Research Article

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A linear regularization method for a nonlinear parameter identification problem

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Abstract: In order to obtain regularized approximations for the solution q of the parameter identification problem $-\nabla.(q\nabla u) = f$ in Ω along with the Neumann boundary condition $q\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$, which is an ill-posed problem, we consider its weak formulation as a linear operator equation with operator as a function of the data $u \in W^{1,\infty}(\Omega)$, and then apply the Tikhonov regularization and a finite-dimensional approximation procedure when the data is noisy. Here, Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary, $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. This approach is akin to the *equation error method* of Al-Jamal and Gockenback (2012) wherein error estimates are obtained in terms of a quotient norm, whereas our procedure facilitates to obtain error estimates in terms of the regularization parameters and data errors with respect to the norms of the spaces under consideration. In order to obtain error estimates when the noisy data belongs to $L^2(\Omega)$ instead of $W^{1,\infty}(\Omega)$, we shall make use of a smoothing procedure using the Clement operator under additional assumptions of Ω and u.

Keywords: Ill-posed, regularization, parameter identification, regularization parameter

MSC 2010: 35R30, 65N30, 65J15, 65J20, 76S05

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. We consider the problem of identifying the parameter function $q(x)$, $x \in \Omega$, from the observations on another function $u(x)$, $x \in \Omega$, which satisfies the PDE

$$-\nabla.(q\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

along with the boundary condition

$$q\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega, \tag{1.2}$$

where $f \in L^2(\Omega)$, $g \in H^{-1/2}(\partial\Omega)$ and ν is the unit outward normal to $\partial\Omega$. It is known that the above problem is ill-posed (cf. [7]).

In [1], Al-Jamal and Gockenback considered an *equation error method* to obtain regularized approximations for the weak formulation of the above problem. The existence and uniqueness of a solution using this method is proved in [12], and stability results and error estimates are obtained in [1] (see also [10]). Here we look into the same problem but with a different approach. We convert the nonlinear problem into a linear operator equation with operator as a function of data and then use a Tikhonov-type regularization and its finite-dimensional realizations. By this procedure, the existence, the uniqueness and stability results follow and error estimates are obtained by applying standard results in regularization theory (cf. [15]). The error estimates are better compared to the results in [1], in the sense that in [1] the bound for the error is in terms of some quotient norm, whereas our estimates for the actual error are in terms of the norms of the space under

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consideration. The abstract formulation of the method of this paper is somewhat similar to the procedure adopted in Cao and Pereverzev [5] and Cao and Nair [4].

2 Operator theoretic formulation

We consider the problem of identifying the parameter function $q(x)$, $x \in \Omega$ from the observations on another function $u(x)$, $x \in \Omega$ which satisfies the PDE (1.1) along with the boundary condition (1.2), where $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. We look for a q in an appropriate function space such as $L^2(\Omega)$ or $H^1(\Omega)$ which satisfies the weak form of (1.1)–(1.2), namely,

$$\int_\Omega q(x)\nabla u(x).\nabla v(x)\,dx = \int_\Omega f(x)v(x)\,dx + \int_{\partial\Omega} g(x)v(x)\,dx \quad \text{for all } v \in H^1(\Omega) \tag{2.1}$$

for $u \in W^{1,\infty}(\Omega)$. Equation (2.1) can be written as

$$T_u(q)(v) = \Phi(v), \tag{2.2}$$

where

$$\begin{aligned} T_u(q)(v) &:= \int_\Omega q(x)\nabla u(x).\nabla v(x)\,dx, && v \in H^1(\Omega),\\ \Phi(v) &:= \int_\Omega f(x)v(x)\,dx + \int_{\partial\Omega} g(x)v(x)\,dx, && v \in H^1(\Omega). \end{aligned}$$

Thus, the inverse problem at hand is the following: given $u \in W^{1,\infty}(\Omega)$, find q in $L^2(\Omega)$ or $H^1(\Omega)$ such that (2.2) is satisfied. Clearly, this is a nonlinear problem, and we shall show that it is also ill-posed (see Theorem 2.3). However, the above operator formulation enables us to use linear regularization methods. We shall use the well-known Tikhonov regularization for this purpose and derive error estimates. First, let us observe some properties of the operator $q \mapsto T_u(q)$. We may also observe that if $|\nabla u| = 0$ a.e., then $T_u = 0$. Thus, the inverse problem is non-trivial only when $|\nabla u| > 0$ a.e. (See Theorem 2.4.)

In the following, we shall use the notations $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$ for the norms $\|\cdot\|_{[L^2(\Omega)]^d}$ and $\|\cdot\|_{[L^\infty(\Omega)]^d}$, respectively. Also, the dual of $H^1(\Omega)$ is denoted by $H^1(\Omega)^*$.

Theorem 2.1. *Let $w \in W^{1,\infty}(\Omega)$ and let*

$$T_w(q)(v) := \int_\Omega q(x)\nabla w(x).\nabla v(x)\,dx \quad \text{for } q \in L^2(\Omega),\ v \in H^1(\Omega). \tag{2.3}$$

Then we have the following:
(i) *$T_w(q) \in H^1(\Omega)^*$ for every $q \in L^2(\Omega)$, and $T_w : L^2(\Omega) \to H^1(\Omega)^*$ is a bounded linear operator such that $\|T_w\| \le \|\nabla w\|_{L^\infty(\Omega)}$.*
(ii) *$T_w : H^1(\Omega) \to H^1(\Omega)^*$ is a compact operator.*

Proof. Let $q \in L^2(\Omega)$. Then we observe that the map $T_w(q) : H^1(\Omega) \to \mathbb{R}$ is linear, and for every $v \in H^1(\Omega)$ we have

$$\begin{aligned} |T_w(q)(v)| &\le \int_\Omega |q(x)\nabla w(x).\nabla v(x)|\,dx\\ &\le \|q\nabla w\|_{L^2}\|\nabla v\|_{L^2}\\ &\le \|\nabla w\|_{L^\infty}\|q\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}. \end{aligned}$$

Thus, $T_w(q) \in H^1(\Omega)^*$ and

$$\|T_w(q)\| \le \|\nabla w\|_{L^\infty}\|q\|_{L^2(\Omega)},$$

so that $T_w : L^2(\Omega) \to H^1(\Omega)^*$ is a bounded linear operator and $\|T_w\| \le \|\nabla w\|_{L^\infty}$.

It is known, under the assumptions on Ω, that the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is compact (cf. [21]). Hence, being a composition of a compact operator with a bounded operator, T_w as an operator from $H^1(\Omega)$ into $H^1(\Omega)^*$ is a compact operator (cf. [14]). □

Theorem 2.2. *For $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$ let Φ be defined by*

$$\Phi(v) := \int_\Omega f(x)v(x)\,dx + \int_{\partial\Omega} g(x)v(x)\,dx, \quad v \in H^1(\Omega).$$

Then $\Phi \in H^1(\Omega)^*$ and

$$\|\Phi\| \le \|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Omega)}.$$

Proof. For $v \in H^1(\Omega)$ we have

$$\begin{aligned}|\Phi(v)| &\le \int_\Omega |f(x)v(x)|\,dx + \int_{\partial\Omega} |g(x)v(x)|\,dx\\ &\le \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}\|v\|_{H^1(\Omega)}\\ &= (\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)})\|v\|_{H^1(\Omega)}.\end{aligned}$$

Hence, $\Phi \in H^1(\Omega)^*$ and $\|\Phi\| \le \|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Omega)}$. □

Theorem 2.3. *Let $\mathcal{H}$ be either $L^2(\Omega)$ or $H^1(\Omega)$. Then the map $q \mapsto u$ from $\mathcal{H}$ to $W^{1,\infty}(\Omega)$ satisfying (2.2) does not have a continuous inverse.*

Proof. Let $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Omega)$. Let q in $\mathcal{H}$ and u in $W^{1,\infty}(\Omega)$ be such that they satisfy equation (2.1). For $n \in \mathbb{N}$ let $u_n := \frac{u}{n}$ and $q_n := nq$. Then for each $n \in \mathbb{N}$ we have $u_n \in W^{1,\infty}(\Omega)$ and $q_n \in \mathcal{H}$, and q_n is the solution of the inverse problem (2.2) with u_n as data, in place of u. Thus, we have

$$T_{u_n}(q_n) = \Phi.$$

Note that

$$\|u_n\|_{W^{1,\infty}(\Omega)} = \frac{1}{n}\|u\|_{W^{1,\infty}(\Omega)} \to 0 \quad \text{as } n \to \infty,$$

whereas

$$\|q_n\|_{\mathcal{H}} = n\|q\|_{\mathcal{H}} \to \infty \quad \text{as } n \to \infty.$$

Thus, we have proved that the map $q \mapsto u$ from $\mathcal{H}$ to $W^{1,\infty}(\Omega)$ satisfying (2.2) does not have a continuous inverse. □

In view of the above theorem, the inverse problem of finding $q \in \mathcal{H}$ from the data $u \in W^{1,\infty}(\Omega)$ satisfying (2.2) is ill-posed.

The next observation is important in the context of applying the Tikhonov regularization and its finite-dimensional realizations.

Theorem 2.4. *Let $w \in W^{1,\infty}(\Omega)$ be such that $|\nabla w| > 0$ a.e. Then the operator $T_w : \mathcal{H} \to H^1(\Omega)^*$ defined in (2.3) is of infinite rank. In particular, T_w as an operator from $H^1(\Omega)$ to $H^1(\Omega)^*$ is a compact operator of infinite rank.*

Proof. For $n \in \mathbb{N}$ let B_n be a sequence of open balls in Ω such that $B_n \cap B_m = \emptyset$ for $m \ne n$. Also for each $n \in \mathbb{N}$ let B_n' and B_n'' be open balls in Ω such that $B_n'' \subset B_n' \subset B_n$, with the inclusions being proper. Let $q_n \in C_c^\infty(\Omega)$ be such that $0 \le q_n \le 1$, $q_n = 1$ on B_n'' and $\operatorname{supp}(q_n) \subseteq B_n'$. We show that $\{T_w(q_n) : n \in \mathbb{N}\}$ is an infinite linearly independent set in $H^1(\Omega)^*$, which will prove that T_w is of infinite rank. For this, let $v_n \in C_c^\infty(\Omega)$ be such that $v_n = 1$ on B_n' and $\operatorname{supp}(v_n) \subseteq B_n$. Then $v_n w \in H^1(\Omega)$ and $\nabla(v_n w) = \nabla w$ on B_n' for all $n \in \mathbb{N}$. Also,

$$T_w(q_n)(v_n w) = \int_{B_n'} q_n|\nabla w|^2 \quad \text{for all } n \in \mathbb{N} \tag{2.4}$$

and

$$T_w(q_m)(v_n w) = 0 \quad \text{for all } n, m \in \mathbb{N} \text{ with } n \ne m. \tag{2.5}$$

Now, let $k \in \mathbb{N}$ and $c_1, \dots, c_k$ in $\mathbb{R}$ be such that $\sum_{n=1}^{k} c_n T_w(q_n) = 0$. Then

$$\sum_{n=1}^{k} c_n T_w(q_n)(v_m w) = 0, \quad m = 1, \dots, k.$$

Hence, using (2.4) and (2.5), we obtain

$$\int_{B'_m} c_m q_m |\nabla w|^2 = 0, \quad m = 1, \dots, k.$$

Now,

$$\int_{B'_m} q_m |\nabla w|^2 \geq \int_{B''_m} |\nabla w|^2 > 0.$$

Therefore, $c_m = 0$ for $m = 1, \dots, k$. Thus, we have proved that $\{T_w(q_n) : n \in \mathbb{N}\}$ is an infinite linearly independent set in $H^1(\Omega)^*$. The particular case follows since the inclusion operator from $H^1(\Omega)$ to $L^2(\Omega)$ is compact. □

3 Regularization with noisy data

We use the notation $\mathcal{H}$ to denote either $L^2(\Omega)$ or $H^1(\Omega)$. The inner product and norm in a Hilbert space H are denoted by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$, respectively, and no subscripts will be used for the operator norms.

We assume that $u \in W^{1,\infty}(\Omega)$ is such that the operator equation (2.2) has a unique solution $q \in \mathcal{H}$. Now, instead of $u \in W^{1,\infty}(\Omega)$, let us suppose that the available data is $z \in W^{1,\infty}(\Omega)$ with

$$\|u - z\|_{W^{1,\infty}(\Omega)} \leq \delta$$

for some $\delta > 0$.

Remark 3.1. In many of the practical situations, one may not be knowing that the noisy data belongs to $W^{1,\infty}(\Omega)$; we may only have a noisy data in $L^2(\Omega)$. In Section 5, we take care of this situation for the case when Ω is a polygonal domain in $\mathbb{R}^2$ and u is in $H^4(\Omega)$ by considering a “smoothing procedure” as in [11] and [6], and make use of the analysis which is being carried out under the assumption of $z \in W^{1,\infty}(\Omega)$.

In the previous section, we saw that the problem of finding $q \in H^1(\Omega)$ from the data $u \in W^{1,\infty}(\Omega)$ satisfying (2.2) is an ill-posed problem. Thus, in order to find a stable approximation of q, we have to use some regularization procedure. For this purpose, we consider a Tikhonov-type regularization of (2.2) with T_z in place of T_u. So let $q_{\alpha,u}$, $q_{\alpha,z}$ in $\mathcal{H}$ be such that

$$(T_u^* T_u + \alpha I) q_{\alpha,u} = T_u^* \Phi \tag{3.1}$$

and

$$(T_z^* T_z + \alpha I) q_{\alpha,z} = T_z^* \Phi, \tag{3.2}$$

respectively. We remark that the above formulation is different from the standard Tikhonov method. In the standard Tikhonov regularization, the right-hand side involves the noisy data, whereas above, the noise is only in the operator. Note that the adjoint operators T_u^* and T_z^* are from $H^1(\Omega)^*$ into $\mathcal{H}$ and the operators $T_u^* T_u$ and $T_z^* T_z$ on $\mathcal{H}$ are self adjoint and positive definite so that (3.1) and (3.2) are well-posed equations. Now, the question is whether

$$\|q_{\alpha,z} - q_{\alpha,u}\|_{\mathcal{H}} \to 0 \quad \text{as } \delta \to 0.$$

To address this issue, we may recall from Theorem 2.1 that, if $w_1, w_2 \in W^{1,\infty}(\Omega)$, then

$$\|T_{w_1} - T_{w_2}\| = \|T_{w_1 - w_2}\| \leq \|\nabla(w_1 - w_2)\|_{L^\infty} \leq \|w_1 - w_2\|_{W^{1,\infty}(\Omega)}.$$

In particular, we have

$$\|T_z - T_u\| \to 0 \quad \text{as } \delta \to 0.$$

Therefore, modifying the proof of [15, Corollary 5.1] (see also [19, Lemma 2.1]) we derive the following stability and convergence results.

Theorem 3.2. *Let $q_{\alpha,u}$ and $q_{\alpha,z}$ be as in* (3.1) *and* (3.2), *respectively. Then*

$$\|q_{\alpha,z} - q_{\alpha,u}\|_{\mathcal{H}} \le \frac{1}{\sqrt{\alpha}}\|q\|_{\mathcal{H}}\|T_z - T_u\|.$$

In particular,

$$\|q - q_{\alpha,z}\|_{\mathcal{H}} \le \|q - q_{\alpha,u}\|_{\mathcal{H}} + \frac{\delta}{\sqrt{\alpha}}\|q\|_{\mathcal{H}}$$

and

$$\|q_{\alpha,z} - q_{\alpha,u}\|_{\mathcal{H}} \to 0 \quad \textit{as } \delta \to 0.$$

Proof. We have already assumed that q is a solution of (2.2). Hence, we have

$$\begin{aligned} q_{\alpha,z} - q_{\alpha,u} &= (T_z^* T_z + \alpha I)^{-1} T_z^* \Phi - (T_u^* T_u + \alpha I)^{-1} T_u^* \Phi \\ &= (T_z^* T_z + \alpha I)^{-1} T_z^* T_u q - (T_u^* T_u + \alpha I)^{-1} T_u^* T_u q \\ &= (T_z^* T_z + \alpha I)^{-1} T_z^* (T_u - T_z) T_u^* T_u (T_u^* T_u + \alpha I)^{-1} q + \alpha (T_z^* T_z + \alpha I)^{-1} (T_z^* - T_u^*)(T_u T_u^* + \alpha I)^{-1} T_u q. \end{aligned}$$

We know that (see [15, Corollary 4.5, Lemma 4.1])

$$\begin{aligned} &\|(T_z^* T_z + \alpha I)^{-1} T_z^*\| \le \frac{1}{2\sqrt{\alpha}}, \quad &&\|(T_u T_u^* + \alpha I)^{-1} T_u\| \le \frac{1}{2\sqrt{\alpha}}, \\ &\|T_u^* T_u (T_u^* + \alpha I)^{-1}\| \le 1, &&\|(T_z^* T_z + \alpha I)^{-1}\| \le \frac{1}{\alpha}. \end{aligned}$$

Thus, we obtain

$$\|q_{\alpha,z} - q_{\alpha,u}\|_{\mathcal{H}} \le \frac{1}{\sqrt{\alpha}}\|q\|_{\mathcal{H}}\|T_z - T_u\|.$$

The particular cases follow since

$$\|q - q_{\alpha,u}\|_{\mathcal{H}} \le \|q - q_{\alpha,u}\|_{\mathcal{H}} + \|q_{\alpha,u} - q_{\alpha,z}\|_{\mathcal{H}}$$

and $\|T_z - T_u\| \le \|z - u\|_{\mathcal{H}} \le \delta$. □

We may recall from the theory of Tikhonov regularization (cf. [7, 15]) that

$$\|q - q_{\alpha,u}\|_{\mathcal{H}} \to 0 \quad \text{as } \alpha \to 0.$$

Thus, for an appropriate choice of the *regularization parameter* α, say $\alpha := \alpha_\delta$, which satisfies $\alpha_\delta \to 0$ and $\delta/\sqrt{\alpha_\delta} \to 0$ as $\delta \to 0$, we have

$$\|q - q_{\alpha_\delta,z}\|_{\mathcal{H}} \to 0 \quad \text{as } \delta \to 0.$$

4 Finite-dimensional realizations

4.1 Regularized projection method

In order to obtain numerical approximations for the solution of (3.1)–(3.2), it is necessary to use some approximation method.

Observe that equation (3.2) is same as

$$\langle (T_z^* T_z + \alpha I) q_{\alpha,z}, \varphi \rangle_{\mathcal{H}} = \langle T_z^* \Phi, \varphi \rangle_{\mathcal{H}} \quad \text{for all } \varphi \in \mathcal{H}. \tag{4.1}$$

In order to obtain a finite-dimensional approximation for $q_{\alpha,z}$, we consider a finite-dimensional subspace X_n of $\mathcal{H}$ and let φ in (4.1) vary over X_n. Thus, we look for $q_{\alpha,z}^{(n)} \in X_n$ such that

$$\langle (T_z^* T_z + \alpha I) q_{\alpha,z}^{(n)}, \varphi \rangle_{\mathcal{H}} = \langle T_z^* \Phi, \varphi \rangle_{\mathcal{H}} \quad \text{for all } \varphi \in X_n,$$

or, equivalently,

$$\langle (T_z q_{\alpha,z}^{(n)}, T_z \varphi \rangle_{H^1(\Omega)^*} + \alpha \langle q_{\alpha,z}^{(n)}, \varphi \rangle_{\mathcal{H}} = \langle \Phi, T_z \varphi \rangle_{H^1(\Omega)^*} \quad \text{for all } \varphi \in X_n. \tag{4.2}$$

We observe that (4.2) can be represented as

$$(P_n T_z^* T_z P_n + \alpha I_n) q_{\alpha,z}^{(n)} = P_n T_z^* \Phi, \tag{4.3}$$

where $P_n : \mathcal{H} \to \mathcal{H}$ is the orthogonal projection whose range is X_n. Since the operator $P_n T_z^* T_z P_n$ on $\mathcal{H}$ is self adjoint and positive definite, equation (4.3) has a unique solution $q_{\alpha,z}^{(n)}$ for each $\alpha > 0$, $z \in W^{1,\infty}(\Omega)$ and $n \in \mathbb{N}$.

We may observe that (4.3) is similar to (3.2) with T_z replaced by $T_z P_n$. Thus, the above formulation of our approximation method is similar to the projection methods used in [13, 15, 20] for an ill-posed operator equation, where the right-hand side involves the noisy data. In fact, while discussing the computational issues, we shall consider a more general situation with T_z replaced by $Q_m T_z P_n$, as has been done in [13, 20].

Let us assume that $\dim(X_n) = n$ and let $\{\varphi_1, \dots, \varphi_n\}$ be a basis of X_n. Expressing the solution $q_{\alpha,z}^{(n)}$ of (4.2) as

$$q_{\alpha,z}^{(n)} = \sum_{j=1}^{n} q_j \varphi_j, \tag{4.4}$$

equation (4.2) takes the form

$$\sum_{j=1}^{n} q_j \langle T_z \varphi_j, T_z \varphi_i \rangle_{H^1(\Omega)^*} + \alpha \sum_{j=1}^{n} q_j \langle \varphi_j, \varphi_i \rangle_{\mathcal{H}} = \langle \Phi, T_z \varphi_i \rangle_{H^1(\Omega)^*}, \quad i = 1, \dots, n.$$

Thus, the approximate problem (4.2) is equivalent to the matrix equation

$$\mathbf{A}\mathbf{q} + \alpha \mathbf{D}\mathbf{q} = \mathbf{b}, \tag{4.5}$$

where

$$\mathbf{A} = [a_{ij}], \quad \mathbf{D} = [d_{ij}], \quad \mathbf{b} = [b_j]$$

with

$$a_{ij} = \langle T_z \varphi_j, T_z \varphi_i \rangle_{H^1(\Omega)^*}, \quad d_{ij} = \langle \varphi_j, \varphi_i \rangle_{\mathcal{H}}, \quad b_i = \langle \Phi, T_z \varphi_i \rangle_{H^1(\Omega)^*}$$

for $i, j = 1, \dots, n$. This procedure can be reversed as well. Thus, to obtain the solution $q_{\alpha,z}^{(n)}$ of (4.2), we may solve (4.5) for $\mathbf{q} := [q_1, \dots, q_m]^T$ and then take $q_{\alpha,z}^{(n)}$ as in (4.4).

At this point we may recall that for $\xi, \eta \in H^1(\Omega)^*$ we have

$$\langle \xi, \eta \rangle_{H^1(\Omega)^*} = \langle R\eta, R\xi \rangle_{H^1(\Omega)},$$

where $R : H^1(\Omega)^* \to H^1(\Omega)$ is the Riesz representation map, that is,

$$\xi(\varphi) = \langle \varphi, R\xi \rangle_{H^1(\Omega)}, \quad \xi \in H^1(\Omega)^*,\ \varphi \in H^1(\Omega). \tag{4.6}$$

In other words, $\psi_\xi := R\xi$ is the unique solution of the equation $\langle \varphi, \psi_\xi \rangle = \xi(\varphi)$ for all $\varphi \in H^1(\Omega)$, i.e.,

$$\int_\Omega \varphi \psi_\xi + \int_\Omega \nabla\varphi . \nabla\psi_\xi = \xi(\varphi) \quad \text{for all } \varphi \in H^1(\Omega).$$

Thus,

$$a_{ij} = \langle T_z \varphi_j, T_z \varphi_i \rangle_{H^1(\Omega)^*} = \langle \psi_i, \psi_j \rangle_{H^1(\Omega)} := \int_\Omega \psi_i \psi_j + \int_\Omega \nabla\psi_i . \nabla\psi_j \tag{4.7}$$

and

$$b_i = \langle \Phi, T_z \varphi_i \rangle_{H^1(\Omega)^*} = \langle \psi_i, \Psi \rangle_{H^1(\Omega)} := \int_\Omega \psi_i \Psi + \int_\Omega \nabla \psi_i . \nabla \Psi,$$

where $\psi_k := R(T_z \varphi_k) \in H^1(\Omega)$ is the unique solution of

$$\int_\Omega \varphi \psi_k + \int_\Omega \nabla \varphi . \nabla \psi_k = (T_z \psi_k)(\varphi) := \int_\Omega \varphi_k \nabla z . \nabla \varphi, \tag{4.8}$$

and $\Psi := R\Phi \in H^1(\Omega)$ is the unique solution of

$$\int_\Omega \varphi \Psi + \int_\Omega \nabla \varphi . \nabla \Psi = \Phi(\varphi) := \int_\Omega f(x)\varphi(x)\,dx + \int_{\partial\Omega} g(x)\varphi(x)\,dx \tag{4.9}$$

for all $\varphi \in H^1(\Omega)$.

Algorithm 4.1. The procedure for obtaining $q_{\alpha,z}^{(n)}$ can be described as follows:
(i) Solve (4.8) and (4.9) to obtain $\psi_1, \dots, \psi_n$ and Ψ.
(ii) Compute

$$a_{ij} = \int_\Omega \psi_i \psi_j + \int_\Omega \nabla \psi_i . \nabla \psi_j,$$
$$b_i = \int_\Omega \varphi_j \Psi + \int_\Omega \nabla \varphi_j . \nabla \Psi.$$

(iii) Solve (4.5) to obtain $\mathbf{q} := [q_1, \dots, q_m]^T$.
(iv) Construct $q_{\alpha,z}^{(n)} := \sum_{i=1}^n q_i \varphi_i$.

Note that in equations (4.8) and (4.9), φ varies over the whole of $H^1(\Omega)$. In the next subsection, we shall consider the situation in which φ varies over a finite-dimensional subspace of $H^1(\Omega)$.

Next, we must obtain a meaningful estimate for the error $\|q - q_{\alpha,z}^{(n)}\|_{\mathcal{H}}$. In order to do this, we assume that $\mathcal{H} = H^1(\Omega)$ and for every $\varphi \in H^1(\Omega)$ we assume that

$$\|\varphi - P_n \varphi\|_{H^1(\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{4.10}$$

By Theorem 2.1, for every $w \in W^{1,\infty}(\Omega)$ we have that $T_w : H^1(\Omega) \to H^1(\Omega)^*$ is a compact operator. Therefore, $T_w^* : H^1(\Omega)^* \to H^1(\Omega)$ is also a compact operator, and hence (see [14])

$$\|T_w P_n - T_w\| = \|(I - P_n) T_w^*\| \to 0 \quad \text{as } n \to \infty. \tag{4.11}$$

It can be shown easily that the requirement (4.10) on (P_n) will be satisfied if $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$ and the closure of $\bigcup_{n=1}^\infty X_n$ is the whole of $\mathcal{H}$.

Now, we have the theorem giving the error estimate.

Theorem 4.2. *Let $q_{\alpha,u}$ and $q_{\alpha,z}^{(n)}$ in $H^1(\Omega)$ be as in* (3.1) *and* (4.2), *respectively. If $\varepsilon_n > 0$ is such that*

$$\|T_z(I - P_n)\| \le \varepsilon_n,$$

then

$$\|q - q_{\alpha,z}^{(n)}\|_{H^1(\Omega)} \le \|q - q_{\alpha,u}\|_{H^1(\Omega)} + \frac{(\delta + \varepsilon_n)\|q\|_{H^1(\Omega)}}{\sqrt{\alpha}}.$$

Proof. As in Theorem 3.2 with $T_z P_n$ in place of T_z, we have

$$\|q_{\alpha,u} - q_{\alpha,z}^{(n)}\|_{H^1(\Omega)} \le \frac{\|q\|_{H^1(\Omega)} \|T_u - T_z P_n\|}{\sqrt{\alpha}}.$$

Then

$$\|T_u - T_z P_n\| \le \|T_u - T_z\| + \|T_z - T_z P_n\| \le \delta + \varepsilon_n.$$

Thus,

$$\|q_{\alpha,u} - q_{\alpha,z}^{(n)}\|_{H^1(\Omega)} \le \frac{\|q\|_{H^1(\Omega)}(\delta + \varepsilon_n)}{\sqrt{\alpha}}.$$

From this, the required estimate follows. □

Remark 4.3. By (4.11), $\|T_z(I - P_n)\| \to 0$ as $n \to \infty$. Hence, the estimate ε_n of $\|T_z(I - P_n)\|$ can be such that $\varepsilon_n \to 0$ as $n \to \infty$.

4.2 Computational issues

We note that for solving the linear system (4.5) we need to compute the inner products

$$a_{ij} = \langle T_z\varphi_j, T_z\varphi_i\rangle_{H^1(\Omega)^*} = \langle \psi_i, \psi_j\rangle_{H^1(\Omega)},$$
$$b_i = \langle \Phi, T_z\varphi_i\rangle_{H^1(\Omega)^*} = \langle \psi_i, \Psi\rangle_{H^1(\Omega)}$$

given in (4.7), where ψ_k and Ψ are the solutions of (4.8) and (4.9), respectively. Observe that φ in these equations varies over the infinite-dimensional space $H^1(\Omega)$. Therefore, to obtain numerical approximations of ψ_k and Ψ, we may vary φ over a finite-dimensional subspace $\widetilde{X}_m$ of $H^1(\Omega)$, say of dimension m. Thus, we look for $\psi_k^{(m)}$ and $\Psi^{(m)}$ in $\widetilde{X}_m$ such that

$$\int_\Omega \tilde{\varphi}\psi_k^{(m)} + \int_\Omega \nabla\tilde{\varphi}.\nabla\psi_k^{(m)} = (T_z\varphi_k)(\tilde{\varphi}) := \int_\Omega \varphi_k \nabla z.\nabla\tilde{\varphi} \tag{4.12}$$

for $k = 1, \dots, n$ and

$$\int_\Omega \tilde{\varphi}\Psi^{(m)} + \int_\Omega \nabla\tilde{\varphi}.\nabla\Psi^{(m)} = \Phi(\tilde{\varphi}) := \int_\Omega f(x)\tilde{\varphi}(x)\,dx + \int_{\partial\Omega} g(x)\tilde{\varphi}(x)\,dx \tag{4.13}$$

for all $\tilde{\varphi} \in \widetilde{X}_m$. In order to solve the equations in (4.12) and (4.13), we may write

$$\psi_k^{(m)} = \sum_{j=1}^m \beta_{k,j}\tilde{\varphi}_j \quad \text{and} \quad \Psi^{(m)} = \sum_{j=1}^m \gamma_j\tilde{\varphi}_j,$$

where $\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_m\}$ is a basis of $\widetilde{X}_m$, and substitute these expressions in (4.12) and (4.13). Thus, $\beta_{k,j}$ and γ_j for $j = 1, \dots, m$ are obtained by solving the equations

$$\sum_{j=1}^m \Big(\int_\Omega \tilde{\varphi}_i\tilde{\varphi}_j + \int_\Omega \nabla\tilde{\varphi}_i.\nabla\tilde{\varphi}_j\Big)\beta_{k,j} = \int_\Omega \varphi_k\nabla z.\nabla\tilde{\varphi}_i$$

and

$$\sum_{j=1}^m \Big(\int_\Omega \tilde{\varphi}_i\tilde{\varphi}_j + \int_\Omega \nabla\tilde{\varphi}_i.\nabla\tilde{\varphi}_j\Big)\gamma_j = \int_\Omega f(x)\tilde{\varphi}_i(x)\,dx + \int_{\partial\Omega} g(x)\tilde{\varphi}_i(x)\,dx,$$

respectively, for $i = 1, \dots, m$. Thus, an approximation

$$\tilde{q}_{\alpha,z}^{(n,m)} := \sum_{i=1}^n q_i\varphi_i \tag{4.14}$$

of $q_{\alpha,z}^{(n)}$ in (4.4) is obtained by taking $\mathbf{q} := [q_1, \dots, q_n]^T$ which is the solution of the matrix equation

$$\tilde{\mathbf{A}}\mathbf{q} + \alpha\mathbf{D}\mathbf{q} = \tilde{\mathbf{b}}, \tag{4.15}$$

where

$$\tilde{\mathbf{A}} = [\tilde{a}_{ij}], \quad \mathbf{D} = [d_{ij}], \quad \tilde{\mathbf{b}} = [\tilde{b}_j]$$

with

$$\tilde{a}_{ij} = \langle \psi_i^{(m)}, \psi_j^{(m)}\rangle_{H^1(\Omega)}, \qquad d_{ij} = \langle \varphi_j, \varphi_i\rangle_{H^1(\Omega)}, \quad \tilde{b}_i = \langle \psi_i^{(m)}, \Psi^{(m)}\rangle_{H^1(\Omega)}$$

for $i, j = 1, \dots, n$.

Algorithm 4.4. The procedure for obtaining $\tilde{q}_{\alpha,z}^{(n,m)}$ can be described as follows:
(i) Solve (4.12) and (4.13) to obtain $\psi_1^{(m)}, \dots, \psi_n^{(m)}$ and $\Psi^{(m)}$.
(ii) Compute

$$\tilde{a}_{ij} = \int_\Omega \psi_i^{(m)} \psi_j^{(m)} + \int_\Omega \nabla \psi_i^{(m)} . \nabla \psi_j^{(m)},$$
$$\tilde{b}_i = \int_\Omega \varphi_j \Psi^{(m)} + \int_\Omega \nabla \varphi_j . \nabla \Psi^{(m)}.$$

(iii) Solve (4.15) to obtain $\mathbf{q} := [q_1, \dots, q_m]^T$.
(iv) Construct $\tilde{q}_{\alpha,z}^{(n,m)} := \sum_{i=1}^n q_i \varphi_i$.

In order to realize the above procedure in the operator theoretic setting, we assume that the orthogonal projection $\Pi_m : H^1(\Omega) \to H^1(\Omega)$ onto $\widetilde{X}_m$ satisfies

$$\|\varphi - \Pi_m \varphi\|_{H^1(\Omega)} \to 0 \quad \text{as } m \to \infty \tag{4.16}$$

for every $\varphi \in H^1(\Omega)$. Now, let $Q_m : H^1(\Omega)^* \to H^1(\Omega)^*$ be the dual of Π_m, that is,

$$(Q_m \xi)(\varphi) = \xi(\Pi_m \varphi) \quad \text{for all } \xi \in H^1(\Omega)^*,\ \varphi \in H^1(\Omega). \tag{4.17}$$

We shall show that $\tilde{q}_{\alpha,z}^{(n,m)}$ constructed in Algorithm 4.4 satisfies the operator equation

$$(P_n T_z^* Q_m T_z P_n + \alpha I) \tilde{q}_{\alpha,z}^{(n,m)} = P_n T_z^* Q_m \Phi. \tag{4.18}$$

Since the operator $P_n T_z^* Q_m T_z P_n$ on $H^1(\Omega)$ is self adjoint and positive definite, equation (4.18) has a unique solution $\tilde{q}_{\alpha,z}^{(n,m)}$ for each $\alpha > 0$, $z \in W^{1,\infty}(\Omega)$ and $n, m \in \mathbb{N}$.

Remark 4.5. If $\widetilde{X}_m = X_m$, then $\Pi_m = P_m$ so that

$$(Q_m \xi)(\varphi) = \xi(P_m \varphi) \quad \text{for all } \xi \in H^1(\Omega)^*,\ \varphi \in H^1(\Omega).$$

Let us observe some properties of Q_m.

Lemma 4.6. *Let $Q_m : H^1(\Omega)^* \to H^1(\Omega)^*$ be as in* (4.17)*. Then Q_m is an orthogonal projection satisfying $RQ_m = \Pi_m R$, where $R : H^1(\Omega)^* \to H^1(\Omega)$ is the Riesz representation map defined as in* (4.6) *and*

$$\|Q_m \xi - \xi\|_{H^1(\Omega)^*} \to 0 \quad \textit{as } m \to \infty.$$

Further, for every $w \in W^{1,\infty}(\Omega)$ we have

$$\|Q_m T_w - T_w\| \to 0 \quad \textit{as } m \to \infty.$$

Proof. We note that

$$(Q_m(Q_m \xi))(\varphi) = (Q_m \xi)(\Pi_m \varphi) = \xi[\Pi_m(\Pi_m \varphi)] = \xi(\Pi_m \varphi) = (Q_m \xi)(\varphi)$$

for $\xi \in H^1(\Omega)^*$ and $\varphi \in H^1(\Omega)$. Thus, Q_m is a projection operator.

For $\xi \in H^1(\Omega)^*$ and $\varphi \in H^1(\Omega)$ we have

$$(Q_m \xi)(\varphi) = \xi(\Pi_m \varphi) = \langle \Pi_m \varphi, R\xi \rangle_{H^1(\Omega)} = \langle \varphi, \Pi_m R\xi \rangle_{H^1(\Omega)},$$

so that $RQ_m \xi = \Pi_m R\xi$ for all $\xi \in H^1(\Omega)^*$, and hence $RQ_m = \Pi_m R$.

Also, for $\xi, \eta \in H^1(\Omega)^*$ we have

$$\langle \xi, \eta \rangle_{H^1(\Omega)^*} = \langle R\eta, R\xi \rangle_{H^1(\Omega)}.$$

Hence, for $\xi, \eta \in H^1(\Omega)^*$, using the fact that P_n is orthogonal, we have

$$\begin{aligned} \langle Q_m \xi, \eta \rangle_{H^1(\Omega)^*} &= \langle R\eta, RQ_m \xi \rangle_{H^1(\Omega)} = \langle R\eta, \Pi_m R\xi \rangle_{H^1(\Omega)} \\ &= \langle \Pi_m R\eta, R\xi \rangle_{H^1(\Omega)} = \langle RQ_m \eta, R\xi \rangle_{H^1(\Omega)} \\ &= \langle \xi, Q_m \eta \rangle_{H^1(\Omega)^*}. \end{aligned}$$

Thus, Q_m is an orthogonal projection.

Since $RQ_m = \Pi_m R$, for each $\xi \in H^1(\Omega)^*$, we have

$$\|Q_m\xi - \xi\|_{H^1(\Omega)^*} = \|R(Q_m\xi - \xi)\|_{H^1(\Omega)} = \|\Pi_m R\xi - R\xi\|_{H^1(\Omega)} \to 0 \quad \text{as } m \to \infty.$$

Let $w \in W^{1,\infty}(\Omega)$. Then the compactness of the operator T_w implies (cf. [14])

$$\|Q_m T_w - T_w\| = \|(Q_m - I)T_w\| \to 0 \quad \text{as } m \to \infty. \qquad \square$$

Theorem 4.7. *Let $Q_m : H^1(\Omega)^* \to H^1(\Omega)^*$ be the orthogonal projection from Lemma 4.6. Then we have*

$$\langle Q_m T_z\varphi_j, Q_m T_z\varphi_i\rangle_{H^1(\Omega)^*} = \langle \psi_i^{(m)}, \psi_j^{(m)}\rangle_{H^1(\Omega)},$$
$$\langle \Phi, Q_m T_z\varphi_i\rangle_{H^1(\Omega)^*} = \langle \psi_i^{(m)}, \Psi^{(m)}\rangle_{H^1(\Omega)}$$

for $i, j = 1, \dots, n$, where $\psi_k^{(m)}$, for $k = 1, \dots, n$, and $\Psi^{(m)}$ are in X_n satisfying (4.12) *and* (4.13), *respectively.*

Proof. By Lemma 4.6, $RQ_m = \Pi_m R$. Hence,

$$\begin{aligned}\langle Q_m T_z\varphi_j, Q_m T_z\varphi_i\rangle_{H^1(\Omega)^*} &= \langle R[Q_m T_z\varphi_i], R[Q_m T_z\varphi_j]\rangle_{H^1(\Omega)}\\ &= \langle \Pi_m R(T_z\varphi_i), \Pi_m R(T_z\varphi_j)\rangle_{H^1(\Omega)}\\ &= \langle \psi_i^{(m)}, \psi_j^{(m)}\rangle_{H^1(\Omega)},\end{aligned}$$

where $\psi_k^{(m)} := \Pi_m R(T_{\tilde z}\varphi_k) \in \widetilde{X}_m$. Note that for every $\varphi \in \widetilde{X}_m$ we have

$$\langle \psi_k^{(m)}, \varphi\rangle_{H^1(\Omega)} = \langle \Pi_m R(T_z\varphi_k), \varphi\rangle_{H^1(\Omega)} = \langle R(T_z\varphi_k), \varphi\rangle_{H^1(\Omega)}.$$

Thus, $\psi_k^{(m)}$ is the unique element in $\widetilde{X}_m$ satisfying (4.12). Also, note that

$$\begin{aligned}\langle \Phi, Q_m T_z\varphi_i\rangle_{H^1(\Omega)^*} &= \langle RQ_m T_z\varphi_i, R\Phi\rangle_{H^1(\Omega)}\\ &= \langle \Pi_m R(T_z\varphi_i), R\Phi\rangle_{H^1(\Omega)}\\ &= \langle \Pi_m R(T_z\varphi_i), \Pi_m R\Phi\rangle_{H^1(\Omega)}\\ &= \langle \psi_i^{(m)}, \Psi^{(m)}\rangle_{H^1(\Omega)},\end{aligned}$$

where $\Psi^{(m)} := \Pi_m R\Phi \in \widetilde{X}_m$. Then for every $\varphi \in \widetilde{X}_m$ we have

$$\langle \Psi^{(m)}, \varphi\rangle_{H^1(\Omega)} = \langle \Pi_m R\Phi, \varphi\rangle_{H^1(\Omega)} = \langle R\Phi, \varphi\rangle_{H^1(\Omega)}.$$

Thus, $\Psi^{(m)}$ is the unique element in $\widetilde{X}_m$ satisfying (4.13). $\square$

Let $Q_m : H^1(\Omega)^* \to H^1(\Omega)^*$ be the orthogonal projection as in Lemma 4.6. Then for every $\varphi \in H^1(\Omega)$ we have

$$\begin{aligned}\langle P_n T_z^* Q_m T_z P_n\varphi, \varphi\rangle_{H^1(\Omega)} &= \langle T_z^* Q_m T_z P_n\varphi, P_n\varphi\rangle_{H^1(\Omega)}\\ &= \langle Q_m T_z P_n\varphi, T_z P_n\varphi\rangle_{H^1(\Omega)^*}\\ &= \langle Q_m T_z P_n\varphi, Q_m T_z P_n\varphi\rangle_{H^1(\Omega)^*}.\end{aligned}$$

Thus, $P_n T_z^* Q_m T_z P_n : H^1(\Omega) \to H^1(\Omega)$ is a positive self adjoint operator so that equation (4.10) has a unique solution in X_n.

Now, we deduce the result that we promised.

Theorem 4.8. *Let $\mathbf{q} := [q_1, \dots, q_n]^T$ be the solution of the matrix equation* (4.15) *and let $\tilde q_{\alpha,z}^{(n,m)} := \sum_{i=1}^n q_i\varphi_i$. Then $\tilde q_{\alpha,z}^{(n,m)}$ satisfies equation* (4.18).

Proof. Observe first that equation (4.18) with $\tilde q_{\alpha,z}^{(n,m)} \in X_n$ is the same as

$$\langle (P_n T_z^* Q_m T_z P_n + \alpha I_n)\tilde q_{\alpha,z}^{(n,m)}, \varphi_i\rangle_{H^1(\Omega)} = \langle P_n T_z^* Q_m\Phi, \varphi_i\rangle_{H^1(\Omega)}, \quad i = 1, \dots, n.$$

But,

$$\begin{aligned}\langle (P_n T_z^* Q_m T_z P_n + \alpha I)\tilde{q}_{\alpha,z}^{(n,m)}, \varphi_i\rangle_{H^1(\Omega)} &= \langle P_n T_z^* Q_m T_z \tilde{q}_{\alpha,z}^{(n,m)}, \varphi_i\rangle_{H^1(\Omega)} + \alpha\langle \tilde{q}_{\alpha,z}^{(n,m)}, \varphi_i\rangle_{H^1(\Omega)}\\ &= \langle Q_m T_z \tilde{q}_{\alpha,z}^{(n,m)}, Q_m T_z P_n \varphi_i\rangle_{H^1(\Omega)^*} + \alpha\langle \tilde{q}_{\alpha,z}^{(n,m)}, \varphi_i\rangle_{H^1(\Omega)}\end{aligned}$$

and

$$\langle P_n T_z^* Q_m \Phi, \varphi_i\rangle_{H^1(\Omega)} = \langle \Phi, Q_m T_z \varphi_i\rangle_{H^1(\Omega)^*}.$$

Writing $\tilde{q}_{\alpha,z}^{(n,m)} := \sum_{j=1}^n q_j \varphi_j$, we have

$$\begin{aligned}\langle Q_m T_z \tilde{q}_{\alpha,z}^{(n,m)}, Q_m T_z P_n \varphi_i\rangle_{H^1(\Omega)^*} &= \sum_{j=1}^n q_j \langle Q_m T_z \varphi_j, Q_m T_z \varphi_i\rangle_{H^1(\Omega)^*},\\ \langle \tilde{q}_{\alpha,z}^{(n,m)}, \varphi_i\rangle_{H^1(\Omega)} &= \sum_{j=1}^n q_j \langle \varphi_j, \varphi_i\rangle_{H^1(\Omega)}.\end{aligned}$$

Thus, in view of Theorem 4.7,

$$\langle (P_n T_z^* Q_m T_z P_n + \alpha I)\tilde{q}_{\alpha,z}^{(n,m)}, \varphi_i\rangle_{H^1(\Omega)} = \sum_{j=1}^n q_j \langle \psi_i^{(m)}, \psi_j^{(m)}\rangle_{H^1(\Omega)} + \alpha \sum_{j=1}^n q_j \langle \varphi_j, \varphi_i\rangle_{H^1(\Omega)}$$

and

$$\langle P_n T_z^* Q_m \Phi, \varphi_i\rangle_{H^1(\Omega)} = \langle \psi_i^{(m)}, \Psi^{(m)}\rangle_{H^1(\Omega)},$$

where $\psi_k^{(m)}$, for $k = 1, \dots, n$, and $\Psi^{(m)}$ are in $\widetilde{X}_m$ satisfying (4.12) and (4.13), respectively. Therefore, the operator equation (4.18) is equivalent to the matrix equation (4.15). □

4.3 Error estimate for the modified approximation

Theorem 4.9. *Let $Q_m : H^1(\Omega)^* \to H^1(\Omega)^*$ be the orthogonal projection from Lemma 4.6 and let $\tilde{q}_{\alpha,z}^{(n,m)}$ be as in (4.14) or, equivalently, the solution of equation (4.18). If $\varepsilon_n > 0$ and $\tilde{\varepsilon}_m > 0$ are such that*

$$\|T_z(I - P_n)\| \le \varepsilon_n, \quad \|(I - Q_m)T_z\| \le \tilde{\varepsilon}_m,$$

then

$$\|q - \tilde{q}_{\alpha,z}^{(n,m)}\|_{H^1(\Omega)} \le \|q - q_{\alpha,u}\|_{H^1(\Omega)} + \frac{(\delta + \varepsilon_n + \tilde{\varepsilon}_m)}{\sqrt{\alpha}} \|q\|_{H^1(\Omega)}. \tag{4.19}$$

Proof. As in Theorem 3.2 with $Q_m T_z P_n$ in place of T_z, we obtain

$$\|q_{\alpha,u} - \tilde{q}_{\alpha,z}^{(n,m)}\|_{H^1(\Omega)} \le \frac{\|q\|_{H^1(\Omega)} \|T_u - Q_m T_z P_n\|}{\sqrt{\alpha}}.$$

Since

$$\|T_u - Q_m T_z P_n\| \le \|T_u - T_z\| + \|T_z - Q_m T_z P_n\| \le \delta + \|T_z - Q_m T_z P_n\|$$

and

$$\begin{aligned}\|T_z - Q_m T_z P_n\| &= \|T_z - T_z P_n + T_z P_n - Q_m T_z P_n\|\\ &\le \|T_z - T_z P_n\| + \|T_z - Q_m T_z\|\\ &\le \varepsilon_n + \tilde{\varepsilon}_m,\end{aligned}$$

we have

$$\|T_u - Q_m T_z P_n\| \le \delta + \varepsilon_n + \tilde{\varepsilon}_m.$$

Thus,

$$\|q_{\alpha,u} - \tilde{q}_{\alpha,z}^{(n,m)}\|_{H^1(\Omega)} \le \frac{\|q\|_{H^1(\Omega)}(\delta + \varepsilon_n + \tilde{\varepsilon}_m)}{\sqrt{\alpha}}.$$

From this we obtain (4.19). □

Remark 4.10. By the assumptions (4.10) and (4.16) (see also Lemma 4.6) on (P_n) and (Π_m), respectively, we know that $\|T_z - T_z P_n\| \to 0$ and $\|T_z - Q_m T_z\| \to 0$ as $n, m \to \infty$. Therefore, (ε_n) and $(\tilde{\varepsilon}_m)$ in Theorem 4.9 can be assumed to be such that $\varepsilon_n \to 0$ and $\tilde{\varepsilon}_m \to 0$ as $m, n \to \infty$. Under this assumption, for $\delta > 0$ let $n_\delta, m_\delta \in \mathbb{N}$ be such that

$$\varepsilon_n + \tilde{\varepsilon}_m \le \delta \quad \text{for all } n \ge n_\delta,\ m \ge m_\delta.$$

Thus, by Theorem 4.9 we have

$$\|q - \tilde{q}_{\alpha,z}^{(n_\delta,m_\delta)}\|_{H^1(\Omega)} \le \|q - q_{u,\alpha}\|_{H^1(\Omega)} + \frac{2\delta}{\sqrt{\alpha}}\|q\|_{H^1(\Omega)}.$$

As mentioned in Section 3, we may recall from the theory of Tikhonov regularization (cf. [7, 15]) that

$$\|q - q_{\alpha,u}\|_{H^1(\Omega)} \to 0 \quad \text{as } \alpha \to 0.$$

Thus, for an appropriate choice of the *regularization parameter* α, say $\alpha := \alpha_\delta$, which satisfies $\delta/\sqrt{\alpha_\delta} \to 0$ as $\delta \to 0$, we have

$$\|q - \tilde{q}_{\alpha_\delta,z}^{(n_\delta,m_\delta)}\|_{H^1(\Omega)} \to 0 \quad \text{as } \delta \to 0.$$

From the theory of Tikhonov regularization (cf.[7, 15]), it follows that further regularity assumptions on the solution q lead to order optimal error estimates with respect to certain source sets. For example, if q belongs to the range of $\phi(T_u^* T_u)$ for some continuous function $\phi : (0, \infty) \to (0, \infty)$ with $\lim_{\lambda\to 0} \phi(\lambda) = 0$, and if $\alpha_\delta > 0$ is chosen a priori by the requirement $\delta = c_0 \sqrt{\alpha_\delta}\phi(\alpha_\delta)$ for some $c_0 > 0$, then

$$\|q - \tilde{q}_{\alpha,z}^{(n_\delta,m_\delta)}\|_{H^1(\Omega)} = O(\phi(\alpha_\delta)).$$

Some of the standard forms of ϕ that occur in the literature on ill-posed operator equations are

$$\phi(\lambda) := \lambda^\nu, \quad 0 < \nu \le 1$$

and

$$\phi(\lambda) := [\log(1/\lambda)]^{-p}, \quad p > 0.$$

(See [17, 18, 23].) There are many a posteriori choices of the regularization parameter which exist in the literature (see [7, 8, 15, 18]), including the recently introduced *balancing principle* (cf. [9, 16]) which can be applied to obtain the above order optimal rate.

Remark 4.11. Let us note that our problem involves perturbation of the operator T_u and that we have found an estimate for the error in the solution which occurs due to this perturbation. This is in contrast with regularized projection methods applied to ill-posed operator equations in the general setting such as those considered in Plato and Vainikko [20], George and Nair [8] and Mathe and Pereverzev [13], where noisy data corresponds to perturbations in the right-hand side of the operator equation. Thus, though the formulation of the regularized equation in the finite-dimensional setting is similar to that of [8, 13, 20], the respective results cannot be compared.

5 Smoothing

Our hitherto analysis involves the perturbed data z to be in $W^{1,\infty}(\Omega)$ and the error in the data is measured with the norm in $W^{1,\infty}(\Omega)$. However, in practical situations, it is too much to demand that the noisy data z is in $W^{1,\infty}(\Omega)$; one may only have $z \in L^2(\Omega)$. To take care of such cases, we may follow a “smoothing procedure” described as in [11] by using a Clement operator (see [6]), with some additional assumptions on Ω and the data u.

Let u be defined as in Section 3, that is, $u \in W^{1,\infty}(\Omega)$ is such that the operator equation (2.2) has a unique solution $q \in \mathcal{H}$. Along with this, we shall also assume that $u \in H^4(\Omega)$. Let $z \in L^2(\Omega)$ be the perturbed data such that

$$\|z - u\|_{L^2(\Omega)} \le \delta. \tag{5.1}$$

We intend to find an element $\tilde{z} \in W^{1,\infty}(\Omega)$ which is a "smoothed version" of z so that we can use the analysis that we carried out so far. For this, we assume that Ω is a polygonal domain in $\mathbb{R}^2$ and we have a quasi-uniform triangulation of Ω (cf. [22]), that is,

$$\min_{G\in\tau} \frac{d(G)}{h} = \gamma_G \geq \gamma_0 > 0, \tag{5.2}$$

where h is the mesh size, τ is the set of all triangles G in the triangulation of Ω, $d(G)$ is the diameter of the largest disc contained in G, and γ_0 is a constant. Let Π be a Clement operator (see [6]), which takes the elements of $L^2(\Omega)$ to the space of all polynomials of degree less than or equal to 3 on this triangulation. Then we claim that Πz is our desired $\tilde{z}$. First we observe the following result (see [6, 11] for its proof).

Lemma 5.1. *For any $v \in L^2(\Omega)$ we have $\Pi v \in W^{1,\infty}(\Omega)$ and*

$$\|v - \Pi v\|_{L^2(\Omega)} \leq C_1 \|v\|_{L^2(\Omega)} \tag{5.3}$$

for some constant $C_1 > 0$. Further, if $v \in H^4(\Omega)$, then

$$\|v - \Pi v\|_{H^3(\Omega)} \leq C_2 h \|v\|_{H^4(\Omega)} \tag{5.4}$$

for some constant $C_2 > 0$.

The following result is crucial.

Theorem 5.2. *For every $v \in H^4(\Omega)$ we have*

$$\|v - \Pi v\|_{W^{1,\infty}(\Omega)} \leq C_0 h \|v\|_{H^4(\Omega)},$$

where $C_0 > 0$ is a constant.

Proof. Let $v \in H^4(\Omega)$. Since $H^4(\Omega) \subseteq H^3(\Omega)$ by a Sobolev imbedding theorem (cf. [21, Corollary 7.19]),

$$\|v - \Pi v\|_{W^{1,\infty}(\Omega)} \leq C_3 \|v - \Pi v\|_{H^3(\Omega)},$$

where C_3 is a positive constant. Thus, using (5.4), we obtain the required estimate with $C_0 = C_3 C_2$, with C_2 as in (5.4). □

The following lemma, which will be used to prove our next theorem, is a consequence of one of the *inverse inequalities* in [2] (see also [3]), namely,

$$\|w\|_{W^{m,q}(G)} \leq (\operatorname{diam}(G))^{n-m+2\min\{0,(1/q-1/p)\}} \|w\|_{W^{n,p}(G)} \tag{5.5}$$

for any $w \in \mathcal{P}_k(G)$ with $k \in \mathbb{N}$, $m, n \in \mathbb{N} \cup \{0\}$, $m, n \leq k$, and $1 \leq p, q \leq \infty$, where G is a triangle in τ, $\operatorname{diam}(G)$ is the diameter of the triangle G and $\mathcal{P}_k(G)$ is the space of all polynomials up to degree k restricted to G, with the convention that $1/\infty = 0$.

Lemma 5.3. *For any $G \in \tau$ and $v \in L^2(G)$ we have*

$$\|\Pi v\|_{W^{1,\infty}(G)} \leq \frac{1}{(\operatorname{diam}(G))^2} \|\Pi v\|_{L^2(G)},$$

where $\operatorname{diam}(G)$ *is the diameter of the triangle G.*

Proof. The result follows from (5.5) since $\Pi v \in \mathcal{P}_3(G)$ by taking $m = 1$, $q = \infty$, $n = 0$, and $p = 2$. □

Theorem 5.4. *Let $v \in L^2(\Omega)$. Then*

$$\|\Pi v\|_{W^{1,\infty}(\Omega)} \leq \frac{C_4}{h^2} \|v\|_{L^2(\Omega)},$$

where $C_4 := (C_1 + 1)/\gamma_0^2$ with γ_0 and C_1 are as in (5.2) and (5.3), respectively.

Proof. Let G be a triangle in the quasi-uniform triangulation τ of Ω. Then, by Lemma 5.3 and the fact that $\operatorname{diam}(G) \geq d(G) \geq h\gamma_0$, we have

$$\|\Pi v\|_{W^{1,\infty}(G)} \leq \frac{1}{(\operatorname{diam}(G))^2}\|\Pi v\|_{L^2(G)} \leq \frac{1}{h^2\gamma_0^2}\|\Pi v\|_{L^2(G)}.$$

Also, by (5.3),

$$\|\Pi v\|_{L^2(\Omega)} \leq \|(I-\Pi)v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq (C_1+1)\|v\|_{L^2(\Omega)}.$$

From the two inequalities above we obtain

$$\|\Pi v\|_{W^{1,\infty}(\Omega)} \leq \max_{G\in\tau}\|\Pi v\|_{W^{1,\infty}(G)} \leq \frac{1}{h^2\gamma_0^2}\max_{G\in\tau}\|\Pi v\|_{L^2(G)} \leq \frac{1}{h^2\gamma_0^2}\|\Pi v\|_{L^2(\Omega)} \leq \frac{(C_1+1)}{h^2\gamma_0^2}\|v\|_{L^2(\Omega)}.$$

Thus, we obtain the required estimate with $C_4 = (C_1+1)/\gamma_0^2$. □

Now, we derive an estimate for $\|u - \Pi z\|_{W^{1,\infty}(\Omega)}$ under the assumption (5.1).

Theorem 5.5. *Let $u \in H^4(\Omega)$ and $z \in L^2(\Omega)$ be the exact and noisy data, respectively, such that $\|u - z\|_{L^2(\Omega)} \leq \delta$. Then*

$$\|u - \Pi z\|_{W^{1,\infty}(\Omega)} \leq C\Big(h\|u\|_{H^4(\Omega)} + \frac{\delta}{h^2}\Big),$$

where $C := \max\{C_0, C_4\}$ with C_0 and C_4 as in Theorem 5.2 and Theorem 5.4, respectively.

Proof. Using the estimates in Theorem 5.2 and Theorem 5.4, we have

$$\begin{aligned}\|u - \Pi z\|_{W^{1,\infty}(\Omega)} &\leq \|u - \Pi u\|_{W^{1,\infty}(\Omega)} + \|\Pi(u-z)\|_{W^{1,\infty}(\Omega)}\\ &\leq C_0 h\|u\|_{H^4(\Omega)} + \frac{C_4}{h^2}\|u - z\|_{L^2(\Omega)}\\ &\leq C\Big(h\|u\|_{H^4(\Omega)} + \frac{\delta}{h^2}\Big),\end{aligned}$$

where $C := \max\{C_0, C_4\}$. □

Now, taking $\tilde z := \Pi z$ instead of z, we carry on with the analysis as in Sections 3 and 4 with

$$\tilde\delta_h := C\Big(h\|u\|_{H^4(\Omega)} + \frac{\delta}{h^2}\Big) \tag{5.6}$$

in place of δ there. Thus, from Theorem 4.9, we obtain the following theorem.

Theorem 5.6. *Let Q_m and P_n be the orthogonal projections as defined in Section 4. Let $\tilde q_{\alpha,\tilde z}^{(n,m)}$ be as in (4.14). If $\varepsilon_n > 0$ and $\tilde\varepsilon_m > 0$ are such that*

$$\|T_{\tilde z}(I - P_n)\| \leq \varepsilon_n, \quad \|(I - Q_m)T_{\tilde z}\| \leq \tilde\varepsilon_m,$$

then

$$\|q - \tilde q_{\alpha,\tilde z}^{(n,m)}\|_{H^1(\Omega)} \leq \|q - q_{\alpha,u}\|_{H^1(\Omega)} + \|q\|\frac{(\tilde\delta_h + \varepsilon_n + \tilde\varepsilon_m)}{\sqrt{\alpha}},$$

where $\tilde\delta_h$ is as in (5.6). In particular, the following hold:

(i) *If $h \geq \delta^{1/3}$, then*

$$\|q - \tilde q_{\alpha,\tilde z}^{(n,m)}\|_{H^1(\Omega)} \leq \|q - q_{\alpha,u}\|_{H^1(\Omega)} + C'_{q,u}\frac{(h + \varepsilon_n + \tilde\varepsilon_m)}{\sqrt{\alpha}}.$$

(ii) *If $h \sim \delta^{1/3}$, then*

$$\|q - \tilde q_{\alpha,\tilde z}^{(n,m)}\|_{H^1(\Omega)} \leq \|q - q_{\alpha,u}\|_{H^1(\Omega)} + C''_{q,u}\frac{(\delta^{1/3} + \varepsilon_n + \tilde\varepsilon_m)}{\sqrt{\alpha}}.$$

Here, $C'_{q,u}$ and $C''_{q,u}$ are positive constants independent of h, n, m, α, δ.

Remark 5.7. It is apparent that the error estimate in Theorem 5.6 for the noisy data $z \in L^2(\Omega)$ is not as sharp as that in Theorem 4.9 under the stronger assumption $z \in W^{1,\infty}(\Omega)$, although the smoothing process of z requires additional requirements on u and Ω. This observation calls for further investigation of the problem while dealing with the noisy data.

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