Research Article

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A linear regularization method for a parameter identification problem in heat equation

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Abstract: Recently, Nair and Roy (2017) considered a linear regularization method for a parameter identification problem in an elliptic PDE. In this paper, we consider similar procedure for identifying the diffusion coefficient in the heat equation, modifying the Sobolev spaces involved appropriately. We derive error estimates under appropriate conditions and also consider the finite-dimensional realization of the method, which is essential for practical application. In the analysis of finite-dimensional realization, we give a procedure to obtain finite-dimensional subspaces of an infinite-dimensional Hilbert space $L^2(0, T; H^1(\Omega))$ by doing double discretization, that is, discretization corresponding to both the space and time domain. Also, we analyze the parameter choice strategy and obtain an a posteriori parameter which is order optimal.

Keywords: Ill-posed, projection, regularization, parameter identification, parameter choice

MSC 2010: 35R30, 65J10, 65N20, 65N21, 65N30

1 Introduction and formulation of the problem

Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. For $T > 0$, we write $Q = \Omega \times [0, T]$ and $\Gamma = \partial\Omega \times [0, T]$. Consider the PDE

$$u_t - \operatorname{div}(q(x)\nabla u) = f(x, t) \quad \text{in } Q \tag{1.1}$$

along with the conditions

$$q(x)\frac{\partial u}{\partial \nu} = g(x, t) \quad \text{on } \Gamma, \tag{1.2}$$

$$u(x, 0) = h(x) \qquad \text{in } \Omega, \tag{1.3}$$

where ν is the unit outward normal to $\partial\Omega$, $f \in L^2(0, T; L^2(\Omega))$, $h \in L^2(\Omega)$, $q \in H^1(\Omega)$ and $g\colon \partial\Omega \times [0, T] \to \mathbb{R}$ is such that the integral

$$\int_\Gamma g(x, t)\psi(x)\,dx\,dt$$

is well-defined for every $\psi \in H^{1/2}(\partial\Omega)$ and the map $\psi \mapsto \int_{\partial\Omega} g(x, t)\psi(x)\,dx$ belongs to $H^{-1/2}(\partial\Omega)$, the dual of $H^{1/2}(\partial\Omega)$, for each $t \in [0, T]$.

In the above, for a Banach space Y, we used the notation $L^2(0, T; Y)$ for the space of all Y-valued measurable functions φ on $[0, T]$ such that $\int_0^T \|\varphi(t)\|_Y^2\,dt < \infty$. In the due course, we take Y as $L^2(\Omega)$ or the Sobolev spaces $H^1(\Omega)$, $H^{-1/2}(\partial\Omega)$ or $W^{1,\infty}(\Omega)$. For details on Sobolev spaces, one may refer to [1, 6, 9]. Further, if Y is a space of functions (or equivalence class of functions) on Ω and if $\phi\colon \Omega \times [0, T] \to \mathbb{R}$, then by $\phi \in L^2(0, T; Y)$ we mean that the function $t \mapsto \phi(t, \cdot)$ belongs to $\in L^2(0, T; Y)$.

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The forward problem associated with (1.1)–(1.3) is to find a solution u satisfying (1.1)–(1.3) in some sense, for a given suitable data f, g, h and q. Under certain conditions on the data, the existence results are well known (see, e.g., [10] or [3, Theorems 2.4 and 3.3]). For instance, the following theorem is a special case of [3, Theorem 3.3].

Theorem 1.1 ([3]). *Let $h \in H^1(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$ and $g \in L^2(0, T; H^{1/2}(\partial\Omega))$ be such that*

$$g_t \in L^2(0, T; H^{-1/2}(\partial\Omega)).$$

If $q \in L^\infty(\Omega)$ is such that $q \geq c_0$ a.e. on Ω for some constant $c_0 > 0$, then there exists a unique $u \in L^2(0, T; H^2(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$ satisfying

$$\int_Q [u_t v + q(x)\nabla u \cdot \nabla v]\, dx\, dt = \int_Q fv\, dx\, dt + \int_\Gamma gv\, dx\, dt \quad \text{for all } v \in L^2(0, T; H^1(\Omega)) \tag{1.4}$$

and

$$u(\cdot\,, 0) = h \quad \text{in } \Omega. \tag{1.5}$$

We observe that if $u(x, t)$ satisfies (1.1)–(1.3), then it also satisfies (1.4)–(1.5). Following [8], for Robin boundary conditions, we call the formulation (1.4)–(1.5) also as the *weak form* of (1.1)–(1.3).

In this paper, we are interested in the inverse problem of determining the *diffusion coefficient* function q satisfying (1.4)–(1.5) from some knowledge of the *temperature distribution* u. This problem is clearly a nonlinear problem and is known to be ill-posed. For problems related to identification of diffusion coefficients in a parabolic PDE, one may refer to [2, 5]. In [5], the authors have considered the identification of a diffusion coefficient in the parabolic PDE (1.1) with homogeneous Dirichlet boundary condition and have used the theory of regularization for nonlinear operator equations to tackle the problem. In [2], the authors have considered a parameter identification problem in the quasi-linear case, that is, with the diffusion coefficient as a function of the temperature distribution u, and carried out the analysis by converting it into an elliptic PDE.

Motivated by the procedure used in [12] for parameter identification in an elliptic problem, we propose a linear regularization method for obtaining stable approximate solutions for the above mentioned inverse problem, and provide error estimates for same. Also, we consider approximations for the regularized solution in a finite-dimensional setting by considering finite-dimensional subspaces of $L^2(0, T; H^1(\Omega))$. We would like to mention that our formulation of the inverse problem involves perturbations in both the operators and the data, and for obtaining stable approximate solutions, we considered Tikhonov regularization with noise in the operator as well as in the right-hand side of the equation. For theory related to Tikhonov regularization of a perturbed operator, one may refer to [4, 11].

Let us first formulate the inverse problem for which the proposed regularization method is going to be applied.

Problem. *From the knowledge of a function $u \in L^2(0, T; W^{1,\infty}(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$, identify $q \in H^1(\Omega)$ or $L^2(\Omega)$ satisfying* (1.4)–(1.5).

We assume that a solution for the above problem exists. The weak form of the inverse problem, namely (1.4)–(1.5), facilitates to state it as a linear operator equation in appropriate setting.

In (1.4), the integral $\int_\Gamma gv\, dx\, dt$ is understood as follows:

$$\int_\Gamma gv\, dx\, dt = \int_0^T \langle g(t), \gamma v(t)\rangle\, dt,$$

where γ is the trace map and $\langle\cdot\,,\cdot\rangle$ denotes the duality action of $H^{-1/2}(\partial\Omega)$ on $H^{1/2}(\partial\Omega)$ as mentioned earlier.

Remark 1.2. We may observe that, in our formulation of the inverse problem associated with (1.4)–(1.5), we assumed that $u \in L^2(0, T; W^{1,\infty}(\Omega))$, whereas Theorem 1.1 guarantees only that $u \in L^2(0, T; H^2(\Omega))$. At this point, we may recall that if $k > \frac{d}{2}$, then $H^k(\Omega)$ is continuously embedded in $C(\bar{\Omega})$ (see [14, Corollary 7.19]).

Therefore, for $k > \frac{d}{2}$, $L^2(0, T; H^{k+1}(\Omega))$ is continuously embedded in $L^2(0, T; W^{1,\infty}(\Omega))$. In particular, if $d = 1$, then $L^2(0, T; H^2(\Omega))$ is continuously embedded in $L^2(0, T; W^{1,\infty}(\Omega))$.

In the next section (Section 2), we represent the inverse problem as an operator equation and prove some of the properties of the operators involved. We also show that the inverse problem is ill-posed. In Section 3, we consider the Tikhonov-type method as the regularization of the ill-posed operator equation and derive error estimates under noisy measurements of u and u_t. Section 4 is devoted to the finite-dimensional realization of the method and the corresponding error estimates. In Section 5, we do the analysis on choosing the regularization parameter, thereby obtaining the parameter a posteriori that will give the order optimal rate. In Section 6, we give numerical illustrations for the feasibility of our method.

2 Operator theoretic formulation

In the following, we denote $\mathcal{X} = L^2(0, T; H^1(\Omega))$ and $\mathcal{W} = \{u \in L^2(0, T; W^{1,\infty}(\Omega)) : u_t \in L^2(0, T; L^2(\Omega))\}$. Note that here u_t is the Banach space valued distributional derivative of u, that is, there exists ψ in $L^1_{\mathrm{loc}}(0, T; L^2(\Omega))$ such that

$$\int_0^T \varphi'(t)u(t) = \int_0^T \varphi(t)\psi(t) \quad \text{for all } \varphi \in C_c^\infty(0, T).$$

Also, it is known that $\mathcal{X}$ is a Hilbert space with the inner product

$$\langle v, w\rangle_{\mathcal{X}} = \int_0^T \langle v(\cdot\,, t), w(\cdot\,, t)\rangle_{H^1(\Omega)}\, dt \quad \text{for all } v, w \in \mathcal{X}$$

and the associated norm

$$\|v\|_{\mathcal{X}}^2 = \int_0^T \|v(\cdot\,, t)\|_{H^1(\Omega)}^2\, dt \quad \text{for all } v, w \in \mathcal{X}.$$

We observe that if $u \in \mathcal{W}$ is a solution of (1.1)–(1.3), then u is also a weak solution (as termed in [8]) of (1.1)–(1.3), in the sense that it satisfies

$$\int_Q q(x)\nabla u \cdot \nabla v\, dx\, dt = \int_Q fv\, dx\, dt + \int_\Gamma gv\, dx\, dt - \int_Q u_t v\, dx\, dt \quad \text{for all } v \in \mathcal{X}. \tag{2.1}$$

Throughout this paper, we take $\mathcal{H} = L^2(\Omega)$ or $H^1(\Omega)$, and use the notation $\|\cdot\|$ for the norm on $\mathcal{H}$ and also for the operator norms.

For $z \in L^2(0, T; W^{1,\infty}(\Omega))$, $w \in L^2(0, T; L^2(\Omega))$ and $q \in \mathcal{H}$, we define the linear functionals $A_z(q): \mathcal{X} \to \mathbb{R}$ and $\Phi_w: \mathcal{X} \to \mathbb{R}$ by

$$A_z(q)(v) = \int_Q q(x)\nabla z \cdot \nabla v\, dx\, dt \qquad \text{for all } v \in \mathcal{X}, \tag{2.2}$$

$$\Phi_w(v) = \int_Q fv\, dx\, dt + \int_\Gamma gv\, dx\, dt - \int_Q wv\, dx\, dt \quad \text{for all } v \in \mathcal{X}, \tag{2.3}$$

respectively. Then equation (2.1) can be written as $A_u(q)(v) = \Phi_{u_t}(v)$ for all $v \in \mathcal{X}$, or more compactly as an operator equation

$$A_u(q) = \Phi_{u_t}. \tag{2.4}$$

Thus, the inverse problem that we investigate can be stated as follows.

Problem. *Given $u \in \mathcal{W}$, find $q \in \mathcal{H}$ such that* (2.4) *is satisfied.*

In order to obtain stable approximations for q, we make use of some regularization methods. Here, we would like to emphasize that our operator theoretic formulation allows us to use regularization theory for linear operators. For this purpose, we shall use the well-known regularization, the Tikhonov regularization.

We now prove some of the properties of the linear functionals $A_u(q)$ and Φ_{u_t}. We shall denote the space of all bounded linear functionals on normed linear space X by X^*.

Theorem 2.1. *Let $z \in L^2(0,T;W^{1,\infty}(\Omega))$ and A_z be as defined in (2.2). Then, for each $q \in \mathcal{H}$, $A_z(q)$ is a bounded linear functional on $\mathcal{X}$ and*

$$\|A_z(q)\|^2 \le \|q\|^2 \int_0^T \|\nabla z(\cdot,t)\|^2_{L^\infty(\Omega)}\,dt. \tag{2.5}$$

In particular, $A_z : \mathcal{H} \to \mathcal{X}^*$ is a bounded linear operator with

$$\|A_z\|^2 \le \int_0^T \|\nabla z(\cdot,t)\|^2_{L^\infty(\Omega)}\,dt. \tag{2.6}$$

Proof. Let $z \in \mathcal{W}$ and $q \in \mathcal{H}$. From (2.2), for all $v \in \mathcal{X}$, we have

$$\begin{aligned}|A_z(q)(v)| &\le \int_Q |q(x)\nabla z(x,t)\cdot\nabla v(x,t)|\,dx\,dt \le \int_0^T \|q(\cdot)\nabla z(\cdot,t)\|_{L^2(\Omega)}\|\nabla v(\cdot,t)\|_{L^2(\Omega)}\,dt\\ &\le \int_0^T \|q\|_{L^2(\Omega)}\|\nabla z(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla v(\cdot,t)\|_{L^2(\Omega)}\,dt \le \|q\|_{\mathcal{H}}\bigg(\int_0^T \|\nabla z(\cdot,t)\|^2_{L^\infty(\Omega)}\,dt\bigg)^{1/2}\|v\|_{\mathcal{X}}.\end{aligned}$$

This shows that $A_z(q)$ is a bounded linear functional on $\mathcal{X}$ and (2.5) is satisfied. Also, from (2.2), we observe that $A_z : \mathcal{H} \to \mathcal{X}^*$ is a linear operator, and from (2.5), we see that $A_z : \mathcal{H} \to \mathcal{X}^*$ is a bounded linear operator with its norm satisfying (2.6). □

If $\mathcal{H} = H^1(\Omega)$, then we have the following result.

Theorem 2.2. *Let $z \in L^2(0,T;W^{1,\infty}(\Omega))$ and A_z be as in (2.2). Then $A_z : H^1(\Omega) \to \mathcal{X}^*$ is a compact operator.*

Proof. We know that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact [1, 9]. Also, by Theorem 2.1, $A_z : L^2(\Omega) \to \mathcal{X}^*$ is a bounded linear operator. Thus, $A_z : H^1(\Omega) \to \mathcal{X}^*$ is a composition of two linear operators, of which one is continuous and the other is compact. Hence $A_z : H^1(\Omega) \to \mathcal{X}^*$ is a compact operator. □

Using arguments similar to those used in the proof of [12, Theorem 2.4], we establish the following theorem.

Theorem 2.3. *Let $z \in L^2(0,T;W^{1,\infty}(\Omega))$ be such that $|\nabla z(\cdot,t)| > 0$ a.e. on Ω for every $t \in [0,T]$. Let $A_z : \mathcal{H} \to \mathcal{X}^*$ be as defined in (2.2). Then A_z is of infinite rank.*

Proof. For each $n \in \mathbb{N}$, let B_n be an open ball in Ω such that $B_n \cap B_m = \phi$ for $m \ne n$. Also, for each $n \in \mathbb{N}$, let B_n' and B_n'' be open balls in Ω such that $B_n'' \subset B_n' \subset B_n$, where the inclusion is strict. For $n \in \mathbb{N}$, let $q_n \in C_c^\infty(\Omega)$ be such that $\operatorname{supp} q_n \subset B_n'$, $q_n = 1$ on B_n'' and $0 \le q_n \le 1$. We now show that the set $\{A_z(q_n) : n \in \mathbb{N}\}$ is linearly independent, which would show that A_z is of infinite rank. For each $n \in \mathbb{N}$, let $v_n \in C_c^\infty(\Omega)$ be such that $\operatorname{supp} v_n \subset B_n$ and $v_n = 1$ on B_n'. Then $w_n = v_n z \in \mathcal{X}$ and $\nabla w_n = \nabla z$ on B_n'. Therefore,

$$A_z(q_n)(w_n) = \int_0^T \int_{B_n'} q_n|\nabla z|^2\,dx\,dt \quad \text{for all } n \in \mathbb{N}, \tag{2.7}$$

$$A_z(q_n)(w_m) = 0 \qquad \text{for all } m,n \in \mathbb{N} \text{ with } m \ne n. \tag{2.8}$$

For $n \in \mathbb{N}$, let c_i, $i = 1,\dots,n$, be scalars such that $\sum_{i=1}^n c_i A_z(q_i) = 0$. Then, using (2.7) and (2.8), we obtain, for $j = 1,\dots,n$, $c_j A_z(q_j)(w_j) = 0$, i.e.,

$$c_j \int_0^T \int_{B_j'} q_j|\nabla z|^2\,dx\,dt = 0. \tag{2.9}$$

Now,

$$\int_0^T \int_{B_j'} q_j |\nabla z|^2 \, dx \, dt \geq \int_0^T \int_{B_j''} q_j |\nabla z|^2 \, dx \, dt = \int_0^T \int_{B_j''} |\nabla z|^2 \, dx \, dt > 0.$$

Thus, from (2.9), we get $c_j = 0$. This shows that $\{A_z(q_n) : n \in \mathbb{N}\}$ is an infinite linearly independent set. Hence A_z is of infinite rank. □

Next, we show that Φ_{u_t}, as defined in (2.3), is a bounded linear functional on $\mathcal{X}$, which in turn will ensure that the operator equation $A_u(q) = \Phi_{u_t}$ is well-defined.

Theorem 2.4. *Let $w \in L^2(0, T; L^2(\Omega))$ and $\Phi_w : \mathcal{X} \to \mathbb{R}$ be as defined in* (2.3)*. Then Φ_w is a bounded linear functional and*

$$\|\Phi_w\| \leq \bigg(\int_0^T \|f(\cdot\,, t)\|^2_{L^2(\Omega)} \, dt\bigg)^{1/2} + C\bigg(\int_0^T \|g(\cdot\,, t)\|^2_{H^{-1/2}(\partial\Omega)} \, dt\bigg)^{1/2} + \bigg(\int_0^T \|w(\cdot\,, t)\|^2_{L^2(\Omega)} \, dt\bigg)^{1/2}$$

for some constant $C > 0$.

Proof. The linearity of Φ_w is obvious; we prove only the continuity. Let $v \in \mathcal{X}$. Then, from (2.3), we have

$$|\Phi_w(v)| \leq \int_Q |f(x, t)v(x, t)| \, dx \, dt + \int_\Gamma |g(x, t)v(x, t)| \, dx \, dt + \int_Q |w(x, t)v(x, t)| \, dx \, dt.$$

Using the Schwarz inequality, we have

$$\int_Q |f(x, t)v(x, t)| \, dx \, dt \leq \bigg(\int_0^T \|f(\cdot\,, t)\|^2_{L^2(\Omega)} \, dt\bigg)^{1/2} \|v\|_{\mathcal{X}}.$$

Similarly, we also have

$$\int_Q |w(x, t)v(x, t)| \, dx \, dt \leq \bigg(\int_0^T \|w(\cdot\,, t)\|^2_{L^2(\Omega)}\bigg)^{1/2} \|v\|_{\mathcal{X}}.$$

Since, for $t \in [0, T]$, $\gamma v(\cdot\,, t)$, i.e., the trace of $v(\cdot\,, t)$ belongs to $H^{1/2}(\partial\Omega)$, using the property of g and the Schwarz inequality, we have

$$\int_\Gamma |g(x, t)v(x, t)| \, dx \, dt \leq \bigg(\int_0^T \|g(\cdot\,, t)\|^2_{H^{-1/2}(\partial\Omega)} \, dt\bigg)^{1/2} \bigg(\int_0^T \|\gamma v(\cdot\,, t)\|^2_{H^{1/2}(\partial\Omega)} \, dt\bigg)^{1/2}.$$

Now, using the continuity of the trace map, there exists a constant $C > 0$ such that

$$\bigg(\int_0^T \|\gamma v(\cdot\,, t)\|^2_{H^{1/2}(\partial\Omega)} \, dt\bigg)^{1/2} \leq C\|v\|_{\mathcal{X}},$$

and hence

$$\int_\Gamma |g(x, t)v(x, t)| \, dx \, dt \leq C\bigg(\int_0^T \|g(\cdot\,, t)\|^2_{H^{-1/2}(\partial\Omega)} \, dt\bigg)^{1/2} \|v\|_{\mathcal{X}}.$$

This completes the proof. □

Remark 2.5. By Theorem 2.2 and Theorem 2.3, we know that A_z is a compact operator of infinite rank for each $z \in \mathcal{W}$. Therefore, if $\mathcal{H} = H^1(\Omega)$, then the ill-posedness of the operator equation (2.4) also follows from the fact that A_z does not have continuous inverse.

In the next section, we consider the regularization of the problem and carry out the error analysis under noisy data.

3 Regularization and error analysis

Let $u \in \mathcal{W}$ be such that $q \in \mathcal{H}$ is the unique solution of (2.4). Suppose that we have an approximate knowledge of u and u_t, i.e., $z \in L^2(0,T;W^{1,\infty}(\Omega))$ and $w \in L^2(0,T;L^2(\Omega))$ such that

$$\int_0^T \|\nabla u(\cdot,t) - \nabla z(\cdot,t)\|^2_{L^\infty(\Omega)}\,dt \le \varepsilon^2, \tag{3.1}$$

$$\int_0^T \|u_t(\cdot,t) - w(\cdot,t)\|^2_{L^2(\Omega)}\,dt \le \delta^2, \tag{3.2}$$

respectively, for some $\varepsilon > 0$ and $\delta > 0$. Note that here we are having perturbations of both A_u and Φ_{u_t}, namely, A_z and Φ_w, respectively, that is, we are dealing with noise in both the operator A_u and the data Φ_{u_t}. So we need to find stable approximation to q from the knowledge of these known data z and w satisfying the above mentioned noise level. So we make use of some regularization method for obtaining stable approximations to q. We use Tikhonov regularization to carry out our error analysis.

For $\alpha > 0$, let $q_{\alpha,u} \in \mathcal{H}$ be the Tikhonov regularized solution corresponding to the operator equation (2.4), that is, $q_{\alpha,u}$ satisfies the equation

$$(A_u^* A_u + \alpha I) q_{\alpha,u} = A_u^* \Phi_{u_t}, \tag{3.3}$$

and let $q_{\alpha,z,w} \in \mathcal{H}$ be the Tikhonov regularized solution when both the operator A_u and the functional Φ_{u_t} are replaced by the noisy operator A_z and noisy functional Φ_w, respectively, that is, $q_{\alpha,z,w}$ satisfies the equation

$$(A_z^* A_z + \alpha I) q_{\alpha,z,w} = A_z^* \Phi_w. \tag{3.4}$$

Note that the linear operators A_u and A_z are from $\mathcal{H}$ into $\mathcal{X}^*$. So $A_u^*A_u$ and $A_z^*A_z$ are positive and self-adjoint linear operators on $\mathcal{H}$. Hence (3.3) and (3.4) are well-posed equations.

Before proceeding further, we make the following observations, which will help us to carry out the error analysis. Let u_1 and u_2 be in $L^2(0,T;W^{1,\infty}(\Omega))$. Then, by Theorem 2.1, we have

$$\|A_{u_1} - A_{u_2}\| \le \left(\int_0^T \|\nabla u_1(\cdot,t) - \nabla u_2(\cdot,t)\|^2_{L^\infty(\Omega)}\,dt \right)^{1/2}. \tag{3.5}$$

Also, let w_1, w_2 be in $L^2(0,T;L^2(\Omega))$. Then, from (2.3),

$$(\Phi_{w_1} - \Phi_{w_2})(v) = \int_Q (w_1 - w_2) v \,dx\,dt \quad \text{for all } v \in \mathcal{X}$$

so that

$$\|\Phi_{w_1} - \Phi_{w_2}\| \le \left(\int_0^T \|w_1(\cdot,t) - w_2(\cdot,t)\|^2_{L^2(\Omega)}\,dt \right)^{1/2}. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\|A_u - A_z\| \le \varepsilon, \tag{3.7}$$

$$\|\Phi_{u_t} - \Phi_w\| \le \delta. \tag{3.8}$$

The following theorem, proved using standard techniques in regularization theory (see, e.g., Nair [11]), shows that the solution of (3.3) is indeed stable under perturbations in u and u_t.

Theorem 3.1. *For $\alpha > 0$, let $q_{\alpha,u}$ and $q_{\alpha,z,w}$ be as in* (3.3) *and* (3.4)*, respectively. Let $\varepsilon, \delta > 0$ be as in* (3.1) *and* (3.2)*, respectively. Then*

$$\|q_{\alpha,u} - q_{\alpha,z,w}\| \le \frac{\varepsilon}{\sqrt{\alpha}}\|q\| + \frac{\delta}{2\sqrt{\alpha}}.$$

In particular,

$$\|q_{\alpha,u} - q_{\alpha,z,w}\| \to 0 \quad \text{as } \varepsilon \to 0 \text{ and } \delta \to 0.$$

Proof. Let $q_{\alpha,z}$ be the unique element in $\mathcal{H}$ such that

$$(A_z^* A_z + \alpha I) q_{\alpha,z} = A_z^* \Phi_{u_t}. \tag{3.9}$$

We have

$$q_{\alpha,u} - q_{\alpha,z,w} = (q_{\alpha,u} - q_{\alpha,z}) + (q_{\alpha,z} - q_{\alpha,z,w}).$$

Now, from (3.3), (3.9) and using the fact that $\Phi_{u_t} = A_u q$, we have

$$q_{\alpha,z} - q_{\alpha,u} = (A_z^* A_z + \alpha I)^{-1} A_z^* A_u q - (A_u^* A_u + \alpha I)^{-1} A_u^* A_u q.$$

By appropriate modification, the above equation can be written as

$$\begin{aligned} q_{\alpha,z} - q_{\alpha,u} &= (A_z^* A_z + \alpha I)^{-1} A_z^* (A_u - A_z) A_u^* A_u (A_u^* A_u + \alpha I)^{-1} q \\ &\quad + \alpha (A_z^* A_z + \alpha I)^{-1} (A_z^* - A_u^*)(A_u A_u^* + \alpha I)^{-1} A_u q. \end{aligned}$$

We now make use of the following estimates (for more details, see [11]):

$$\|(A_u^* A_u + \alpha I)^{-1}\| \le \frac{1}{\alpha}, \qquad \|A_u^* A_u (A_u^* A_u + \alpha I)^{-1}\| \le 1,$$
$$\|(A_u^* A_u + \alpha I)^{-1} A_u^*\| \le \frac{1}{2\sqrt{\alpha}}, \qquad \|(A_u A_u^* + \alpha I)^{-1} A_u\| \le \frac{1}{2\sqrt{\alpha}}.$$

Thus,

$$\|q_{\alpha,u} - q_{\alpha,z}\| \le \frac{\|A_u - A_z\|}{\sqrt{\alpha}} \|q\|.$$

Also, from (3.4) and (3.9), we have

$$q_{\alpha,z} - q_{\alpha,z,w} = (A_z^* A_z + \alpha I)^{-1} A_z^* (\Phi_{u_t} - \Phi_w).$$

Thus,

$$\|q_{\alpha,z} - q_{\alpha,z,w}\| \le \frac{\|\Phi_w - \Phi_{u_t}\|}{2\sqrt{\alpha}}.$$

Therefore, using the inequalities in (3.7) and (3.8), we have

$$\begin{aligned} \|q_{\alpha,u} - q_{\alpha,z,w}\| &\le \|q_{\alpha,u} - q_{\alpha,z}\| + \|q_{\alpha,z} - q_{\alpha,z,w}\| \\ &\le \frac{\|A_u - A_z\|}{\sqrt{\alpha}} \|q\| + \frac{\|\Phi_{u_t} - \Phi_w\|}{2\sqrt{\alpha}} \\ &\le \frac{\varepsilon}{\sqrt{\alpha}} \|q\| + \frac{\delta}{2\sqrt{\alpha}}. \end{aligned}$$

The particular case follows immediately. □

From the above theorem, we obtain the following.

Theorem 3.2. *Let $q_{\alpha,u}$ and $q_{\alpha,z,w}$ be as in* (3.3) *and* (3.4)*, respectively. Let $\varepsilon, \delta > 0$ be as in* (3.1) *and* (3.2)*, respectively. Then*

$$\|q - q_{\alpha,z,w}\| \le \|q - q_{\alpha,u}\| + c_q \frac{\varepsilon + \delta}{\sqrt{\alpha}},$$

where $c_q := \max\{\frac{1}{2}, \|q\|\}$.

Remark 3.3. Since $q \in \mathcal{H}$ is the unique solution of (2.4), from the theory of Tikhonov regularization, we know that $\|q - q_{\alpha,u}\| \to 0$ as $\alpha \to 0$ (cf. [4, 11]). Thus, choosing α depending on ε, δ such that $\alpha \to 0$ and $\frac{\varepsilon+\delta}{\sqrt{\alpha}} \to 0$ as $\varepsilon, \delta \to 0$, we have the convergence $\|q - q_{\alpha,z,w}\| \to 0$ as $\varepsilon, \delta \to 0$. For instance, if $\alpha := \varepsilon + \delta$, then $\|q - q_{\varepsilon+\delta,z,w}\| \to 0$ as $\varepsilon, \delta \to 0$.

It is known that the best rate possible for the quantity $\|q - q_{\alpha,u}\|$ is $O(\alpha)$, and that is achieved if the function q belongs to the range of the operator $A_u^* A_u$ (see [4, 11]).

4 Finite-dimensional realization

In practical applications, one would like to obtain approximate solutions using numerical methods. For this purpose, we look for a Galerkin approximation for the solutions of (3.4) by using finite-dimensional spaces.

Let $\langle \cdot , \cdot \rangle$ denote the inner product on $\mathcal{H}$. Let $\alpha > 0$, $z \in L^2(0, T; W^{1,\infty}(\Omega))$ and $w \in L^2(0, T; L^2(\Omega))$. Then (3.4) holds if and only if

$$\langle (A_z^* A_z + \alpha I) q_{\alpha,z,w}, \varphi \rangle = \langle A_z^* \Phi_w, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{H}. \tag{4.1}$$

For obtaining a finite-dimensional approximation of $q_{\alpha,z,w}$, we consider equation (4.1) by varying φ in a subspace of $\mathcal{H}$. For this purpose, we consider a sequence (X_n) of finite-dimensional subspaces of $\mathcal{H}$. Let $P_n \colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto X_n. In applications, one may have pointwise convergence of (P_n) to I, the identity operator. This property of (P_n) is satisfied if, for example, X_n, $n \in \mathbb{N}$, are such that
(a) $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$,
(b) $\bigcup_{n=1}^{\infty} X_n$ is dense in $\mathcal{H}$.
Thus, we look for $q_{\alpha,z,w}^{(n)} \in X_n$ such that

$$\langle (A_z^* A_z + \alpha I) q_{\alpha,z,w}^{(n)}, \varphi \rangle = \langle A_z^* \Phi_w, \varphi \rangle \quad \text{for all } \varphi \in X_n. \tag{4.2}$$

Also, (4.2) can be written as

$$\langle (P_n A_z^* A_z P_n + \alpha I) q_{\alpha,z,w}^{(n)}, \varphi \rangle = \langle P_n A_z^* \Phi_w, \varphi \rangle \quad \text{for all } \varphi \in X_n. \tag{4.3}$$

We observe that (4.3) can be represented as the operator equation

$$(P_n A_z^* A_z P_n + \alpha I) q_{\alpha,z,w}^{(n)} = P_n A_z^* \Phi_w. \tag{4.4}$$

Note that the operator $P_n A_z^* A_z P_n$ is self-adjoint and positive definite. Therefore, equation (4.4) has a unique solution $q_{\alpha,z,w}^{(n)}$ for each $\alpha > 0$, $n \in \mathbb{N}$ and $z \in L^2(0, T; W^{1,\infty}(\Omega))$, $w \in L^2(0, T; L^2(\Omega))$. Let $\dim(X_n) = n$ and $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ be a basis of X_n. Then the solution $q_{\alpha,z,w}^{(n)}$ of (4.2), equivalently of (4.4), can be written as

$$q_{\alpha,z,w}^{(n)} = \sum_{i=1}^{n} q_i \varphi_i \tag{4.5}$$

for some scalars q_i, $i = 1, 2, \dots, n$. Then, from (4.2), we have

$$\sum_{j=i}^{n} q_j \langle A_z^* A_z \varphi_j, \varphi_i \rangle + \alpha \sum_{j=1}^{n} q_j \langle \varphi_j, \varphi_i \rangle = \langle A_z^* \Phi_w, \varphi_i \rangle, \quad i = 1, 2, \dots, n,$$

equivalently,

$$\sum_{j=i}^{n} q_j \langle A_z \varphi_j, A_z \varphi_i \rangle_{\mathcal{X}^*} + \alpha \sum_{j=1}^{n} q_j \langle \varphi_j, \varphi_i \rangle = \langle \Phi_w, A_z \varphi_i \rangle_{\mathcal{X}^*}, \quad i = 1, 2, \dots, n. \tag{4.6}$$

Let

$$a_{ij} = \langle A_z \varphi_j, A_z \varphi_i \rangle_{\mathcal{X}^*}, \quad d_{ij} = \langle \varphi_j, \varphi_i \rangle, \quad b_i = \langle \Phi_w, A_z \varphi_i \rangle_{\mathcal{X}^*} \quad \text{for } i, j = 1, 2, \dots, n.$$

Then (4.6) is the same as the matrix equation

$$A\mathbf{q} + \alpha D\mathbf{q} = \mathbf{b} \tag{4.7}$$

where

$$A = [a_{ij}]_{n\times n}, \quad D = [d_{ij}]_{n\times n}, \quad \mathbf{b} = [b_1, b_2, \dots, b_n]^T, \quad \mathbf{q} = [q_1, q_2, \dots, q_n]^T.$$

The above discussion also shows that (4.7) has a unique solution $\mathbf{q} = [q_1, q_2, \dots, q_n]^T$, and in that case, $q_{\alpha,z,w}^{(n)}$ as given in (4.5) is the solution of (4.2). Thus, for obtaining the solution $q_{\alpha,z,w}^{(n)}$ of (4.2), we first solve the matrix equation (4.7), and then obtain $q_{\alpha,z,w}^{(n)}$ as given in (4.5).

We now write these a_{ij} and b_i explicitly. For doing that, we make use of the following well-known result from functional analysis.

Lemma 4.1. *Let H be a real Hilbert space, and let $\mathfrak{R}\colon H^* \to H$ be the Riesz representation map, that is, for $\xi \in H^*$, $\xi(x) = \langle x, \mathfrak{R}\xi\rangle_H$ for all $x \in H$. Then $\langle\cdot,\cdot\rangle_{H^*}$ defined by $\langle \xi, \eta\rangle_{H^*} = \langle \mathfrak{R}\eta, \mathfrak{R}\xi\rangle_H$, $\xi, \eta \in H^*$, is an inner product on H^* inducing the norm on H^*, and the space H^* with this inner product is a Hilbert space.*

Let $\mathfrak{R}\colon \mathcal{X}^* \to \mathcal{X}$ be the Riesz representation map. Then, for $\xi \in \mathcal{X}^*$, $\mathfrak{R}\xi$ is the unique solution of the equation $\langle \Phi, \mathfrak{R}\xi\rangle_{\mathcal{X}} = \xi(\Phi)$ for all $\Phi \in \mathcal{X}$, that is,

$$\int_0^T \langle \Phi(\cdot,t), (\mathfrak{R}\xi)(\cdot,t)\rangle_{H^1(\Omega)}\,dt = \xi(\Phi).$$

Thus,

$$a_{ij} = \langle \mathfrak{R}A_z\varphi_j, \mathfrak{R}A_z\varphi_i\rangle_{\mathcal{X}} := \int_0^T \langle (\mathfrak{R}A_z\varphi_j)(\cdot,t), (\mathfrak{R}A_z\varphi_i)(\cdot,t)\rangle_{H^1(\Omega)}\,dt,$$

$$b_i = \langle \mathfrak{R}\Phi_w, \mathfrak{R}A_z\varphi_i\rangle_{\mathcal{X}} := \int_0^T \langle (\mathfrak{R}\Phi_w)(\cdot,t), (\mathfrak{R}A_z\varphi_i)(\cdot,t)\rangle_{H^1(\Omega)}\,dt,$$

where $\mathfrak{R}A_z\varphi_k \in \mathcal{X}$ is the unique solution of

$$\int_0^T \langle \Phi(\cdot,t), (\mathfrak{R}A_z\varphi_k)(\cdot,t)\rangle_{H^1(\Omega)}\,dt = (A_z\varphi_k)(\Phi) \tag{4.8}$$

and $\mathfrak{R}\Phi_w \in \mathcal{X}$ is the unique solution of

$$\int_0^T \langle \Phi(\cdot,t), (\mathfrak{R}\Phi_w)(\cdot,t)\rangle_{H^1(\Omega)}\,dt = \Phi_w(\Phi) \tag{4.9}$$

for all $\Phi \in \mathcal{X}$ and A_z, Φ_w are as defined in (2.2) and (2.3), respectively.

We now have the following result about the estimate for the error $\|q - q^{(n)}_{\alpha,z,w}\|$.

Theorem 4.2. *Let $q_{\alpha,u}$ and $q^{(n)}_{\alpha,z,w}$ be as in (3.3) and (4.2), respectively. If $\eta_n > 0$ is such that $\|A_z(I - P_n)\| \le \eta_n$, then*

$$\|q - q^{(n)}_{\alpha,z,w}\| \le \|q - q_{\alpha,u}\| + \frac{\delta}{2\sqrt{\alpha}} + \frac{(\varepsilon + \eta_n)}{\sqrt{\alpha}}\|q\|.$$

Proof. Following the similar calculations, as done in the proof of Theorem 3.1, with A_z being replaced by A_zP_n, we have

$$\|q_{\alpha,u} - q^{(n)}_{\alpha,z,w}\| \le \frac{\delta}{2\sqrt{\alpha}} + \frac{\|A_u - A_zP_n\|}{\sqrt{\alpha}}\|q\|.$$

Since $\|A_u - A_zP_n\| \le \|A_u - A_z\| + \|A_z - A_zP_n\| \le \varepsilon + \eta_n$, we obtain

$$\|q - q^{(n)}_{\alpha,z,w}\| \le \frac{\delta}{2\sqrt{\alpha}} + \frac{(\varepsilon + \eta_n)}{\sqrt{\alpha}}\|q\|. \qquad \square$$

Remark 4.3. By Theorem 2.2, $A_z\colon H^1(\Omega) \to \mathcal{X}^*$ is a compact operator, and hence its adjoint $A_z^*\colon \mathcal{X}^* \to H^1(\Omega)$ is also a compact operator. Then, using the pointwise convergence of P_n, we have (see [11, Theorem 2.13]) $\|A_z - A_zP_n\| = \|(I - P_n)A_z^*\| \to 0$ as $n \to \infty$. Hence if we take $\mathcal{H} = H^1(\Omega)$, the estimate η_n of $\|A_z - A_zP_n\|$ can be such that $\eta_n \to 0$ as $n \to \infty$.

Remark 4.4. From Theorem 4.2, we have

$$\|q - q^{(n)}_{\alpha,z,w}\| \le \|q - q_{\alpha,u}\| + c_q\frac{(\delta + \varepsilon + \eta_n)}{\sqrt{\alpha}},$$

where $c_q := \max\{\frac{1}{2}, \|q\|\}$. From the theory of Tikhonov regularization, we know that $\|q - q_{\alpha,u}\| \to 0$ as $\alpha \to 0$. Thus, if we choose the parameter α, depending on δ, ε, n, such that $\alpha \to 0$ and $\frac{\delta+\varepsilon+\eta_n}{\sqrt{\alpha}} \to 0$ as $\delta \to 0$, $\varepsilon \to 0$ and $n \to \infty$, we have the convergence $\|q - q^{(n)}_{\alpha,z,w}\| \to 0$ as $\delta \to 0$, $\varepsilon \to 0$ and $n \to \infty$.

We have already remarked that the best rate possible for the quantity $\|q - q_{\alpha,u}\|$ is $O(\alpha)$, and that is achieved if q belongs to the range of the operator $A_u^*A_u$ (see Remark 3.3). Thus, choosing $\alpha = (\delta + \varepsilon)^{2/3}$ and taking n sufficiently large such that $\eta_n \le \delta + \varepsilon$, we obtain the rate $\|q - q_{\alpha,z,w}^{(n)}\| \le O((\delta + \varepsilon)^{2/3})$ whenever q is *smooth enough* such that it belongs to the range of the operator $A_u^*A_u$. We have considered the issue of parameter choice strategies more elaborately in Section 5.

4.1 Further discretization

We have seen that, in order to obtain $q_{\alpha,z,w}^{(n)}$, as given in (4.5), one has to solve the matrix equation (4.7) and find the solution $\mathbf{q}$. Equation (4.7) involve

$$a_{ij} = \langle A_z\varphi_j, A_z\varphi_i\rangle_{\mathcal{X}^*} = \langle \Re A_z\varphi_j, \Re A_z\varphi_i\rangle_{\mathcal{X}},$$
$$b_i = \langle \Phi_w, A_z\varphi_i\rangle_{\mathcal{X}^*} = \langle \Re\Phi_w, \Re A_z\varphi_i\rangle_{\mathcal{X}},$$

where $\Re A_z\varphi_i$ and $\Re\Phi_w$ are the unique solutions of (4.8) and (4.9), respectively. We note that, in these equations, Φ varies over the infinite-dimensional Hilbert space $\mathcal{X}$. Thus, in order to obtain a numerical approximation of $\Re A_z\varphi_i$ and $\Re\Phi_w$, we may vary Φ over a finite-dimensional subspace $\mathcal{X}_N$ of $\mathcal{X}$ for some $N \in \mathbb{N}$. In other words, we look for $(\Re A_z\varphi_k)^{(N)}$ and $(\Re\Phi_w)^{(N)}$ in $\mathcal{X}_N$ such that

$$\int_0^T \langle\tilde{\Phi}(\cdot\,,t), (\Re A_z\varphi_k)^{(N)}(\cdot\,,t)\rangle_{H^1(\Omega)}\,dt = (A_z\varphi_k)(\tilde{\Phi}) \quad \text{for } k = 1,2,\dots,n, \tag{4.10}$$

$$\int_0^T \langle\tilde{\Phi}(\cdot\,,t), (\Re\Phi_w)^{(N)}(\cdot\,,t)\rangle_{H^1(\Omega)}\,dt = \Phi_w(\tilde{\Phi}) \tag{4.11}$$

for all $\tilde{\Phi} \in \mathcal{X}_N$, where $A_z\varphi_k$ and Φ_w are as defined in (2.2) and (2.3), respectively. Let $\dim(\mathcal{X}_N) = N$ and $\{\Psi_1, \Psi_2, \dots, \Psi_N\}$ be a basis of $\mathcal{X}_N$. Then $(\Re A_z\varphi_k)^{(N)} = \sum_{i=1}^N c_i^k\Psi_i$ and $(\Re\Phi_w)^{(N)} = \sum_{i=1}^N d_i\Psi_i$, where the scalars c_i^k and d_i are obtained by solving the equations

$$\sum_{j=1}^N c_j^k \int_0^T \langle\Psi_i(\cdot\,,t), \Psi_j(\cdot\,,t)\rangle_{H^1(\Omega)}\,dt = (A_z\varphi_k)(\Psi_i),$$
$$\sum_{j=1}^N d_j \int_0^T \langle\Psi_i(\cdot\,,t), \Psi_j(\cdot\,,t)\rangle_{H^1(\Omega)}\,dt = \Phi_w(\Psi_i),$$

respectively, for $i = 1, \dots, N$. Let

$$\tilde{a}_{ij} = \langle(\Re A_z\varphi_j)^{(N)}, (\Re A_z\varphi_i)^{(N)}\rangle_{\mathcal{X}},$$
$$\tilde{b}_i = \langle(\Re\Phi_w)^{(N)}, (\Re A_z\varphi_i)^{(N)}\rangle_{\mathcal{X}},$$
$$d_{ij} = \langle\varphi_j, \varphi_i\rangle$$

for $i, j = 1, \dots, n$. Thus, an approximation to $q_{\alpha,z,w}^{(n)}$ is obtained as

$$\tilde{q}_{\alpha,z,w}^{(n)} := \sum_{i=1}^n \tilde{q}_i\varphi_i,$$

where $\tilde{\mathbf{q}} := [\tilde{q}_1, \dots, \tilde{q}_n]$ is a solution of the matrix equation

$$\tilde{A}\tilde{\mathbf{q}} + \alpha D\tilde{\mathbf{q}} = \tilde{\mathbf{b}}$$

with $\tilde{A} = [\tilde{a}_{ij}]$, $D = [d_{ij}]$, $\tilde{\mathbf{b}} = [\tilde{b}_i]$.

We now propose a procedure for obtaining finite-dimensional subspaces of the space $\mathcal{X}$. For this purpose, let $(\tilde{P}_n)$ be a sequence of finite rank orthogonal projections on $H^1(\Omega)$ such that it converges pointwise to the identity operator on $H^1(\Omega)$. We assume that rank $\tilde{P}_n = n$ for each $n \in \mathbb{N}$. Let $\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_n\}$ be an orthonormal basis of $\tilde{X}_n = R(\tilde{P}_n)$, the range of $\tilde{P}_n$. Further, let $I = [0, T]$, and let (Π_n) be a sequence of finite rank orthogonal projections on $L^2(I)$ with $\operatorname{rank}(\Pi_n) = n$ for each $n \in \mathbb{N}$ such that it converges pointwise to the identity operator on $L^2(I)$. For $k, m \in \mathbb{N}$, we define $Q_k^m : \mathcal{X} \to \mathcal{X}$ as follows:

$$(Q_k^m \Psi)(t) = \sum_{i=1}^{k} (\Pi_m \psi_i)(t) \tilde{\varphi}_i, \quad \Psi \in \mathcal{X},\ t \in I, \tag{4.12}$$

where $\psi_i(t) := \langle \Psi(t), \tilde{\varphi}_i \rangle_{H^1}$ for $\Psi \in \mathcal{X}$, $t \in I$, $i = 1, \dots, k$. It can be easily seen that $\psi_i \in L^2(I)$ for all $\Psi \in \mathcal{X}$ and $i = 1, \dots, k$ so that the operator Q_k^m is indeed well-defined. Also, Q_k^m is a linear operator. We now show that Q_k^m is, in fact, a finite rank orthogonal projection.

Theorem 4.5. *For $k, m \in \mathbb{N}$, the linear operator $Q_k^m : \mathcal{X} \to \mathcal{X}$ defined as in (4.12) is an orthogonal projection of rank mk. In fact, if $\{g_1, \dots, g_m\} \subseteq L^2(I)$ is any orthonormal basis of $R(\Pi_m)$ and $\Psi^{ij}(t) = g_j(t)\tilde{\varphi}_i$, $t \in I$, for $i = 1, \dots, k$, $j = 1, \dots, m$, then $\{\Psi^{ij} : i = 1, \dots, k,\ j = 1, \dots, m\}$ is an orthonormal set in $\mathcal{X}$ and*

$$Q_k^m \Psi = \sum_{i=1}^{k} \sum_{j=1}^{m} \langle \Psi, \Psi^{ij} \rangle_{\mathcal{X}} \Psi^{ij}, \quad \Psi \in \mathcal{X}.$$

Proof. Let $\{g_1, \dots, g_m\}$ in $L^2(I)$ be an orthonormal basis of $R(\Pi_m)$. Then Π_m is given by

$$\Pi_m h = \sum_{j=1}^{m} \langle h, g_j \rangle_{L^2(I)}\, g_j, \quad h \in L^2(I).$$

Let $\Psi^{ij}(t) = g_j(t)\tilde{\varphi}_i$ for $i = 1, \dots, k$, $j = 1, \dots, m$. Then it can be easily seen that $\Psi^{ij} \in \mathcal{X}$ for all $i = 1, \dots, k$, $j = 1, \dots, m$, and

$$\langle \Psi^{ij}, \Psi^{rs} \rangle_{\mathcal{X}} = \langle \tilde{\varphi}_i, \tilde{\varphi}_r \rangle_{H^1(\Omega)} \langle g_j, g_s \rangle_{L^2(I)} = \delta_{ir}\delta_{js},$$

where $\delta_{pq} = 1$ when $p = q$ and $\delta_{pq} = 0$ for $p \neq q$. Thus, $\{\Psi^{ij} : i = 1, \dots, k,\ j = 1, \dots, m\}$ is an orthonormal set in $\mathcal{X}$. Next, we observe that, for $\Psi \in \mathcal{X}$ and $t \in I$,

$$(Q_k^m \Psi)(t) = \sum_{i=1}^{k} (\Pi_m \psi_i)(t) \tilde{\varphi}_i = \sum_{i=1}^{k} \left(\sum_{j=1}^{m} \langle \psi_i, g_j \rangle_{L^2(I)} g_j(t) \right) \tilde{\varphi}_i = \sum_{i=1}^{k} \sum_{j=1}^{m} a_{ij} \Psi^{ij}(t),$$

where $a_{ij} = \langle \psi_i, g_j \rangle_{L^2(I)}$. By the definition of ψ_i, we have

$$\begin{aligned} a_{ij} &= \int_I \psi_i(t) g_j(t)\, dt = \int_I \langle \Psi(t), \tilde{\varphi}_i \rangle_{H^1} g_j(t)\, dt \\ &= \int_I \langle \Psi(t), g_j(t)\tilde{\varphi}_i \rangle_{H^1}\, dt = \int_I \langle \Psi(t), \Psi^{ij}(t) \rangle_{H^1}\, dt = \langle \Psi, \Psi^{ij} \rangle_{\mathcal{X}}. \end{aligned}$$

Thus,

$$Q_k^m \Psi = \sum_{i=1}^{k} \sum_{j=1}^{m} \langle \Psi, \Psi^{ij} \rangle_{\mathcal{X}} \Psi^{ij}, \quad \Psi \in \mathcal{X}.$$

In particular, Q_k^m is an orthogonal projection of rank mk. □

Lemma 4.6. *Let (Q_n) be a uniformly bounded sequence of projection operators on $H^1(\Omega)$ such that $Q_n \to I$ pointwise as $n \to \infty$. For $\Psi \in \mathcal{X}$, let $\Psi_n(t) = Q_n \Psi(t)$, $t \in [0, T]$. Then $\Psi_n \in \mathcal{X}$ and $\|\Psi_n - \Psi\|_{\mathcal{X}} \to 0$ as $n \to \infty$ for all $\Psi \in \mathcal{X}$.*

Proof. Let $M > 0$ be such that $\|Q_n\| \leq M$ for all $n \in \mathbb{N}$. Since $\Psi \in \mathcal{X}$, we obtain that $\Psi_n \in \mathcal{X}$. Now,

$$\|\Psi_n - \Psi\|_{\mathcal{X}}^2 = \int_0^T \|\Psi_n(t) - \Psi(t)\|_{H^1(\Omega)}^2\, dt = \int_0^T \|Q_n \Psi(t) - \Psi(t)\|_{H^1(\Omega)}^2\, dt.$$

Since $\|Q_n \Psi(t) - \Psi(t)\|_{H^1(\Omega)}^2 \to 0$ as $n \to \infty$ for all $t \in I$ and

$$\|Q_n \Psi(t) - \Psi(t)\|_{H^1(\Omega)} \leq (M + 1)\|\Psi(t)\|_{H^1(\Omega)}, \quad \text{where} \quad \int_0^T \|\Psi(t)\|_{H^1(\Omega)}^2\, dt < \infty,$$

by the dominated convergence theorem, we have $\lim_{n\to\infty} \|\Psi_n - \Psi\|_{\mathcal{X}} = 0$. □

Theorem 4.7. *Let Q_k^m be as defined in* (4.12). *Then, for each* $\Psi \in \mathcal{X}$,

$$\lim_{k\to\infty}\lim_{m\to\infty}\|Q_k^m\Psi - \Psi\|_{\mathcal{X}} = 0$$

Proof. Let $\Psi \in \mathcal{X}$. Note that, for $t \in I$, $\psi_i(t)\tilde{\varphi}_i \in H^1(\Omega)$ for all $i = 1, 2, \dots, k$. Equivalently, the map $t \mapsto \psi_i(t)\tilde{\varphi}_i$ is an element of $\mathcal{X}$. We denote this map by $\psi_i(\cdot)\tilde{\varphi}_i$ for all $i = 1, 2, \dots, k$. Since $\{\tilde{\varphi}_i : i = 1, \dots, k\}$ is an orthonormal set, we have

$$\begin{aligned}\|Q_k^m\Psi - \Psi\|_{\mathcal{X}} = \Big\|\sum_{i=1}^k(\Pi_m\psi_i)(\cdot)\tilde{\varphi}_i - \Psi\Big\|_{\mathcal{X}} &\le \Big\|\sum_{i=1}^k[(\Pi_m\psi_i)(\cdot) - \psi_i(\cdot)]\tilde{\varphi}_i\Big\|_{\mathcal{X}} + \Big\|\sum_{i=1}^k \psi_i(\cdot)\tilde{\varphi}_i - \Psi\Big\|_{\mathcal{X}}\\ &\le \sum_{i=1}^k\|\tilde{\varphi}_i\|_{H^1(\Omega)}\|\Pi_m\psi_i - \psi_i\|_{L^2(I)} + \Big\|\sum_{i=1}^k \psi_i(\cdot)\tilde{\varphi}_i - \Psi\Big\|_{\mathcal{X}}\\ &= \sum_{i=1}^k\|\Pi_m\psi_i - \psi_i\|_{L^2(I)} + \Big\|\sum_{i=1}^k \psi_i(\cdot)\tilde{\varphi}_i - \Psi\Big\|_{\mathcal{X}}.\end{aligned}$$

Since $\Pi_m \to I$ pointwise in $L^2(I)$, for each $k \in \mathbb{N}$, we have

$$\lim_{m\to\infty}\|Q_k^m\Psi - \Psi\|_{\mathcal{X}} \le \Big\|\sum_{i=1}^k \psi_i(\cdot)\tilde{\varphi}_i - \Psi\Big\|_{\mathcal{X}}.$$

Since $\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_k\}$ is an orthonormal basis of $R(\tilde{P}_k)$, for each $t \in I$, we have

$$\tilde{P}_k\Psi(t) = \sum_{i=1}^k\langle\Psi(t), \tilde{\varphi}_i\rangle_{H^1}\tilde{\varphi}_i = \sum_{i=1}^k \psi_i(t)\tilde{\varphi}_i.$$

Since $\tilde{P}_n \to I$ pointwise in $H^1(\Omega)$, taking $Q_n = \tilde{P}_n$ in Lemma 4.6, we have

$$\lim_{k\to\infty}\Big\|\sum_{i=1}^k \psi_i(\cdot)\tilde{\varphi}_i - \Psi\Big\|_{\mathcal{X}} = 0,$$

and hence

$$\lim_{k\to\infty}\lim_{m\to\infty}\|Q_k^m\Psi - \Psi\|_{\mathcal{X}} = 0$$

for each $\Psi \in \mathcal{X}$. This completes the proof. □

From the above theorem, we obtain the following corollary.

Corollary 4.8. *Let Q_k^m be as defined in* (4.12). *Then, for every $\varepsilon > 0$ and for every $\Psi \in \mathcal{X}$, there exist $N \in \mathbb{N}$ and $m_k \in N$ for every $k \ge N$ such that $\|Q_k^m\Psi - \Psi\|_{\mathcal{X}} < \varepsilon$ for all $m \ge m_k$, $k \ge N$. In fact, m_k for $k \ge N$ can be chosen such that (m_k) is increasing and $m_k \ge k$ for all $k \ge N$.*

Let $\mathcal{X}_k^m$ denote the range space of Q_k^m for $m, k \in \mathbb{N}$. We have already seen that $\mathcal{X}_k^m$ is finite dimensional and $\{\Psi^{ij} : i = 1, \dots, k, j = 1, \dots, m\}$ is a basis of $\mathcal{X}_k^m$. Then, following the arguments as given in the beginning of this subsection, we may consider an approximation $q_{\alpha,z,w}^{mkn}$ to $q_{\alpha,z,w}^{(n)}$ by defining

$$q_{\alpha,z,w}^{mkn} := \sum_{i=1}^n q_i^{mk}\varphi_i \tag{4.13}$$

with $\mathbf{q}^{mk} := [q_1^{mk}, \dots, q_n^{mk}]$ satisfying the matrix equation

$$A^{mk}\mathbf{q}^{mk} + \alpha D\mathbf{q}^{mk} = b^{mk}, \tag{4.14}$$

where $A^{mk} = [a_{ij}^{mk}]$, $D = [d_{ij}]$, $\mathbf{b}^{mk} = [b_i^{mk}]$ with

$$a_{ij}^{mk} = \langle(\Re A_z\varphi_j)^{mk}, (\Re A_z\varphi_i)^{mk}\rangle_{\mathcal{X}}, \quad d_{ij} = \langle\varphi_j, \varphi_i\rangle, \quad b_i^{mk} = \langle(\Re\Phi_w)^{mk}, (\Re A_z\varphi_i)^{mk}\rangle_{\mathcal{X}}$$

for $i, j = 1, \dots, n$.

We now write the above considered discussion in operator theoretic setting and show that the matrix equation (4.14) does, indeed, have a unique solution. More precisely, we show that $q_{\alpha,z,w}^{mkn}$, as defined in (4.13), is a solution of some operator equation. For this, we shall make use of the following lemma associated with Riesz representation theorem in Hilbert spaces.

Lemma 4.9. *Let X be a real Hilbert space, and let $\tilde{X}$ be a finite-dimensional subspace of X. Let $P\colon X \to X$ be an orthogonal projection onto $\tilde{X}$. Let $P'\colon X^* \to X^*$ be the transpose of P, that is, $(P'\xi)(x) = \xi(Px)$ for all $\xi \in X^*$ and $x \in X$. Let $\mathfrak{R}\colon X^* \to X$ be the Riesz representation map as defined in Lemma 4.1. Then*
(1) *P' is an orthogonal projection;*
(2) *$\mathfrak{R}P' = P\mathfrak{R}$.*

Proof. From the definition, we have $P'(P'\xi)(x) = (P'\xi)(Px) = \xi(P^2x) = \xi(Px) = (P'\xi)(x)$ for all $\xi \in X^*$ and $x \in X$. Thus, $P'^2\xi = P'\xi$ for all $\xi \in X^*$. This shows that P' is a projection operator.

Now, using the orthogonality of P, we have

$$\langle \mathfrak{R}P'\xi, x\rangle = (P'\xi)(x) = \xi(Px) = \langle \mathfrak{R}\xi, Px\rangle = \langle P\mathfrak{R}\xi, x\rangle \quad \text{for all } \xi \in X^* \text{ and } x \in X.$$

Thus, $\mathfrak{R}P' = P\mathfrak{R}$.

Let $\xi, \eta \in X^*$. Then, using Lemma 4.1, we have

$$\langle P'\xi, \eta\rangle = \langle \mathfrak{R}P'\xi, \mathfrak{R}\eta\rangle = \langle P\mathfrak{R}\xi, \mathfrak{R}\eta\rangle = \langle \mathfrak{R}\xi, P\mathfrak{R}\eta\rangle = \langle \mathfrak{R}\xi, \mathfrak{R}P'\eta\rangle = \langle \xi, P'\eta\rangle.$$

This shows that P' is an orthogonal projection. □

By Theorem 4.5 and Lemma 4.9, $(Q_k^m)'$ is an orthogonal projection and $\mathfrak{R}(Q_k^m)' = (Q_k^m)\mathfrak{R}$. Hence we have that $P_nA_z^*(Q_k^m)'A_zP_n$ is a positive self-adjoint operator so that the equation

$$(P_nA_z^*(Q_k^m)'A_zP_n + \alpha I)\zeta = P_nA_z^*(Q_k^m)'\Phi_w \tag{4.15}$$

has a unique solution $\zeta \in R(P_n)$ for each $n \in \mathbb{N}$.

Theorem 4.10. *For $m, n, k \in \mathbb{N}$ and $\alpha > 0$, $\zeta := \sum_{i=1}^n q_i^{mk}\varphi_i$ is the solution of the operator equation* (4.15) *if and only if $\mathbf{q}^{mk} := [q_1^{mk}, \dots, q_n^{mk}]$ is the solution of the matrix equation* (4.14).

Proof. Note that $\zeta \in R(P_n)$ is a solution of (4.15) if and only if

$$\langle (P_nA_z^*(Q_k^m)'A_zP_n + \alpha I)\zeta, \varphi_i\rangle = \langle P_nA_z^*(Q_k^m)'\Phi_w, \varphi_i\rangle \quad \text{for all } i = 1, \dots, n.$$

We note that

$$\begin{aligned}\langle P_nA_z^*(Q_k^m)'A_zP_n\zeta, \varphi_i\rangle &= \langle P_nA_z^*(Q_k^m)'A_zP_n\zeta, \varphi_i\rangle\\ &= \langle (Q_k^m)'A_zP_n\zeta, A_z\varphi_i\rangle_{\mathcal{X}^*}\\ &= \langle (Q_k^m)'A_zP_n\zeta, (Q_k^m)'A_z\varphi_i\rangle_{\mathcal{X}^*}\\ &= \langle \mathfrak{R}(Q_k^m)'A_zP_n\zeta, \mathfrak{R}(Q_k^m)'A_z\varphi_i\rangle_{\mathcal{X}}\\ &= \langle Q_k^m\mathfrak{R}A_zP_n\zeta, Q_k^m\mathfrak{R}A_z\varphi_i\rangle_{\mathcal{X}}.\end{aligned}$$

Also,

$$\begin{aligned}\langle P_nA_z^*(Q_k^m)'\Phi_w, \varphi_i\rangle &= \langle (Q_k^m)'\Phi_w, A_z\varphi_i\rangle_{\mathcal{X}^*} = \langle (Q_k^m)'\Phi_w, (Q_k^m)'A_z\varphi_i\rangle_{\mathcal{X}^*}\\ &= \langle \mathfrak{R}(Q_k^m)'\Phi_w, \mathfrak{R}(Q_k^m)'A_z\varphi_i\rangle_{\mathcal{X}} = \langle Q_k^m\mathfrak{R}\Phi_w, Q_k^m\mathfrak{R}A_z\varphi_i\rangle_{\mathcal{X}}.\end{aligned}$$

Now, using (4.10) and (4.11), we have

$$\begin{aligned}\langle (P_nA_z^*(Q_k^m)'A_zP_n + \alpha I)\zeta, \varphi_i\rangle &= \sum_{j=1}^n q_j^{mk}\langle (\mathfrak{R}A_z\varphi_j)^{mk}, (\mathfrak{R}A_z\varphi_i)^{mk}\rangle_{\mathcal{X}} + \alpha\sum_{j=1}^n q_j^{mk}\langle \varphi_j, \varphi_i\rangle,\\ \langle P_nA_z^*(Q_k^m)'\Phi_w, \varphi_i\rangle &= \langle (\mathfrak{R}\Phi_w)^{mk}, (\mathfrak{R}A_z\varphi_i)^{mk}\rangle_{\mathcal{X}}\end{aligned}$$

for all $i = 1, \dots, n$. Thus, $\zeta = \sum_{i=1}^n q_i^{mk}\varphi_i$ is the solution of (4.15) if and only if $\mathbf{q}^{mk} := [q_1^{mk}, \dots, q_n^{mk}]$ is the solution of (4.14). □

Hereafter, the solution of (4.15) is denoted by $q_{\alpha,z,w}^{mkn}$, that is,

$$q_{\alpha,z,w}^{mkn} = \sum_{i=1}^n q_i^{mk}\varphi_i,$$

where $\mathbf{q}^{mk} := [q_1^{mk}, \dots, q_n^{mk}]$ is the solution of the matrix equation (4.14).

Our next result gives an estimate for the quantity $\|q - q^{mkn}_{\alpha,z,w}\|$.

Theorem 4.11. *Let $q_{\alpha,u}$ and $q^{mkn}_{\alpha,z,w}$ be as defined in* (3.3) *and* (4.13)*, respectively. Let $\eta_n > 0$ and $\eta_{mk} > 0$ be such that*

$$\|A_z - A_z P_n\| \le \eta_n, \quad \|(Q^m_k)' A_z - A_z\| \le \eta_{mk}.$$

Let $\varepsilon > 0$ and $\delta > 0$ be as in (3.1) *and* (3.2)*, respectively. Then*

$$\|q - q^{mkn}_{\alpha,z,w}\| \le \|q - q_{\alpha,u}\| + c_q \frac{\varepsilon + \delta + \eta_{mk} + \eta_n}{2\sqrt{\alpha}},$$

where $c_q = \max\{\|q\|, \frac{1}{2}\}$.

Proof. Following the similar calculations, as done in the proof of Theorem 3.1, with A_z being replaced by $(Q^m_k)' A_z P_n$, we have

$$\|q_{\alpha,u} - q^{mkn}_{\alpha,z,w}\| \le \frac{\|A_u - (Q^m_k)' A_z P_n\|}{\sqrt{\alpha}} \|q\| + \frac{\delta}{2\sqrt{\alpha}}.$$

Now,

$$A_u - (Q^m_k)' A_z P_n = A_u - A_z + [I - (Q^m_k)'] A_z + (Q^m_k)' [A_z (I - P_n)].$$

Therefore,

$$\|A_u - (Q^m_k)' A_z P_n\| \le \varepsilon + \eta_{mk} + \eta_n.$$

Thus, we have

$$\|q - q^{mkn}_{\alpha,z,w}\| \le \|q - q_{\alpha,u}\| + c_q \frac{\varepsilon + \delta + \eta_{mk} + \eta_n}{2\sqrt{\alpha}}. \qquad \square$$

If $\mathcal{H} = H^1(\Omega)$, then we know from Theorem 2.2 that A_z is a compact operator. Using this fact and Corollary 4.8, we deduce the following result.

Theorem 4.12. *Let $A_z : H^1(\Omega) \to \mathcal{X}^*$ and $Q^m_k : \mathcal{X} \to \mathcal{X}$ be as defined in* (2.2) *and* (4.12)*, respectively, and let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ and $m_k \in N$ for every $k \ge N$ such that*

$$\|(Q^m_k)' A_z - A_z\| < \varepsilon \quad \textit{for all } m \ge m_k,\ k \ge N.$$

Proof. Since $A_z : H^1(\Omega) \to \mathcal{X}^*$ is a compact operator, $\mathfrak{R} A_z$ is also a compact operator. Hence

$$S = \mathrm{cl}\{\mathfrak{R} A_z q : q \in H^1(\Omega),\ \|q\| \le 1\}$$

is a compact subset of $\mathcal{X}$. Since $\mathfrak{R}(Q^m_k)' = (Q^m_k)\mathfrak{R}$, we also observe that, for any $q \in H^1(\Omega)$,

$$\|(Q^m_k)' A_z q - A_z q\|_{\mathcal{X}^*} = \|\mathfrak{R}(Q^m_k)' A_z q - \mathfrak{R} A_z q\|_{\mathcal{X}} = \|(Q^m_k)\mathfrak{R} A_z q - \mathfrak{R} A_z q\|_{\mathcal{X}}.$$

Therefore,

$$\|(Q^m_k)' A_z - A_z\| = \sup_{\|q\| \le 1} \|(Q^m_k)' A_z q - A_z q\|_{\mathcal{X}^*} = \sup_{\xi \in S} \|(Q^m_k)\xi - \xi\|_{\mathcal{X}}.$$

Let $\xi \in S$ and $\varepsilon > 0$. Since S is a compact subset of $\mathcal{X}$, there exists $\xi_1, \dots, \xi_p \in S$ such that $S \subset \bigcup_{i=1}^p B(\xi_i, \frac{\varepsilon}{4})$, where $B(\xi_i, \frac{\varepsilon}{4})$ denotes the open ball in $\mathcal{X}$ centered at ξ_i and radius $\frac{\varepsilon}{4}$. Let $j \in \{1, \dots, p\}$ be such that $\|\xi - \xi_j\|_{\mathcal{X}} < \frac{\varepsilon}{4}$. Now, using Corollary 4.8, for each $i \in \{1, \dots, p\}$, there exists $N_i \in \mathbb{N}$ and $m_{i,k} \in \mathbb{N}$ for each $k \ge N_i$ such that $\|(Q^m_k)\xi_i - \xi_i\|_{\mathcal{X}} < \frac{\varepsilon}{2}$ for all $m \ge m_{i,k}$, $k \ge N_i$. Let $N = \max\{N_i : i = 1, \dots, p\}$, and for $k \ge N$, let $m_k = \max\{m_{i,k} : i = 1, \dots, p\}$. Then, for every $k \ge N$ and $m \ge m_k$, we have $\|(Q^m_k)\xi_i - \xi_i\|_{\mathcal{X}} < \frac{\varepsilon}{2}$ for all $i \in \{1, \dots, p\}$. Thus, for every $k \ge N$, there exists $m_k \in \mathbb{N}$ such that, for all $m \ge m_k$,

$$\begin{aligned} \|(Q^m_k)\xi - \xi\|_{\mathcal{X}} &\le \|(Q^m_k)\xi - Q^m_k \xi_j\|_{\mathcal{X}} + \|(Q^m_k)\xi_j - \xi_j\|_{\mathcal{X}} + \|\xi_j - \xi\|_{\mathcal{X}} \\ &\le 2\|\xi_j - \xi\|_{\mathcal{X}} + \|(Q^m_k)\xi_j - \xi_j\|_{\mathcal{X}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \qquad \square$$

Remark 4.13. From the above theorem, it follows that, for the case $\mathcal{H} = H^1(\Omega)$, η_{mk} can be small for some large enough m, k. Thus, combining the above theorem and Remark 4.3, it follows that, for large enough $m, k, n \in \mathbb{N}$, η_n and η_{mk} can be small. Suppose that $\eta_n + \eta_{mk} < \varepsilon + \delta$ for some large enough $m, k, n \in \mathbb{N}$. Then the estimate in Theorem 4.11 takes the form

$$\|q - q_{\alpha,z,w}^{mkn}\| \le \|q - q_{\alpha,u}\| + c_q \frac{\varepsilon + \delta}{\sqrt{\alpha}},$$

where $c_q = \max\{\|q\|, \frac{1}{2}\}$.

5 Parameter choice strategy

We have seen in Theorem 4.11, our bound for the error $\|q - q_{\alpha,z,w}^{mkn}\|$ is given by quantities depending on the parameter α along with the noise levels ε, δ, and the estimates η_n, η_{mk} depending on the natural numbers k, m and n. Also, in Remark 4.3 and Theorem 4.12, we have observed that, in certain cases, η_n and η_{mk} can be small enough for some large $n, m, k \in \mathbb{N}$. In our subsequent analysis, we will show that if we can choose $k, m, n \in \mathbb{N}$ depending on the noise level ε, δ appropriately, then we can choose the parameter α depending on ε, δ such that $\|q - q_{\alpha,z,w}^{mkn}\| \to 0$ as $\varepsilon, \delta \to 0$.

Since $q_{\alpha,u}$ is a Tikhonov regularized solution, it is known that $\|q - q_{\alpha,u}\| \to 0$ as $\alpha \to 0$. In order to obtain an estimate for $\|q - q_{\alpha,u}\|$, it is necessary to assume some source condition on q. As in the classical theory of Tikhonov regularization, we assume the following general source condition on q.

Source condition: Assume that

$$q = \varphi(A_u^* A_u)\xi, \quad \|\xi\| \le \rho, \tag{5.1}$$

for some $\rho > 0$ and for some monotonically increasing positive function φ defined on $(0, \gamma]$, where $\gamma \ge \|A_u\|^2$, such that $\lim_{\lambda \to 0} \varphi(\lambda) = 0$ and

$$\sup_{0 \le \lambda \le \gamma} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \le \varphi(\alpha) \quad \text{for all } \alpha > 0.$$

Note that, under these assumptions, $\|q\| \le \rho\varphi(\gamma)$.

Under the above assumptions, we have

$$q - q_{\alpha,u} = \alpha(A_u^* A_u + \alpha I)^{-1} q = \alpha(A_u^* A_u + \alpha I)^{-1} \varphi(A_u^* A_u)\xi$$

so that

$$\|q - q_{\alpha,u}\| = \|\alpha(A_u^* A_u + \alpha I)^{-1} \varphi(A_u^* A_u)\xi\| \le \sup_{0 \le \lambda \le \gamma} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \|\xi\| \le \rho\varphi(\alpha).$$

A typical case of such a situation is when q is in the range of $\varphi(A_u^* A_u)$, where $\varphi(\lambda) := \lambda^\nu$ for some $\nu \in (0, 1]$ or $\varphi(\lambda) := [\log(\frac{1}{\lambda})]^{-p}$ for some $p > 0$ (see, for example, [4, 11]).

Under the above source condition on q, by Theorem 4.11, we obtain

$$\|q - q_{\alpha,z,w}^{mkn}\| \le \rho\varphi(\alpha) + c_q \frac{\varepsilon + \delta + \eta_{mk} + \eta_n}{2\sqrt{\alpha}}. \tag{5.2}$$

For the simplicity of presentation, we use the notation $\omega := \varepsilon + \delta$.

Now, suppose there exists n_ω, m_ω and $k_\omega \in \mathbb{N}$ such that

$$\eta_{m_\omega k_\omega} + \eta_{n_\omega} < \omega. \tag{5.3}$$

Then, from (5.2), we have

$$\|q - q_{\alpha,z,w}^{m_\omega k_\omega n_\omega}\| \le \rho\varphi(\alpha) + c_q \frac{\omega}{\sqrt{\alpha}}. \tag{5.4}$$

Next, we observe that the estimate $\rho\varphi(\alpha) + c_q \frac{\omega}{\sqrt{\alpha}}$ in (5.4) attains a minimum at α_ω for which

$$\varphi(\alpha_\omega) = \frac{c_q}{\rho} \frac{\omega}{\sqrt{\alpha_\omega}}.$$

Let $\psi(\lambda) = \frac{\rho\lambda\sqrt{\varphi^{-1}(\lambda)}}{c_q}$, $0 < \lambda \leq \gamma$. Then we have

$$\omega = \frac{\rho\sqrt{\alpha_\omega}\varphi(\alpha_\omega)}{c_q} = \psi(\varphi(\alpha_\omega))$$

so that $\alpha_\omega = \varphi^{-1}[\psi^{-1}(\omega)]$. Therefore, from (5.4), it follows that $\|q - q_{\alpha_\omega,z,w}^{m_\omega k_\omega n_\omega}\| \leq 2\rho\varphi(\alpha_\omega) = 2\rho\psi^{-1}(\omega)$. Thus, we have the following result with an *a priori* choice of the parameter α.

Theorem 5.1. *For $\omega = \varepsilon + \delta$, let $k_\omega, m_\omega, n_\omega \in \mathbb{N}$ be as in* (5.3). *Let φ be as in the source condition* (5.1) *and*

$$\psi(\lambda) = \frac{\rho\lambda\sqrt{\varphi^{-1}(\lambda)}}{c_q} \quad \textit{for } 0 < \lambda \leq \gamma,$$

where $\gamma := \|A_u\|^2$ and $c_q = \max\{\|q\|, \frac{1}{2}\}$. Then, with the choice of $\alpha := \varphi^{-1}[\psi^{-1}(\omega)]$,

$$\|q - q_{\alpha,z,w}^{m_\omega k_\omega n_\omega}\| \leq 2\rho\psi^{-1}(\omega).$$

Note that, in the above theorem, for the choice of the regularization parameter α, an a priori knowledge of q through the function φ is required. But, in practical situations, we may not have an a priori knowledge of φ. In such a case, we need to apply some *a posteriori* parameter choice strategies which do not need any prior information about such a function φ. An adaptive method for *a posteriori* parameter choice is given in [7, 13]. In the following, we use the same technique for choosing the regularization parameter a posteriori.

5.1 A posteriori parameter choice

In order to obtain an a posteriori parameter choice, we assume that $C > \frac{1}{2}$ is a constant such that

$$\rho\varphi(\gamma) \leq C, \tag{5.5}$$

where γ, ρ and φ are as given in (5.1). Further assume that $\omega^2 \leq \gamma$, where $\omega = \varepsilon + \delta$.

Let $\alpha_0 = \omega^2$, and let $\mu > 1$ be any fixed real number. Let

$$\alpha_i = \mu^{2i}\alpha_0, \quad i = 1, 2, \dots, N \quad \text{for some } N \in \mathbb{N}. \tag{5.6}$$

Clearly, we have $0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N$ and $\sqrt{\alpha_i} \leq \mu\sqrt{\alpha_{i-1}}$ for all $i = 1, 2, \dots, N$. Since φ is an increasing function and by our assumption $\alpha_0 \leq \gamma$, we have $\rho\varphi(\alpha_0) \leq \rho\varphi(\gamma) \leq C$. Let

$$l = \max\{i \in \{0, 1, \dots, N-1\} : \rho\mu^i\varphi(\alpha_i) \leq C\}. \tag{5.7}$$

Then, for all $j \in \{0, 1, \dots, l\}$, we have

$$\rho\mu^j\varphi(\alpha_j) \leq \rho\mu^l\varphi(\alpha_l). \tag{5.8}$$

We now prove the following lemma.

Lemma 5.2. *Let l be as in* (5.7). *Then, for any $j \in \{0, 1, \dots, l\}$, we have*

$$\|q_{\alpha_l,z,w}^{m_\omega k_\omega n_\omega} - q_{\alpha_j,z,w}^{m_\omega k_\omega n_\omega}\| \leq \frac{4C}{\mu^j}.$$

Proof. Note that, by our assumption in (5.5), we have $c_q \leq C$. Now, using the definition of l, (5.4) and (5.8), we have, for any $j \leq l$,

$$\begin{aligned}\|q_{\alpha_l,z,w}^{m_\omega k_\omega n_\omega} - q_{\alpha_j,z,w}^{m_\omega k_\omega n_\omega}\| &\leq \|q_{\alpha_l,z,w}^{m_\omega k_\omega n_\omega} - q\| + \|q - q_{\alpha_j,z,w}^{m_\omega k_\omega n_\omega}\| \\ &\leq \rho\varphi(\alpha_l) + \frac{c_q}{\mu^l} + \rho\varphi(\alpha_j) + \frac{c_q}{\mu^j} \leq \frac{2C}{\mu^l} + \frac{2C}{\mu^j} \leq \frac{4C}{\mu^j}.\end{aligned}$$

This completes the proof. $\square$

Let

$$\hat{k} = \max\left\{i : \|q^{m_\omega k_\omega n_\omega}_{\alpha_i,z,w} - q^{m_\omega k_\omega n_\omega}_{\alpha_j,z,w}\| \le \frac{4C}{\mu^j},\ j = 0, 1, \dots, i\right\} \quad \text{for } i \in \{0, 1, \dots, N\}. \tag{5.9}$$

Then, by Lemma 5.2, it is clear that $l \le \hat{k}$. Thus, $\hat{k}$ is indeed well-defined.

Following the idea used in the proof of [7, Theorem 4.3], we now prove the following theorem.

Theorem 5.3. *Let $\hat{k}$ be as in (5.9). Then, with the notations as in Theorem 5.1,*

$$\|q - q^{m_\omega k_\omega n_\omega}_{\alpha_{\hat{k}},z,w}\| \le 12C\mu\rho\psi^{-1}(\omega),$$

where $C > \frac{1}{2}$ and $\mu > 1$ are as in (5.5) and (5.6), respectively.

Proof. Since $l \le k$, by (5.4), (5.7), (5.9) and Lemma 5.2, we have

$$\begin{aligned}
\|q - q^{m_\omega k_\omega n_\omega}_{\alpha_{\hat{k}},z,w}\| &\le \|q - q^{m_\omega k_\omega n_\omega}_{\alpha_l,z,w}\| + \|q^{m_\omega k_\omega n_\omega}_{\alpha_l,z,w} - q^{m_\omega k_\omega n_\omega}_{\alpha_{\hat{k}},z,w}\| \\
&\le \rho\varphi(\alpha_l) + \frac{c_q}{\mu^l} + \|q^{m_\omega k_\omega n_\omega}_{\alpha_l,z,w} - q^{m_\omega k_\omega n_\omega}_{\alpha_{\hat{k}},z,w}\| \\
&\le \rho\varphi(\alpha_l) + \frac{C}{\mu^l} + \frac{4C}{\mu^l} \le \frac{6C}{\mu^l}.
\end{aligned}$$

Let α_ω be such that $\rho\varphi(\alpha_\omega) = c_q \frac{\omega}{\sqrt{\alpha_\omega}}$. Then $\rho\varphi(\alpha) + c_q \frac{\omega}{\sqrt{\alpha}}$ attains its minimum at α_ω. Therefore, from the definition of l, we have

$$\varphi(\alpha_\omega)\sqrt{\alpha_\omega} = \frac{c_q}{\rho}\omega \le \frac{C\omega}{\rho} < \omega\mu^{l+1}\varphi(\alpha_{l+1}) = \varphi(\alpha_{l+1})\sqrt{\alpha_{l+1}}.$$

Since φ is an increasing function, from the above inequality, it follows that $\alpha_\omega < \alpha_{l+1}$. Hence

$$\sqrt{\alpha_\omega} < \sqrt{\alpha_{l+1}} = \mu\sqrt{\alpha_l} = \omega\mu^{l+1}.$$

Thus, we have

$$\|q - q^{m_\omega k_\omega n_\omega}_{\alpha_{\hat{k}},z,w}\| \le \frac{6C}{\mu^l} \le 6C\frac{\mu\omega}{\sqrt{\alpha_\omega}} = 6C\frac{\mu\rho\varphi(\alpha_\omega)}{c_q} = 6C\frac{\mu\rho\psi^{-1}(\omega)}{c_q}.$$

Recall that $c_q \ge \frac{1}{2}$, so we have

$$\|q - q^{m_\omega k_\omega n_\omega}_{\alpha_{\hat{k}},z,w}\| \le 12C\rho\mu\psi^{-1}(\omega).$$

This completes the proof. □

Remark 5.4. Note that the parameter $\alpha_{\hat{k}}$ is chosen without any prior knowledge of φ. Also, Theorem 5.3 shows that the parameter $\alpha_{\hat{k}}$ gives an order optimal rate guaranteed by the a priori case.

6 Numerical illustration

In this section, we numerically compute an approximation for the diffusion coefficient q, applying our method for two simple examples. In Section 4, we have seen in detail how to obtain $q^{mkn}_{\alpha,z,w}$, where z and w are the noisy data corresponding to u and u_t, respectively. Also, if ε and δ are the corresponding noise levels as described in (3.1) and (3.2), then Theorem 4.11 gives us an estimate for the quantity $\|q - q^{mkn}_{\alpha,z,w}\|_{\mathcal{H}}$. We first write the algorithm for obtaining $q^{mkn}_{\alpha,z,w}$ and then, as an illustration, consider two examples to see how the functions q and $q^{mkn}_{\alpha,z,w}$ are close to each other, and also to see the decrease in the computed values of error $\|q - q^{mkn}_{\alpha,z,w}\|_{\mathcal{H}}$ by appropriately choosing α, m, k, n.

Algorithm 6.1. The procedure for obtaining $q^{mkn}_{\alpha,z,w}$:
(a) Construct finite-dimensional subspaces X_n of $\mathcal{H}$, $\tilde{X}_k$ of $H^1(\Omega)$ and Y_m of $L^2(0, T)$ such that

$$\overline{\bigcup_n X_n} = \mathcal{H}, \quad \overline{\bigcup_k \tilde{X}_k} = H^1(\Omega) \quad \text{and} \quad \overline{\bigcup_m Y_m} = L^2(0, T).$$

(b) Consider orthonormal bases $\{\varphi_1, \dots, \varphi_n\}$, $\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_k\}$ and $\{g_1, \dots, g_m\}$ of X_n, $\tilde{X}_k$ and Y_m, respectively, and construct the orthonormal basis $\{\Psi^{ij} : 1 \le i \le k, 1 \le j \le m\}$ for the finite-dimensional subspace $\mathcal{X}_k^m$ of $\mathcal{X}$ using the formula $\Psi^{ij} = g_j \tilde{\varphi}_i$.

(c) Compute $(\mathfrak{R}A_z\varphi_l)^{mk} := \sum_{i=1}^k \sum_{j=1}^m c_{ij}^l \Psi^{ij}$ for each $l \in \{1, \dots, n\}$ by solving the equations

$$\sum_{i=1}^k \sum_{j=1}^m c_{ij}^l \int_0^T \langle \Psi^{pq}(\cdot, t), \Psi^{ij}(\cdot, t)\rangle_{H^1(\Omega)}\, dt = (A_z\varphi_l)(\Psi^{pq}) \quad \text{for } 1 \le p \le k,\ 1 \le q \le m.$$

(d) Compute $(\mathfrak{R}\Phi_w)^{mk} := \sum_{i=1}^k \sum_{j=1}^m e_{ij}\Psi^{ij}$ by solving the equations

$$\sum_{i=1}^k \sum_{j=1}^m e_{ij} \int_0^T \langle \Psi^{pq}(\cdot, t), \Psi^{ij}(\cdot, t)\rangle_{H^1(\Omega)}\, dt = \Phi_w(\Psi^{pq}) \quad \text{for } 1 \le p \le k,\ 1 \le q \le m.$$

(e) For $1 \le i, j \le n$, compute

$$\begin{aligned} a_{ij}^{mk} &:= \langle (\mathfrak{R}A_z\varphi_j)^{mk}, (\mathfrak{R}A_z\varphi_i)^{mk}\rangle_{\mathcal{X}} = \sum_{p=1}^k \sum_{q=1}^m c_{pq}^j c_{pq}^i, \\ b_i^{mk} &:= \langle (\mathfrak{R}\Phi_w)^{mk}, (\mathfrak{R}A_z\varphi_i)^{mk}\rangle_{\mathcal{X}} = \sum_{p=1}^k \sum_{q=1}^m e_{pq} c_{pq}^i, \\ d_{ij} &:= \langle \varphi_j, \varphi_i\rangle_{\mathcal{H}}. \end{aligned}$$

(f) Solve the matrix equation $(A^{mk} + \alpha D)\mathbf{q}^{mk} = b^{mk}$, where $A^{mk} = [a_{ij}^{mk}]_{n\times n}$, $D = [d_{ij}]_{n\times n}$, $b^{mk} = [b_i^{mk}]_{n\times 1}$.

Having obtained $\mathbf{q}^{mk} = [\mathrm{q}_1^{mk}, \dots, \mathrm{q}_n^{mk}]$, $q_{\alpha,z,w}^{mkn}$ is given by

$$q_{\alpha,z,w}^{mkn} = \sum_{i=1}^n \mathrm{q}_i^{mk}\varphi_i.$$

In the following two examples, we take $Q = (0, 1) \times (0, 1)$ and ε as the noise level for both u and u_t, and we take $k = n$. All numerical computations are done using MATLAB.

Example 6.2. Let $u(x, t) = \sin \pi xt$. Then, for $q(x) = x(1 - x)$ in $H^1(0, 1)$, u is the solution of (1.1)–(1.3) with $g = 0$, $h = 0$ and

$$f(x, t) = \pi x \cos \pi xt - (1 - 2x)\pi t \cos \pi xt + x(1 - x)\pi^2 t^2 \sin \pi xt.$$

Let

$$z(x, t) = u(x, t) + \frac{\varepsilon}{3\pi} \sin \pi xt, \quad w(x, t) = u_t(x, t) + \frac{\varepsilon}{3} x \cos \pi xt.$$

Example 6.3. Let $u(x, t) = \sin \pi xt$. Then, for $q(x) = \sin 2\pi x$ in $H^1(0, 1)$, u is the solution of (1.1)–(1.3) with $g = 0$, $h = 0$ and

$$f(x, t) = \pi x \cos \pi xt - 2\pi^2 t \cos 2\pi x \cos \pi xt + \pi^2 t^2 \sin 2\pi x \sin \pi xt.$$

Let

$$z(x, t) = u(x, t) + \frac{\varepsilon}{3\pi} \sin \pi xt, \quad w(x, t) = u_t(x, t) + \frac{\varepsilon}{3} x \cos \pi xt.$$

Illustration: We now give a brief illustration of the numerical procedure for the above two examples. Let

$$\tilde{\varphi}_1(x) = 1, \quad \tilde{\varphi}_n(x) = \frac{\sqrt{2}}{\sqrt{(n-1)^2\pi^2 + 1}} \cos(n - 1)\pi x, \quad n \ge 2,\ x \in (0, 1),$$

and

$$g_m(t) := \sqrt{2} \sin m\pi t, \quad m \in \mathbb{N},\ t \in (0, 1).$$

Then it can be seen that $\{\tilde{\varphi}_n : n \in \mathbb{N}\}$ is an orthonormal basis of $H^1(0, 1)$ and $\{g_m : m \in \mathbb{N}\}$ is an orthonormal basis of $L^2(0, 1)$. If $\mathcal{H} = H^1(\Omega)$, then we take $\varphi_n = \tilde{\varphi}_n$, $n \in \mathbb{N}$, as above, and if $\mathcal{H} = L^2(\Omega)$, then we take $\{\varphi_n : n \in \mathbb{N}\}$ as $\{1\} \cup \{\sqrt{2} \cos(n - 1)\pi x : n > 1\}$.

	Error							
	$n = k = 30$		$n = k = 40$		$n = k = 50$		$n = k = 60$	
ε	L^2	H^1	L^2	H^1	L^2	H^1	L^2	H^1
9×10^{-4}	0.0029	0.2911	0.0028	0.2850	0.0028	0.2810	0.0028	0.2781
9×10^{-5}	0.0012	0.2107	0.0009	0.2011	0.0008	0.1945	0.0008	0.1897
9×10^{-6}	0.0011	0.1698	0.0007	0.1572	0.0006	0.1475	0.0005	0.1416
9×10^{-7}	0.0011	0.1462	0.0007	0.1328	0.0006	0.1214	0.0005	0.1180
9×10^{-8}	0.0011	0.1321	0.0007	0.1178	0.0006	0.1070	0.0005	0.1060
9×10^{-9}	0.0011	0.1269	0.0007	0.1108	0.0006	0.1004	0.0005	0.0971
9×10^{-10}	0.0011	0.1256	0.0007	0.1088	0.0006	0.0983	0.0005	0.0928
9×10^{-11}	0.0011	0.1253	0.0007	0.1084	0.0006	0.0979	0.0005	0.0917
9×10^{-12}	0.0011	0.1253	0.0007	0.1083	0.0006	0.0978	0.0005	0.0914

Table 1: Values of $\|q - q^{mkn}_{\alpha,z,w}\|_{L^2(0,1)}$ and $\|q - q^{mkn}_{\alpha,z,w}\|_{H^1(0,1)}$ for $m = 20$, $k = n$ and for different values of ε, n corresponding to Example 6.2.

	Error							
	$n = k = 30$		$n = k = 40$		$n = k = 50$		$n = k = 60$	
ε	L^2	H^1	L^2	H^1	L^2	H^1	L^2	H^1
9×10^{-4}	0.0105	2.2842	0.0095	2.2520	0.0092	2.2314	0.0090	2.2165
9×10^{-5}	0.0065	1.4271	0.0047	1.3670	0.0040	1.3258	0.0036	1.2978
9×10^{-6}	0.0063	1.0908	0.0043	1.0054	0.0035	0.9384	0.0031	0.9055
9×10^{-7}	0.0063	0.9471	0.0043	0.8520	0.0035	0.7629	0.0031	0.7509
9×10^{-8}	0.0063	0.8505	0.0043	0.7627	0.0035	0.6756	0.0031	0.6830
9×10^{-9}	0.0063	0.7948	0.0043	0.7017	0.0035	0.6321	0.0031	0.6280
9×10^{-10}	0.0063	0.7782	0.0043	0.6785	0.0035	0.6163	0.0031	0.5883
9×10^{-11}	0.0063	0.7744	0.0043	0.6729	0.0035	0.6124	0.0031	0.5745
9×10^{-12}	0.0063	0.7736	0.0043	0.6716	0.0035	0.6115	0.0031	0.5713

Table 2: Values of $\|q - q^{mkn}_{\alpha,z,w}\|_{L^2(0,1)}$ and $\|q - q^{mkn}_{\alpha,z,w}\|_{H^1(0,1)}$ for $m = 20$, $k = n$ and different values of ε, n corresponding to Example 6.3.

To obtain integrals numerically, we have used *Simpson's 1/3 rule*. For that, we have considered partitions with uniform mesh for the space and time variables as

$$0 = x_0 < x_1 < \cdots < x_{N1} = 1 \quad \text{and} \quad 0 = t_0 < t_1 < \cdots < t_{N2} = 1.$$

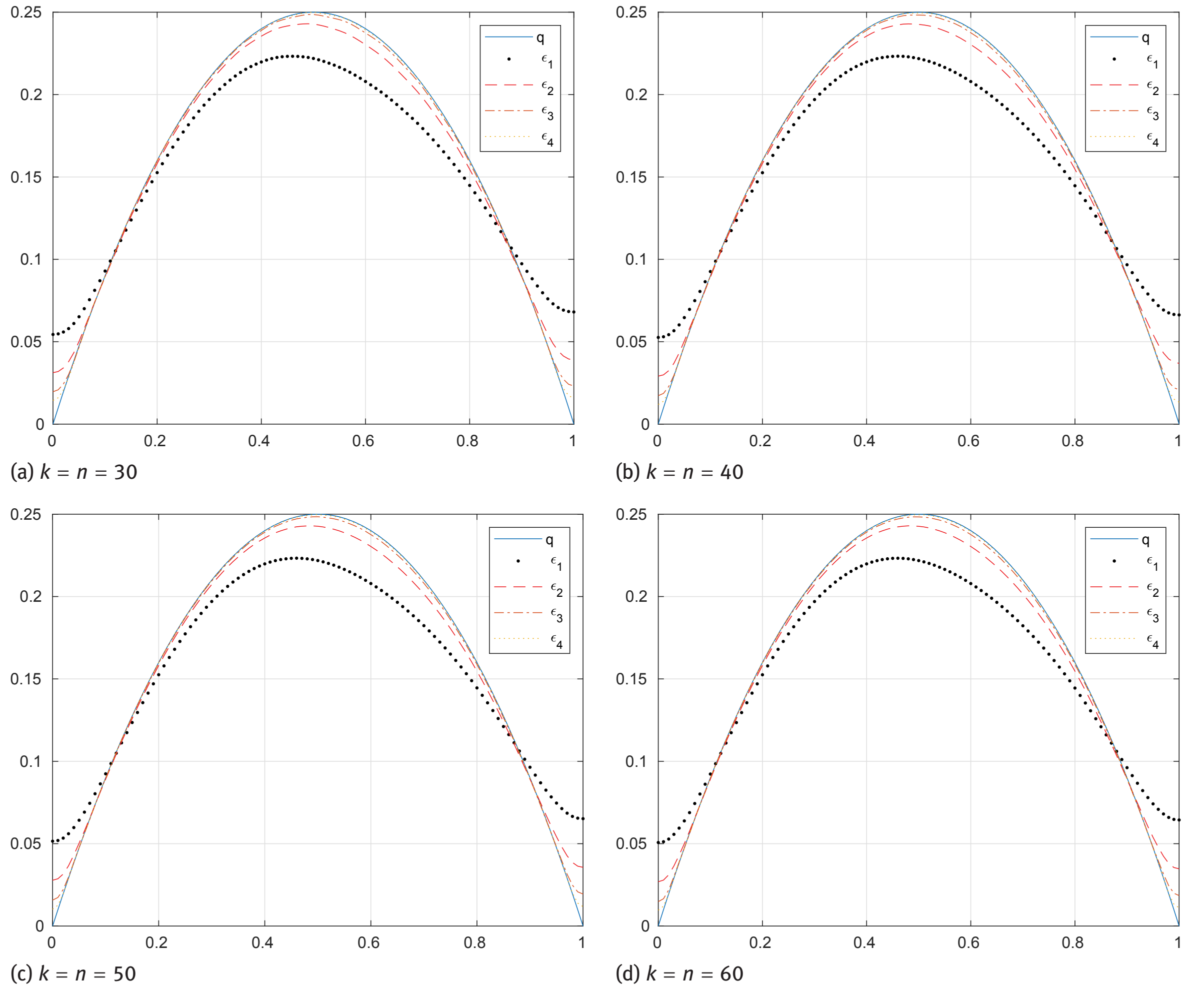
We have taken $N1 = N2 = 100$, $x_i = ih$, $t_j = jh$ with $h = \frac{1}{100}$.

After obtaining $q^{mkn}_{\alpha,z,w}$ as described in Algorithm 6.1, for $k = n$, we have calculated $\|q - q^{mkn}_{\alpha,z,w}\|_{H^1(0,1)}$ and $\|q - q^{mkn}_{\alpha,z,w}\|_{L^2(0,1)}$ numerically using *Simpson's 1/3 rule* for nine different values of the noise level ε which are given in Table 1 and Table 2, namely, for $\varepsilon = 9 \times 10^{-s}$, $s \in \{4, \dots, 12\}$.

From the theory of Tikhonov regularization, it is known that if q is *smooth enough* and m, k, n are large enough such that $\eta_n + \eta_{mk} < \varepsilon$ and if we choose $\alpha = \varepsilon^{2/3}$, then the estimate $\|q - q^{mkn}_{\alpha,z,w}\|_{\mathcal{H}}$ is of the order $\varepsilon^{2/3}$. So we have done the computations by taking $\alpha = \varepsilon^{2/3}$ for different values of ε. From both the tables, we observe that errors in L^2-norm are much smaller than errors in H^1-norm.

Remark 6.4. While performing numerical computations, we have observed that if m is increased for a fixed n and k, then the change in the value of $\|q - q^{mkn}_{\alpha,z,w}\|_{H^1(0,1)}$ is very negligible. So, in both the examples, we have presented all the numerical computations by taking $m = 20$.

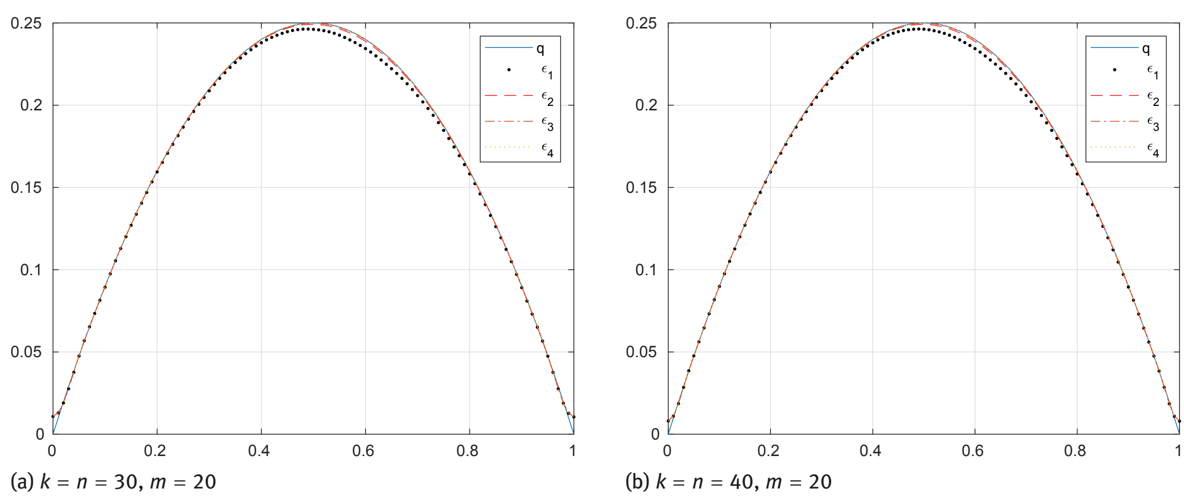
Figure 1 and Figure 2 correspond to Example 6.2 for the cases $H^1(0,1)$ and $L^2(0,1)$, respectively. Figure 3 and Figure 4 correspond to Example 6.3 for the cases $H^1(0,1)$ and $L^2(0,1)$, respectively. In all the figures, the curve corresponding to the label "q" is for the function $q(x)$, and for $i = 1, 2, 3, 4$, the curves with labels ϵ_i correspond to the function q^{mkn}_{α,z_i,w_i}, where q^{mkn}_{α,z_i,w_i} is as obtained using the Algorithm 6.1 for the noise level



(a) $k = n = 30$ (b) $k = n = 40$

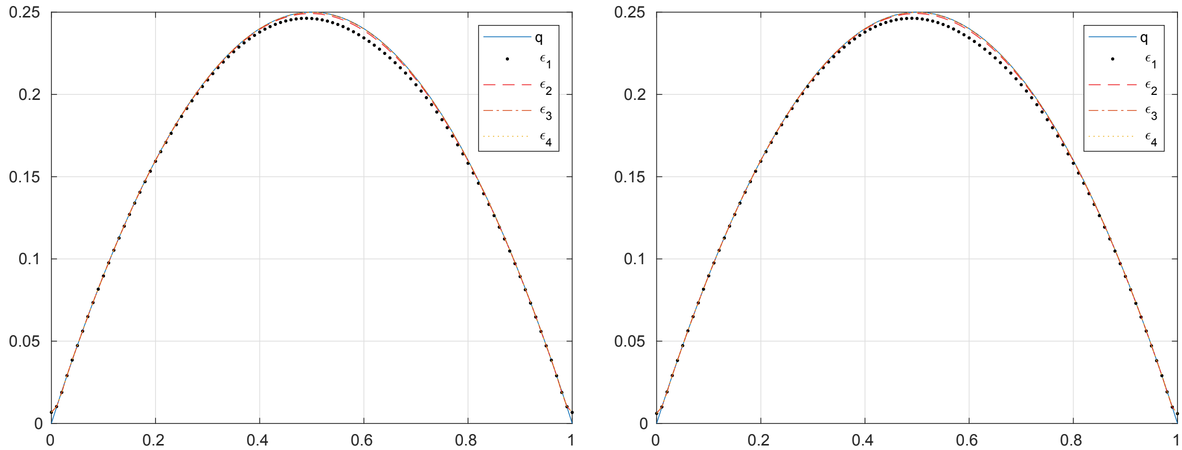
(c) $k = n = 50$ (d) $k = n = 60$

Figure 1: Graphs of q and $q_{\alpha,z,w}^{mkn}$ for different noise levels corresponding to Example 6.2 for the case $\mathcal{H} = H^1(0, 1)$.



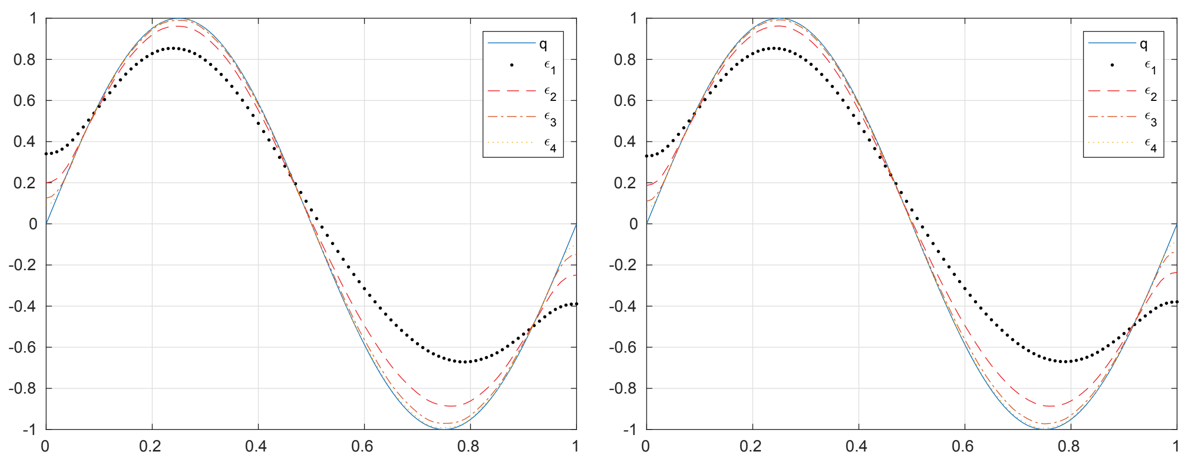
(a) $k = n = 30$, $m = 20$ (b) $k = n = 40$, $m = 20$

Figure 2: Graphs of q and $q_{\alpha,z,w}^{mkn}$ for different noise levels corresponding to Example 6.2 for the case $\mathcal{H} = L^2(0, 1)$.

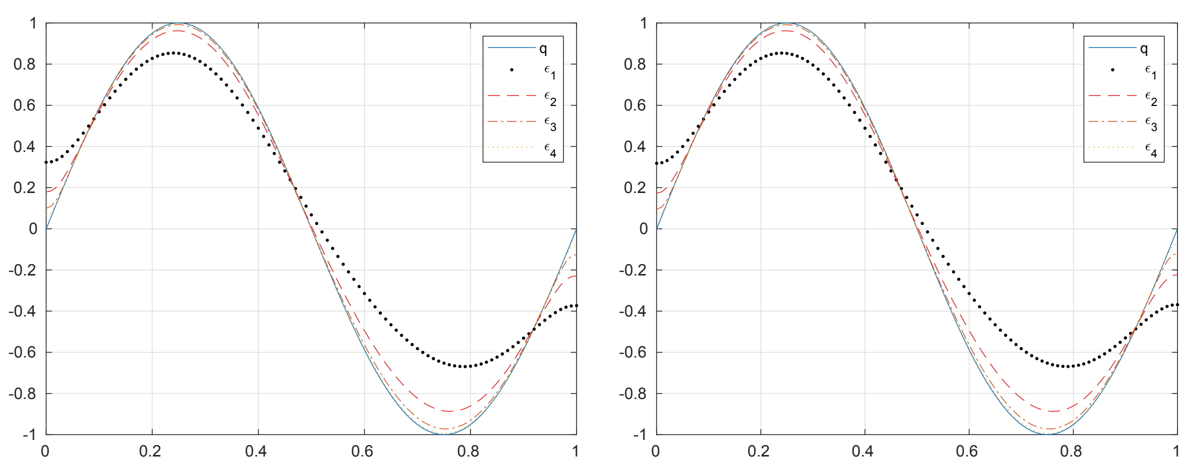


(c) $k = n = 50$, $m = 20$ (d) $k = n = 60$, $m = 20$

Figure 2 (continued)

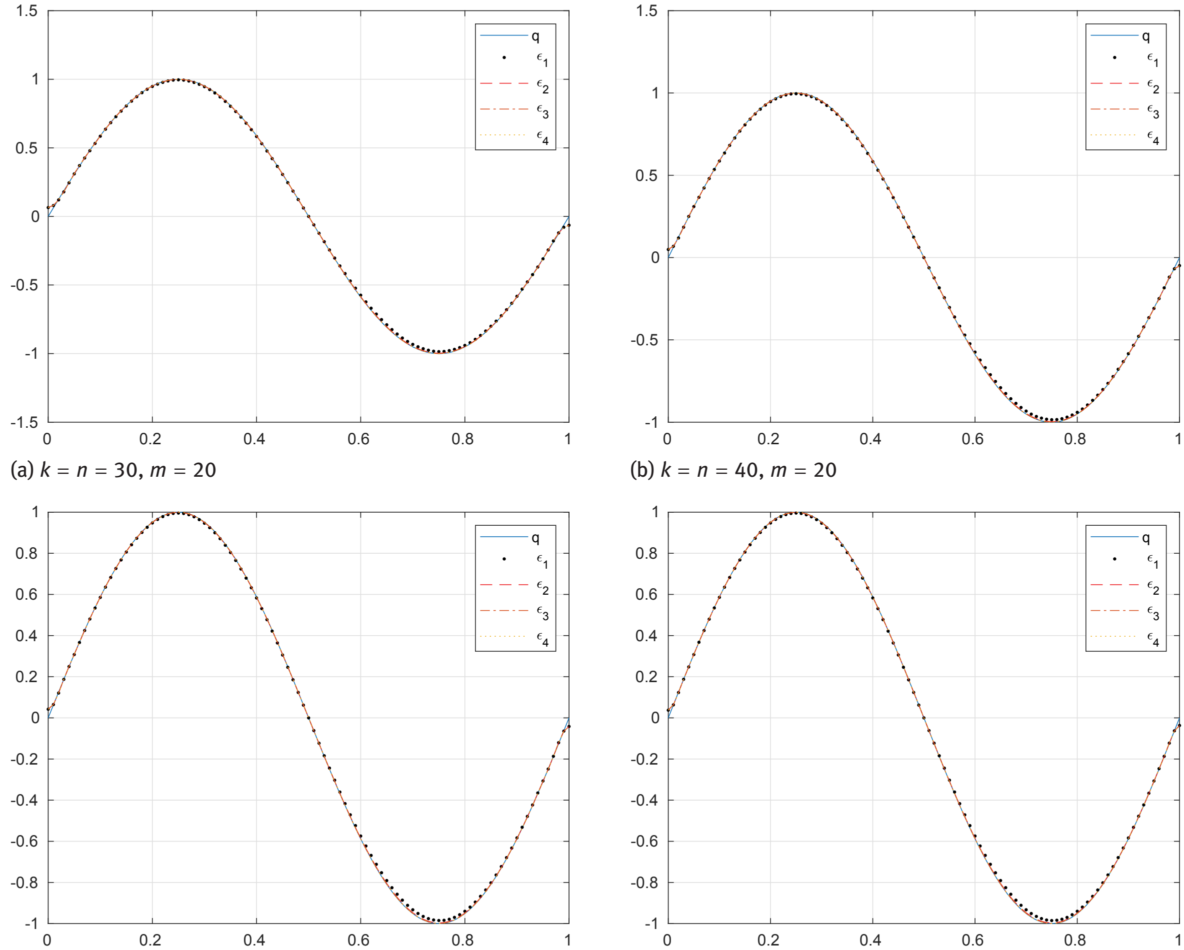


(a) $k = n = 30$, $m = 20$ (b) $k = n = 40$, $m = 20$



(c) $k = n = 50$, $m = 20$ (d) $k = n = 60$, $m = 20$

Figure 3: Graphs of q and $q^{mkn}_{\alpha,z,w}$ for different noise levels corresponding to Example 6.3 for the case $\mathcal{H} = H^1(0, 1)$.



(a) $k = n = 30$, $m = 20$

(b) $k = n = 40$, $m = 20$

(c) $k = n = 50$, $m = 20$

(d) $k = n = 60$, $m = 20$

Figure 4: Graphs of q and $q_{\alpha,z,w}^{mkn}$ for different noise levels corresponding to Example 6.3 for the case $\mathcal{H} = L^2(0,1)$.

$\varepsilon = 9 \times 10^{-(i+3)}$. From the figures, it is evident that, as the noise level ε gets smaller, the computed function $q_{\alpha,z,w}^{mkn}$ gets closer and closer to the actual function q.

7 Concluding remarks

Motivated by the idea used in [12], we have converted our inverse problem of determining the parameter function q (also known as diffusion coefficient) from an approximate knowledge of the solution u and u_t into an operator equation by means of a weak formulation of the given parabolic PDE. We have formulated the inverse problem as an operator equation with data as the operator and the right-hand side and used Tikhonov regularization for obtaining stable approximations to q.

Also, we have given a procedure to obtain finite-dimensional subspaces of the infinite-dimensional Hilbert space $\mathcal{X} = L^2(0,T; H^1(\Omega))$ from the knowledge of finite-dimensional subspaces of $H^1(\Omega)$ and $L^2(0,T)$ in a natural way. Using these finite-dimensional subspaces and finite-dimensional subspaces of $\mathcal{H}$, we have obtained approximations to q, which are obtained using solutions of certain matrix equations.

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