k|>n}|c_k|^2 \\
&\to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

where $e_k(\xi) = e^{2\pi i\langle k,\xi\rangle}$. □

Lemma 3.2. *Suppose* $\varphi \in L^2(\mathbb{R}^{2n})$. *Then the following are equivalent:*

(i) *For any* $\{c_{k,l}\} \in \ell^2(\mathbb{Z}^{2n})$,

$$\sum_{k,l\in\mathbb{Z}^n} c_{k,l} T^t_{(k,l)}\varphi$$

converges to a continuous function.

(ii) $\varphi \in C(\mathbb{R}^{2n})$ *and*

$$\sup_{x,y\in\mathbb{R}^n}\sum_{k,l\in\mathbb{Z}^n}|\varphi(x-k,y-l)|^2 < \infty.$$

Proof. We first prove that (i) implies (ii).

Taking $c_{0,0} = 1$ and $c_{k,l} = 0$ for all $(k, l) \neq (0, 0)$, we see that $\varphi \in C(\mathbb{R}^{2n})$. Now

$$\int_{T^n}\int_{T^n} \sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2\,dx\,dy = \int_{R^n}\int_{R^n} |\varphi(x, y)|^2\,dx\,dy < \infty.$$

Hence $\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2 < \infty$ a.e. $x, y \in \mathbb{R}^n$. Since $\varphi \in C(\mathbb{R}^{2n})$, we have $\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2 < \infty$ for all $x, y \in \mathbb{R}^n$. For $x, y \in \mathbb{T}^n$, we define an operator $\Lambda_{(x,y)}$ on $\ell^2(\mathbb{Z}^{2n})$ as

$$\Lambda_{(x,y)}(\{c_{k,l}\}) = \sum_{k,l\in\mathbb{Z}^n} c_{k,l} T^t_{(k,l)}\varphi(x, y).$$

Then

$$\begin{aligned}
|\Lambda_{(x,y)}(\{c_{k,l}\})| &\leq \sum_{k,l\in\mathbb{Z}^n} |c_{k,l}||T^t_{(k,l)}\varphi(x, y)| \\
&= \sum_{k,l\in\mathbb{Z}^n} |c_{k,l}||e^{\pi i(\langle x,l\rangle - \langle y,k\rangle)}\varphi(x-k, y-l)| \\
&\leq \Big(\sum_{k,l\in\mathbb{Z}^n} |c_{k,l}|^2\Big)^{\frac{1}{2}} \Big(\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2\Big)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Hence

$$\|\Lambda_{(x,y)}\| \leq \Big(\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2\Big)^{\frac{1}{2}}.$$

In particular, taking

$$c_{k,l} = \overline{\varphi(x-k, y-l)}e^{-\pi i(\langle x,l\rangle - \langle y,k\rangle)}$$

for all $k, l \in \mathbb{Z}^n$, we observe that

$$\Lambda_{x,y}(c_{k,l}) = \sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2.$$

Thus

$$\|\Lambda_{(x,y)}\| = \Big(\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2\Big)^{\frac{1}{2}} \quad \text{for all } x, y \in \mathbb{T}^n.$$

Define

$$f(x, y) = \sum_{k,l\in\mathbb{Z}^n} c_{k,l} T^t_{(k,l)}\varphi(x, y),$$

where $\{c_{k,l}\} \in \ell^2(\mathbb{Z}^{2n})$, $x, y \in \mathbb{R}^n$. By our assumption, f is continuous on $\mathbb{R}^{2n}$, hence also when restricted to $\mathbb{T}^{2n}$. Thus it follows that

$$\sup_{x,y\in\mathbb{T}^n} |\Lambda_{(x,y)}(\{c_{k,l}\})| = \sup_{x,y\in\mathbb{T}^n} \Big|\sum_{k,l\in\mathbb{Z}^n} c_{k,l} T^t_{(k,l)}\varphi(x, y)\Big| = \sup_{x,y\in\mathbb{T}^n} |f(x, y)| < \infty.$$

Now, applying the uniform boundedness principle, we have $\sup_{x,y\in\mathbb{T}^n} \|\Lambda_{(x,y)}\| \leq M$ for some $M > 0$. In other words,

$$\sup_{x,y\in\mathbb{T}^n} \Big(\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2\Big)^{\frac{1}{2}} \leq M.$$

Since the function $(x, y) \to \sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2$ is 1×1 periodic on $\mathbb{R}^{2n}$, we get

$$\sup_{x,y\in\mathbb{R}^n} \sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k, y-l)|^2 \leq M^2,$$

from which (ii) follows.

Now we prove that (ii) implies (i). For all $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}\sum_{k,l\in\mathbb{Z}^n} |c_{k,l} T^t_{(k,l)}\varphi(x,y)| &= \sum_{k,l\in\mathbb{Z}^n} |c_{k,l} e^{\pi i(\langle x,l\rangle - \langle y,k\rangle)}\varphi(x-k, y-l)| \\ &\leq \Big(\sum_{k,l\in\mathbb{Z}^n} |c_{k,l}|^2\Big)^{\frac{1}{2}}\Big(\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k,y-l)|^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{k,l\in\mathbb{Z}^n} |c_{k,l}|^2\Big)^{\frac{1}{2}} \sup_{x,y\in\mathbb{R}^n}\Big(\sum_{k,l\in\mathbb{Z}^n} |\varphi(x-k,y-l)|^2\Big)^{\frac{1}{2}} < \infty.\end{aligned}$$

Hence the convergence is uniform on $\mathbb{R}^{2n}$. Since $\varphi \in C(\mathbb{R}^{2n})$, the limit function must be continuous. □

Lemma 3.3. *Let $\varphi, \psi \in L^2(\mathbb{R}^{2n})$ and satisfying condition* C. *Assume that the collections $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ and $\{T^t_{(k,l)}\psi : (k,l) \in \mathbb{Z}^{2n}\}$ are frames for $V^t(\varphi)$. Suppose $\varphi \in C(\mathbb{R}^{2n})$, $\psi \in V^t_0(\varphi)$ and $\sum_{k\in\mathbb{Z}^n}|\varphi(x+k,y)|^2 \leq M$ for all $x, y \in \mathbb{R}^n$, where $V^t_0(\varphi) = \overline{\operatorname{span}}\{T^t_{(k,0)}\varphi : k \in \mathbb{Z}^n\}$. Then there exists a constant $M' > 0$ such that*

$$\sum_{k\in\mathbb{Z}^n} |\psi(x+k,y)|^2 \leq M' \quad \text{for all } x, y \in \mathbb{R}^n.$$

Proof. Let $f \in V^t_0(\varphi)$. Since $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$, there exists $\{c_{k,0}\} \in \ell^2(\mathbb{Z}^n)$ such that

$$f = \sum_{k\in\mathbb{Z}^n} c_{k,0} T^t_{(k,0)}\varphi.$$

Taking $l = 0$ in Lemma 3.2, we get $f \in C(\mathbb{R}^{2n})$. Thus $V^t_0(\varphi) \subset C(\mathbb{R}^{2n})$. Let

$$\psi(x,y) = \sum_{k\in\mathbb{Z}^n} d_{k,0} T^t_{(k,0)}\varphi(x,y)$$

for some $\{d_{k,0}\} \in \ell^2(\mathbb{Z}^n)$. It can be easily shown that the above series converges uniformly to the function ψ on $\mathbb{R}^{2n}$. Moreover, using (2.3), we have

$$\begin{aligned}K_\psi(\xi,\eta) &= \sum_{k\in\mathbb{Z}^n} d_{k,0} K_{T^t_{(k,0)}\varphi}(\xi,\eta) \\ &= \sum_{k\in\mathbb{Z}^n} d_{k,0} e^{2\pi i\langle k,\xi\rangle} K_\varphi(\xi,\eta) \\ &= C(\xi)K_\varphi(\xi,\eta),\end{aligned}$$

where $C(\xi) = \sum_{k\in\mathbb{Z}^n} d_{k,0} e^{2\pi i\langle k,\xi\rangle}$, and

$$\begin{aligned}w_\psi(\xi) &= \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\psi(\xi+m,\eta)|^2\, d\eta \\ &= \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |C(\xi)|^2 |K_\varphi(\xi+m,\eta)|^2\, d\eta \\ &= |C(\xi)|^2 w_\varphi(\xi).\end{aligned}$$

Since φ, ψ satisfy condition C and $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$, $\{T^t_{(k,l)}\psi : (k,l) \in \mathbb{Z}^{2n}\}$ are frames for $V^t(\varphi)$, using Theorem 2.7, we obtain that $C(\xi)$ is bounded on Ω_φ, where

$$\Omega_\varphi = \{\xi \in \mathbb{T}^n : w_\varphi(\xi) \neq 0\}.$$

Let $\tilde{C}(\xi) = C(\xi)\chi_{\Omega_\varphi}(\xi)$. Then $\tilde{C}(\xi)$ is bounded on $\mathbb{T}^n$. Let

$$\tilde{C}(\xi) = \sum_{k\in\mathbb{Z}^n} \tilde{c_k} e^{2\pi i\langle k,\xi\rangle}$$

for some $\{\tilde{c_k}\} \in \ell^2(\mathbb{Z}^n)$. Since $C(\xi)K_\varphi(\xi,\eta) = \tilde{C}(\xi)K_\varphi(\xi,\eta)$ a.e. $\xi, \eta \in \mathbb{R}^n$, we have

$$\psi(x,y) = \sum_{k\in\mathbb{Z}^n} \tilde{c_k} T^t_{(k,0)}\varphi(x,y), \quad (x,y) \in \mathbb{R}^{2n}.$$

The above series converges both in $L^2(\mathbb{R}^{2n})$ and pointwise on $\mathbb{R}^{2n}$. Let τ_y denote the translation operator

$\tau_y f(x) = f(x-y)$, $x, y \in \mathbb{R}^n$. Then for all $x, y \in \mathbb{R}^n$, using Lemma 2.9, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}^n} |\psi(x+n,y)|^2 &= \sum_{n\in\mathbb{Z}^n}\Big|\sum_{k\in\mathbb{Z}^n} \tilde{c_k} T^t_{(k,0)}\varphi(x+n,y)\Big|^2 \\ &= \sum_{n\in\mathbb{Z}^n}\Big|\sum_{k\in\mathbb{Z}^n} \tilde{c_k} e^{-\pi i\langle y,k\rangle}\varphi(x+n-k,y)\Big|^2 \\ &= \int_{\mathbb{T}^n} |(\tau_{\frac{y}{2}}\tilde{C})(\xi)|^2 \Big|\sum_{n\in\mathbb{Z}^n}\varphi(x+n,y)e^{2\pi i\langle n,\xi\rangle}\Big|^2\,d\xi \\ &\le \|\tau_{\frac{y}{2}}\tilde{C}\|^2_\infty \int_{T^n}\Big|\sum_{n\in\mathbb{Z}^n}\varphi(x+n,y)e^{2\pi i\langle n,\xi\rangle}\Big|^2\,d\xi \\ &= \|\tilde{C}\|^2_\infty \sum_{n\in\mathbb{Z}^n}|\varphi(x+n,y)|^2 \\ &\le \|\tilde{C}\|^2_\infty . M,\end{aligned}$$

thus proving the lemma. □

Theorem 3.4. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ and let it satisfy condition C. Assume that $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$. Define $\tilde{\varphi} \in L^2(\mathbb{R}^{2n})$ such that*

$$K_{\tilde{\varphi}}(\xi,\eta) = \begin{cases} \dfrac{1}{w_\varphi(\xi)} K_\varphi(\xi,\eta), & \xi \in \Omega_\varphi, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.1}$$

Then $\{T^t_{(k,l)}\tilde{\varphi} : (k,l) \in \mathbb{Z}^{2n}\}$ is the canonical dual frame of $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$.

Proof. Since $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$, we have

$$Sf = \sum_{k,l\in\mathbb{Z}^n} \langle f, T^t_{(k,l)}\varphi\rangle T^t_{(k,l)}\varphi \quad \text{for all } f \in V^t(\varphi). \tag{3.2}$$

Now, (3.1) can be written as $K_{\tilde{\varphi}}(\xi,\eta) = \sum_{l\in\mathbb{Z}^n} \rho_l(\xi) K_\varphi(\xi+l,\eta)$, where

$$\rho_0(\xi) = \begin{cases} \dfrac{1}{w_\varphi(\xi)}, & \xi \in \Omega_\varphi, \\ 0, & \text{otherwise}, \end{cases}$$

and $\rho_l(\xi) = 0$ a.e. $\xi \in \mathbb{T}^n$ for $l \neq 0$. Let $\rho(\xi) = \{\rho_l(\xi)\}_{l\in\mathbb{Z}^n}$ for $\xi \in \mathbb{T}^n$. Then

$$\|\rho\|^2_{L^2(\mathbb{T}^n,\ell^2(\mathbb{Z}^n),w_\varphi)} = \int_{T^n} \|\rho(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\varphi(\xi)\,d\xi = \int_{\Omega_\varphi} \frac{1}{w_\varphi(\xi)}\,d\xi \le \frac{1}{A} < \infty,$$

using Theorem 2.7. From equation (2.4), it follows that $\tilde{\varphi} \in V^t(\varphi)$. By Theorem 2.8, the canonical dual of $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is given by $\{S^{-1}T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\} = \{T^t_{(k,l)}S^{-1}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$. Thus, in order to prove the theorem, we need to show that $S\tilde{\varphi} = \varphi$. Now, we have

$$\begin{aligned}\langle \tilde{\varphi}, T^t_{(k,l)}\varphi\rangle &= \langle K_{\tilde{\varphi}}, K_{T^t_{(k,l)}\varphi}\rangle \\ &= \int_{R^n}\int_{R^n} K_{\tilde{\varphi}}(\xi,\eta)\overline{K_{T^t_{(k,l)}\varphi}(\xi,\eta)}\,d\xi\,d\eta \\ &= \int_{R^n}\int_{\Omega_\varphi} \frac{1}{w_\varphi(\xi)} K_\varphi(\xi,\eta) e^{-\pi i\langle k,l\rangle} e^{-2\pi i\langle k,\xi\rangle}\overline{K_\varphi(\xi+l,\eta)}\,d\xi\,d\eta,\end{aligned}$$

using (3.1) and (2.3). Then

$$\begin{aligned}\langle \tilde{\varphi}, T^t_{(k,l)}\varphi\rangle &= e^{-\pi i\langle k,l\rangle}\int_{R^n}\int_{R^n} \chi_{\Omega_\varphi}(\xi)\frac{1}{w_\varphi(\xi)} K_\varphi(\xi,\eta)\overline{K_\varphi(\xi+l,\eta)}e^{-2\pi i\langle k,\xi\rangle}\,d\xi\,d\eta \\ &= e^{-\pi i\langle k,l\rangle}\int_{T^n} \chi_{\Omega_\varphi}(\xi)\frac{1}{w_\varphi(\xi)}\sum_{m\in\mathbb{Z}^n}\int_{R^n} K_\varphi(\xi+m,\eta)\overline{K_\varphi(\xi+m+l,\eta)}\,d\eta\, e^{-2\pi i\langle k,\xi\rangle}\,d\xi.\end{aligned}$$

Hence

$$\langle \tilde{\varphi}, T^t_{(k,o)}\varphi\rangle = \int_{T^n} \chi_{\Omega_\varphi}(\xi)\frac{1}{w_\varphi(\xi)} w_\varphi(\xi)e^{-2\pi i\langle k,\xi\rangle}\, d\xi = \widehat{\chi_{\Omega_\varphi}}(k),$$

and for $l \neq 0$, we have $\langle \tilde{\varphi}, T^t_{(k,l)}\varphi\rangle = 0$, as φ satisfies condition C. Now, using Lemma 3.1, we get

$$\begin{aligned}\sum_{k,l\in\mathbb{Z}^n} \langle \tilde{\varphi}, T^t_{(k,l)}\varphi\rangle K_{T^t_{(k,l)}\varphi}(\xi,\eta) &= \sum_{k\in\mathbb{Z}^n} \langle \tilde{\varphi}, T^t_{(k,0)}\varphi\rangle K_{T^t_{(k,0)}\varphi}(\xi,\eta)\\ &= \sum_{k\in\mathbb{Z}^n} \widehat{\chi_{\Omega_\varphi}}(k) K_{T^t_{(k,0)}\varphi}(\xi,\eta)\\ &= \Big(\sum_{k\in\mathbb{Z}^n} \widehat{\chi_{\Omega_\varphi}}(k)e^{2\pi i\langle k,\xi\rangle}\Big)K_\varphi(\xi,\eta)\\ &= \chi_{\Omega_\varphi}(\xi)K_\varphi(\xi,\eta).\end{aligned}$$

It can be easily shown that $\chi_{\Omega_\varphi}(\xi)K_\varphi(\xi,\eta) = K_\varphi(\xi,\eta)$. This implies that

$$\sum_{k,l\in\mathbb{Z}^n} \langle \tilde{\varphi}, T^t_{(k,l)}\varphi\rangle T^t_{(k,l)}\varphi = \varphi.$$

Then it follows from (3.2) that $S\tilde{\varphi} = \varphi$, thus proving the theorem. □

4 A sampling theorem on a subspace of $V^t(\varphi)$

The following theorem gives a necessary and sufficient condition for obtaining a reconstruction formula for functions belonging to a subspace of $V^t(\varphi)$ from their samples $\{f(k,j) : k \in \mathbb{Z}^n\}$ for each fixed $j \in \mathbb{Z}^n$.

Theorem 4.1. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ and satisfying condition C. Assume that $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$. Then the following two statements are equivalent:*

(i) *$\varphi \in C(\mathbb{R}^{2n})$, $\sum_{k\in\mathbb{Z}^n}|\varphi(x-k,y)|^2$ is bounded on $\mathbb{R}^{2n}$ and there exist constants $A_j, B_j > 0$ such that*

$$A_j\chi_{\Omega_\varphi}(\xi) \le |\Phi_j(\xi)| \le B_j\chi_{\Omega_\varphi}(\xi) \quad \text{a.e. } \xi \in \mathbb{T}^n,\ \text{for all } j \in \mathbb{Z}^n, \tag{4.1}$$

where

$$\Phi_j(\xi) = \sum_{k\in\mathbb{Z}^n} \varphi(k,j)e^{\pi i\langle k,j\rangle}e^{2\pi i\langle k,\xi\rangle}. \tag{4.2}$$

(ii) *$\sum_{k\in\mathbb{Z}^n} c_{k,0}T^t_{(k,0)}\varphi$ converges to a continuous function for any $\{c_{k,0}\} \in \ell^2(\mathbb{Z}^n)$, and there exists a countable collection of functions $\{\psi_j \in V^t(\varphi) : j \in \mathbb{Z}^n\}$ such that for all $j \in \mathbb{Z}^n$, ψ_j satisfies condition C and $\{T^t_{(k,l)}\psi_j : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$. Further, for all $j \in \mathbb{Z}^n$,*

$$f(x,y) = \sum_{k\in\mathbb{Z}^n} e^{\pi i\langle j,k\rangle}f(k,j)T^t_{(k,0)}\psi_j(x,y) \quad \text{for all } f \in V^t_0(\varphi), \tag{4.3}$$

where the convergence is both in $L^2(\mathbb{R}^{2n})$ and uniform on $\mathbb{R}^{2n}$.

Proof. First, we shall prove that (i) implies (ii). Taking $l = 0$ in Lemma 3.2, we see that $\sum_{k\in\mathbb{Z}^n} c_{k,0}T^t_{(k,0)}\varphi$ converges to a continuous function for any $\{c_{k,0}\} \in \ell^2(\mathbb{Z}^n)$. For $j \in \mathbb{Z}^n$, define $\psi_j \in L^2(\mathbb{R}^{2n})$ such that

$$K_{\psi_j}(\xi,\eta) = \begin{cases} \dfrac{1}{\Phi_j(\xi)}K_\varphi(\xi,\eta), & \xi \in \Omega_\varphi,\\ 0, & \text{otherwise.}\end{cases} \tag{4.4}$$

Then

$$\begin{aligned} w_{\psi_j}(\xi) &= \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} |K_{\psi_j}(\xi+m,\eta)|^2\, d\eta\\ &= \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} \Big|\frac{1}{\Phi_j(\xi)}K_\varphi(\xi+m,\eta)\Big|^2\, d\eta\\ &= \frac{1}{|\Phi_j(\xi)|^2}w_\varphi(\xi), \quad \xi \in \Omega_\varphi. \end{aligned} \tag{4.5}$$

For $\xi \in \Omega_\varphi^c$, we have $w_{\psi_j}(\xi) = 0$. Thus we observe that $\Omega_\varphi = \Omega_{\psi_j}$ for all $j \in \mathbb{Z}^n$. Since φ satisfies condition C and $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$, using Theorem 2.7 and (4.1), we get constants $A_j, B_j > 0$ such that $A_j \le w_{\psi_j}(\xi) \le B_j$ a.e. $\xi \in \Omega_\varphi$ for all $j \in \mathbb{Z}^n$. Now, for $l \ne 0$,

$$\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_{\psi_j}(\xi+m,\eta)\overline{K_{\psi_j}(\xi+m+l,\eta)}\,d\eta = \frac{1}{|\Phi_j(\xi)|^2} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_\varphi(\xi+m,\eta)\overline{K_\varphi(\xi+m+l,\eta)}\,d\eta = 0,$$

as φ satisfies condition C. Thus ψ_j satisfies condition C for all $j \in \mathbb{Z}^n$. Hence $\{T^t_{(k,l)}\psi_j : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\psi_j)$ for all $j \in \mathbb{Z}^n$ by Theorem 2.7. Now, using an argument similar to the one in Theorem 3.4, it follows from (4.4) that $\psi_j \in V^t(\varphi)$. In fact, $\psi_j \in V^t_0(\varphi)$. Again, since (4.4) can be written as

$$K_\varphi(\xi,\eta) = \begin{cases} \Phi_j(\xi)K_{\psi_j}(\xi,\eta), & \xi \in \Omega_\varphi, \\ 0, & \text{otherwise,} \end{cases}$$

it follows that $\varphi \in V^t(\psi_j)$. Hence $V^t(\varphi) = V^t(\psi_j)$ for all $j \in \mathbb{Z}^n$. For $j \in \mathbb{Z}^n$, define $\tilde{\psi}_j \in L^2(\mathbb{R}^{2n})$ such that

$$K_{\tilde{\psi}_j}(\xi,\eta) = \begin{cases} \dfrac{1}{w_{\psi_j}(\xi)} K_{\psi_j}(\xi,\eta), & \xi \in \Omega_\varphi, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

By Theorem 3.4, $\{T^t_{(k,l)}\tilde{\psi}_j : (k,l) \in \mathbb{Z}^{2n}\}$ is the canonical dual of $\{T^t_{(k,l)}\psi_j : (k,l) \in \mathbb{Z}^{2n}\}$. Using (4.4) and (4.5), equation (4.6) can be rewritten as

$$K_{\tilde{\psi}_j}(\xi,\eta) = \begin{cases} \dfrac{\overline{\Phi_j(\xi)}}{w_\varphi(\xi)} K_\varphi(\xi,\eta), & \xi \in \Omega_\varphi, \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

Let $f \in V^t(\varphi)$. Then

$$\begin{aligned} \langle f, T^t_{(k,l)}\tilde{\psi}_j\rangle &= \langle K_f, K_{T^t_{(k,l)}\tilde{\psi}_j}\rangle \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_f(\xi,\eta)\overline{K_{T^t_{(k,l)}\tilde{\psi}_j}(\xi,\eta)}\,d\xi\,d\eta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{l'\in\mathbb{Z}^n} \rho_{l'}(\xi)K_\varphi(\xi+l',\eta)e^{-\pi i\langle 2\xi+l,k\rangle}\overline{K_{\tilde{\psi}_j}(\xi+l,\eta)}\,d\xi\,d\eta, \end{aligned}$$

using (2.4) and (2.3). Now substituting for $K_{\tilde{\psi}_j}(\xi+l,\eta)$ from (4.7), we get

$$\begin{aligned} \langle f, T^t_{(k,l)}\tilde{\psi}_j\rangle &= e^{-\pi i\langle l,k\rangle} \int_{\mathbb{R}^n}\int_{\Omega_\varphi} \sum_{l'\in\mathbb{Z}^n} \rho_{l'}(\xi)K_\varphi(\xi+l',\eta)\frac{\Phi_j(\xi)}{w_\varphi(\xi)}\overline{K_\varphi(\xi+l,\eta)}e^{-2\pi i\langle k,\xi\rangle}\,d\xi\,d\eta \\ &= e^{-\pi i\langle l,k\rangle} \int_{\mathbb{T}^n} \frac{\Phi_j(\xi)}{w_\varphi(\xi)} \sum_{l'\in\mathbb{Z}^n}\rho_{l'}(\xi) \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} K_\varphi(\xi+m+l',\eta)\overline{K_\varphi(\xi+m+l,\eta)}\,d\eta e^{-2\pi i\langle k,\xi\rangle}\,d\xi \\ &= e^{-\pi i\langle l,k\rangle} \int_{\mathbb{T}^n} \frac{\Phi_j(\xi)}{w_\varphi(\xi)}\rho_l(\xi) \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} |K_\varphi(\xi+m+l,\eta)|^2\,d\eta e^{-2\pi i\langle k,\xi\rangle}\,d\xi, \end{aligned}$$

as φ satisfies condition C. Thus

$$\begin{aligned} \langle f, T^t_{(k,l)}\tilde{\psi}_j\rangle &= e^{-\pi i\langle l,k\rangle} \int_{\mathbb{T}^n} \frac{\Phi_j(\xi)}{w_\varphi(\xi)}\rho_l(\xi) \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2\,d\eta e^{-2\pi i\langle k,\xi\rangle}\,d\xi \\ &= e^{-\pi i\langle l,k\rangle} \int_{\mathbb{T}^n} \frac{\Phi_j(\xi)}{w_\varphi(\xi)}\rho_l(\xi)w_\varphi(\xi)e^{-2\pi i\langle k,\xi\rangle}\,d\xi \\ &= e^{-\pi i\langle l,k\rangle} \int_{\mathbb{T}^n} \Phi_j(\xi)\rho_l(\xi)e^{-2\pi i\langle k,\xi\rangle}\,d\xi, \end{aligned}$$

where $\Phi_j(\xi)$ is given by (4.2) and $\rho_l(\xi) = \sum_{m\in\mathbb{Z}^n} c_{m,l}e^{\pi i\langle l,m\rangle}e^{2\pi i\langle m,\xi\rangle}$. Hence

$$\begin{aligned}\langle f, T^t_{(k,l)}\tilde{\psi}_j\rangle &= e^{-\pi i\langle l,k\rangle}\sum_{m\in\mathbb{Z}^n} c_{m,l}e^{\pi i\langle l,m\rangle}\varphi(k-m,j)e^{\pi i\langle k-m.j\rangle}\\ &= e^{-\pi i\langle l,k\rangle}e^{\pi i\langle j,k\rangle}\sum_{m\in\mathbb{Z}^n} c_{m,l}e^{\pi i\langle l,m\rangle}T^t_{(m,0)}\varphi(k,j).\end{aligned} \tag{4.8}$$

Since $\{T^t_{(k,l)}\tilde{\psi}_j : (k,l)\in\mathbb{Z}^{2n}\}$ is the canonical dual frame of $\{T^t_{(k,l)}\psi_j : (k,l)\in\mathbb{Z}^{2n}\}$, using (2.2), we get

$$f = \sum_{k,l\in\mathbb{Z}^n}\langle f, T^t_{(k,l)}\tilde{\psi}_j\rangle T^t_{(k,l)}\psi_j \quad \text{for all } f\in V^t(\varphi). \tag{4.9}$$

Let $f\in V^t_0(\varphi)$, i.e., $f = \sum_{m\in\mathbb{Z}^n} c_{m,0}T^t_{(m,0)}\varphi$ for some $\{c_{m,o}\}\in\ell^2(\mathbb{Z}^n)$. Then $c_{m,l}=0$ for all $m\in\mathbb{Z}^n$ and all $l\in\mathbb{Z}^n\setminus\{0\}$. It follows from (4.8) that $\langle f, T^t_{(k,l)}\tilde{\psi}_j\rangle = 0$ for all $l\neq 0$ and

$$\langle f, T^t_{(k,0)}\tilde{\psi}_j\rangle = e^{\pi i\langle j,k\rangle}\sum_{m\in\mathbb{Z}^n} c_{m,0}T^t_{(m,0)}\varphi(k,j) = e^{\pi i\langle j,k\rangle}f(k,j).$$

Hence from (4.9), we get

$$f = \sum_{k\in\mathbb{Z}^n}\langle f, T^t_{(k,0)}\tilde{\psi}_j\rangle T^t_{(k,0)}\psi_j = \sum_{k\in\mathbb{Z}^n} e^{\pi i\langle j,k\rangle}f(k,j)T^t_{(k,0)}\psi_j.$$

Since $\psi_j\in V^t_0(\varphi)$, by Lemma 3.3 and the Cauchy–Schwarz inequality, the above series converges uniformly on $\mathbb{R}^{2n}$, and hence we obtain the reconstruction formula

$$f(x,y) = \sum_{k\in\mathbb{Z}^n} e^{\pi i\langle j,k\rangle}f(k,j)T^t_{(k,0)}\psi_j(x,y)$$

in the sense of uniform convergence.

Now, we prove that (ii) implies (i). Taking $l=0$ in Lemma 3.2, we get $\varphi\in C(\mathbb{R}^{2n})$, and $\sum_{k\in\mathbb{Z}^n}|\varphi(x-k,y)|^2$ is bounded on $\mathbb{R}^{2n}$. Now, fix $j\in\mathbb{Z}^n$. Then from (4.3) we have

$$\varphi(x,y) = \sum_{k\in\mathbb{Z}^n} e^{\pi i\langle j,k\rangle}\varphi(k,j)T^t_{(k,0)}\psi_j(x,y).$$

Using (2.3) and (4.2), we get

$$\begin{aligned}K_\varphi(\xi,\eta) &= \sum_{k\in\mathbb{Z}^n} e^{\pi i\langle j,k\rangle}\varphi(k,j)K_{T^t_{(k,0)}\psi_j}(\xi,\eta)\\ &= \sum_{k\in\mathbb{Z}^n} e^{\pi i\langle j,k\rangle}\varphi(k,j)e^{2\pi i\langle k,\xi\rangle}K_{\psi_j}(\xi,\eta)\\ &= \Phi_j(\xi)K_{\psi_j}(\xi,\eta)\end{aligned}$$

and

$$\begin{aligned}w_\varphi(\xi) &= \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|K_\varphi(\xi+m,\eta)|^2\,d\eta\\ &= \sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\Phi_j(\xi)K_{\psi_j}(\xi+m,\eta)|^2\,d\eta\\ &= |\Phi_j(\xi)|^2 w_{\psi_j}(\xi).\end{aligned} \tag{4.10}$$

Then $\Omega_\varphi\subseteq\Omega_{\psi_j}$. Since φ and ψ_j satisfy condition C and $\{T^t_{(k,l)}\varphi : (k,l)\in\mathbb{Z}^{2n}\}$ and $\{T^t_{(k,l)}\psi_j : (k,l)\in\mathbb{Z}^{2n}\}$ are frames for $V^t(\varphi)$, using Theorem 2.7, we obtain from (4.10) that there exist constants $A_j, B_j > 0$ such that $A_j\le|\Phi_j(\xi)|\le B_j$ a.e. $\xi\in\Omega_\varphi$.

Now we shall show that $\Phi_j(\xi)$ is equal to 0 a.e. $\xi\in\Omega^c_\varphi$ for all $j\in\mathbb{Z}^n$. For fixed $j\in\mathbb{Z}^n$, we have

$$\int_{\Omega^c_\varphi}|\Phi_j(\xi)|^2\,d\xi = \int_{\Omega^c_\varphi}\Big|\sum_{k\in\mathbb{Z}^n}\varphi(k,j)e^{\pi i\langle k,j\rangle}e^{2\pi i\langle k,\xi\rangle}\Big|^2\,d\xi,$$

using (4.2). Then using Lemma 2.9, we get

$$\int_{\Omega_\varphi^c} |\Phi_j(\xi)|^2\,d\xi = \int_{T^n} \Big|\chi_{\Omega_\varphi^c}(\xi) \sum_{k\in\mathbb{Z}^n} \varphi(k,j)e^{\pi i\langle k,j\rangle} e^{2\pi i\langle k,\xi\rangle}\Big|^2\,d\xi$$
$$= \sum_{k\in\mathbb{Z}^n}\Big|\sum_{r\in\mathbb{Z}^n} c_{r,0}e^{\pi i\langle j,k-r\rangle}\varphi(k-r,j)\Big|^2,$$

where $\chi_{\Omega_\varphi^c}(\xi) = \sum_{r\in\mathbb{Z}} c_{r,0}e^{2\pi i\langle r,\xi\rangle}$ with $\{c_{r,0}\} \in \ell^2(\mathbb{Z}^n)$. In order to prove that $\Phi_j(\xi)$ is equal to 0 a.e. $\xi \in \Omega_\varphi^c$ for all $j \in \mathbb{Z}^n$, it is enough to prove that

$$\int_{\Omega_\varphi^c} |\Phi_j(\xi)|^2\,d\xi = 0 \quad \text{for all } j \in \mathbb{Z}^n,$$

which is equivalent to show that

$$\sum_{r\in\mathbb{Z}^n} c_{r,0}e^{\pi i\langle j,k-r\rangle}\varphi(k-r,j) = 0 \quad \text{for all } k,j \in \mathbb{Z}^n.$$

In other words, we need to show that $\sum_{r\in\mathbb{Z}^n} c_{r,0}e^{-\pi i\langle j,r\rangle}\varphi(k-r,j) = 0$ for all $k,j \in \mathbb{Z}^n$. In fact, we will show that

$$\sum_{r\in\mathbb{Z}^n} c_{r,0}e^{-\pi i\langle y,r\rangle}\varphi(x-r,y) = 0 \quad \text{for all } x,y \in \mathbb{R}^n$$

i.e., to show that $\sum_{r\in\mathbb{Z}^n} c_{r,0}T^t_{(r,0)}\varphi(x,y) = 0$ for all $x,y \in \mathbb{R}^n$. Since $\chi_{\Omega_\varphi^c}(\xi)K_\varphi(\xi,\eta) = 0$ a.e. $\xi,\eta \in \mathbb{R}^n$, we have

$$\Big(\sum_{r\in\mathbb{Z}^n} c_{r,0}e^{2\pi i\langle r,\xi\rangle}\Big)K_\varphi(\xi,\eta) = 0$$

in $L^2(\mathbb{R}^{2n})$. Using Lemma 3.1, we see that the series

$$\sum_{r\in\mathbb{Z}^n} c_{r,0}K_{T^t_{(r,0)}\varphi}(\xi,\eta)$$

converges to 0 in $L^2(\mathbb{R}^{2n})$, which implies that $\sum_{r\in\mathbb{Z}^n} c_{r,0}T^t_{(r,0)}\varphi(x,y)$ converges to 0 in $L^2(\mathbb{R}^{2n})$. But, by assumption, $\sum_{r\in\mathbb{Z}^n} c_{r,0}T^t_{(r,0)}\varphi(x,y)$ converges pointwise and hence it converges pointwise to 0 for all $x,y \in \mathbb{R}^n$, thus proving our claim. □

In the following theorem, we provide a necessary condition for obtaining a reconstruction formula for functions belonging to a subspace of $V^t(\varphi)$ from their samples $\{f(k,j) : k,j \in \mathbb{Z}^n\}$. However, we are not able to get the sufficient condition of this theorem. We leave this as an open problem to the interested reader.

Theorem 4.2. *Let $\varphi \in L^2(\mathbb{R}^{2n})$ and satisfying condition* C. *Assume that $\{T^t_{(k,l)}\varphi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$. Suppose $\sum_{k\in\mathbb{Z}^n} c_{k,0}T^t_{(k,0)}\varphi$ converges to a continuous function for any $\{c_{k,0}\} \in \ell^2(\mathbb{Z}^n)$, and there exists a function $\psi \in V^t(\varphi)$ which satisfies condition* C *and $\{T^t_{(k,l)}\psi : (k,l) \in \mathbb{Z}^{2n}\}$ is a frame for $V^t(\varphi)$ such that*

$$f(x,y) = \sum_{k,l\in\mathbb{Z}^n} f(k,l)T^t_{(k,l)}\psi(x,y) \quad \textit{for all } f \in V^t_0(\varphi), \tag{4.11}$$

where the convergence is both in $L^2(\mathbb{R}^{2n})$ and uniform on $\mathbb{R}^{2n}$.

Then $\varphi \in C(\mathbb{R}^{2n})$, $\sum_{k\in\mathbb{Z}^n}|\varphi(x-k,y)|^2$ is bounded on $\mathbb{R}^{2n}$ and there exist constants A, B such that

$$A\chi_{\Omega_\varphi}(\xi) \le \|\Phi(\xi)\|_{\ell^2(\mathbb{Z}^n)} \le B\chi_{\Omega_\varphi}(\xi) \quad \textit{a.e. } \xi \in \mathbb{T}^n,$$

where $\Phi(\xi) = \{\Phi_j(\xi)\}_{j\in\mathbb{Z}^n}$ and $\Phi_j(\xi)$ is given by (4.2).

Proof. Taking $l = 0$ in Lemma 3.2, we get $\varphi \in C(\mathbb{R}^{2n})$ and $\sum_{k\in\mathbb{Z}^n}|\varphi(x-k,y)|^2$ is bounded on $\mathbb{R}^{2n}$. Now, from (4.11), we have $\varphi(x,y) = \sum_{k,l\in\mathbb{Z}^n} \varphi(k,l)T^t_{(k,l)}\psi(x,y)$. Then proceeding as in Theorem 4.1, we get

$$K_\varphi(\xi,\eta) = \sum_{l\in\mathbb{Z}^n} \Phi_l(\xi)K_\psi(\xi+l,\eta)$$

and

$$\begin{aligned} w_\varphi(\xi) &= \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |K_\varphi(\xi+m,\eta)|^2\, d\eta \\ &= \sum_{l_1,l_2\in\mathbb{Z}^n} \Phi_{l_1}(\xi)\overline{\Phi_{l_2}(\xi)} \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} K_\psi(\xi+m+l_1,\eta)\overline{K_\psi(\xi+m+l_2,\eta)}\, d\eta \\ &= \sum_{l\in\mathbb{Z}^n} |\Phi_l(\xi)|^2 \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{T}^n} |K_\psi(\xi+m+l,\eta)|^2\, d\eta, \end{aligned}$$

as ψ satisfies condition C. Hence

$$w_\varphi(\xi) = \|\Phi(\xi)\|^2_{\ell^2(\mathbb{Z}^n)} w_\psi(\xi).$$

Using arguments as in Theorem 4.1, we can find constants $A, B > 0$ such that $A \le \|\Phi(\xi)\|_{\ell^2(\mathbb{Z}^n)} \le B$ a.e. $\xi \in \Omega_\varphi$.

Now

$$\begin{aligned} \int_{\Omega_\varphi^c} \|\Phi(\xi)\|^2_{\ell^2(\mathbb{Z}^n)}\, d\xi &= \int_{\Omega_\varphi^c} \sum_{j\in\mathbb{Z}^n} |\Phi_j(\xi)|^2\, d\xi \\ &= \sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\Big|\sum_{r\in\mathbb{Z}^n} c_{r,0} e^{\pi i\langle j,k-r\rangle}\varphi(k-r,j)\Big|^2, \end{aligned}$$

where $\{c_{r,0}\} \in \ell^2(\mathbb{Z}^n)$. In order to prove that $\|\Phi(\xi)\|_{\ell^2(\mathbb{Z}^n)}$ is equal to 0 a.e. $\xi \in \Omega_\varphi^c$, it is enough to prove that

$$\int_{\Omega_\varphi^c} \|\Phi(\xi)\|^2_{\ell^2(\mathbb{Z}^n)}\, d\xi = 0,$$

which is equivalent to show that

$$\sum_{r\in\mathbb{Z}^n} c_{r,0} e^{\pi i\langle j,k-r\rangle}\varphi(k-r,j) = 0 \quad \text{for all } k, j \in \mathbb{Z}^n.$$

This can be shown in lines similar to the ones in the proof of Theorem 4.1, from which the theorem will follow. □

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