A Decode and Forward Protocol for Two-Stage Gaussian Relay Networks

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***Abstract*—We propose a multihopping decode and forward relaying protocol for two-stage Gaussian relay networks with half-duplex nodes. We analytically show that the achievable rates in suitably defined strong and weak interference regimes are close to the cut-set bound.**

***Index Terms*—Two-stage relay network, decode and forward protocol, half-duplex relays.**

I. Introduction

The diamond channel (DC) [1], where the source and destination are connected by two relays, has been an important example in the study of relay networks. DC with practical constraints like half-duplex, non-cooperating, interfering relays and finite SNR has been studied in [2]–[4]. In these studies, decode and forward protocols have been shown to be close to capacity in some channel regimes. In this paper, we are concerned with extending ideas from DC to multistage relay networks, where source and destination are connected through multiple stages of relays.

Two-stage relay network: We consider the two-stage Gaussian relay network shown in Fig. 1(a) with Node 1 as source (S) and Node 6 as destination (D). Nodes $2, 3, 4, 5$ are *half-duplex*, *interfering* relays that enable communication from S to D. A link (i, j) indicates that Nodes i and j are connected by an additive white Gaussian noise (AWGN) channel with constant gain denoted as h_{ij}. Also $h_{ij} = h_{ji}$. Every node has a power constraint P at the transmitter and a noise variance σ^2 at the receiver.

The two-stage relay network studied by us is a natural extension of the diamond channel. It is also partly motivated by the multistage relaying example for 4G networks in [5], [6]. Further, a three-hop network is chosen where transmission by the second stage of relays will interfere with reception by the first stage of relays. This is a crucial factor that affects the capacity of the relay network, and cannot be observed in two-hop networks that have been studied extensively in the literature. Though we discuss a specific two-stage network in this article, the proposed protocol can be generalized to an arbitrary topology as long as there are two non-overlapping paths from the source to destination using similar ideas.

We propose a multihopping decode and forward (MDF) protocol that specifies the scheduling and coding strategies to maximize the information flow in two-stage relay networks.

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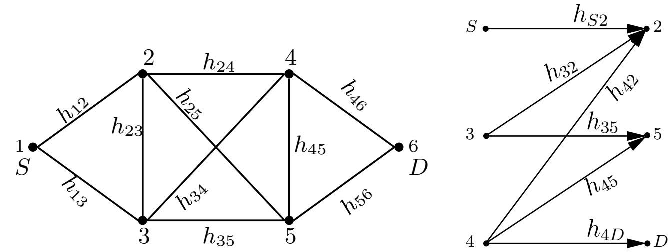

(a) Network (b) State S_1

Fig. 1. Two stage relay network.

The schedule decides the time-sharing between the states of the half-duplex network. The coding strategy decides the rate of information flow in each states. Comparison with the cut-set bound shows that the performance of the proposed MDF scheme is good for several channel conditions. Under suitably defined strong and weak interference conditions, the achieved rate by the proposed MDF protocol is shown to be close to capacity. The specific contributions are as follows: (i) we have used information-theoretic rate regions for interference networks in the optimization of multistage relay communications, (ii) we propose a heuristic two path two state (2P2S) schedule, (iii) we design a coding strategy within a state for appropriate information flow in the 2P2S schedule using dirty paper coding (DPC), superposition coding (SC) and successive interference cancellation (SIC), (iv) we prove that the MDF scheme has a gap to capacity of at most 0.5 bits in the low rate regime, when the links satisfy certain strong and weak interference conditions.

Related work and comparisons: Gaussian relay networks with arbitrary topology have been studied in [7], [8]. The constant gap to capacity in [7], [8] is proportional to the number of nodes in a network and is not optimized for specific topologies like the diamond channel or the two-stage relay network. The authors of [7] have elaborated on the low rate regime in their paper where they provide a closeness to cut-set bound result based on orthogonalization, i.e., interference avoidance. We operate in the low rate regime and our numerical results show significant improvement over interference avoidance.

II. Multihopping Decode and Forward Protocol

We are interested in maximizing the rate $R_{S\to D}$ relayed from the source S to the sink D. This relaying consists of two aspects: (1) scheduling transmissions and receptions by nodes, and (2) coding and decoding methods employed by nodes during transmissions and receptions. Optimal scheduling is known to be a hard problem in most scenarios. So, we propose a heuristic schedule and coding methods suited to the schedule.

A. Two-path two-state (2P2S) schedule

We propose a simple heuristic schedule for information flow in the network of Fig. 1(a). The heuristics used are as follows: (1) S always transmits and D always receives, (2) information is forwarded by relays over at least two node-disjoint shortest paths. The shortest (three-hop) paths connecting S and D are: (i) Path P_1: $S \to 2 \to 4 \to D$, (ii) Path P_2: $S \to 3 \to 5 \to D$, (iii) Path P_3: $S \to 2 \to 5 \to D$, and (iv) Path P_4: $S \to 3 \to 4 \to D$. Among these four paths, there are only two pairs of node-disjoint paths: (i) P_1, P_2 and (ii) P_3, P_4. We describe the 2P2S schedule for the choice P_1, P_2. A similar schedule for P_3, P_4 is also possible. First, we construct two states S_1 and S_2 that enable information forwarding along paths P_1, P_2. In both states, S will transmit and D will receive. In state S_1, we activate the first link $(S, 2)$, and the third link $(4, D)$ of path P_1. This fixes Node 2 as a receiver and Node 4 as a transmitter. We add link $(3, 5)$ to state S_1 for information forwarding along path P_2. Analogously, in state S_2, we activate link $(2, 4)$ of path P_1 and links $(S, 3)$, $(4, D)$ of path P_2. The states are: (i) State S_1: Nodes $S, 3, 4$ are transmitters, and Nodes $2, 5, D$ are receivers and (ii) State S_2: Nodes $S, 2, 5$ are transmitters, and Nodes $3, 4, D$ are receivers. These states are similar in structure, and State S_1 is shown in Fig. 1(b). In the proposed MDF protocol, we use the 2P2S schedule.

B. Coding scheme

For a link (i, j) in a state, let R_{ij} denote the rate of information flow. We now describe a coding scheme for S_1 that fixes the rate region i.e., the possible values for R_{ij}. For computing the rate region, we assume Gaussian codebooks at transmitters and successive interference cancellation (SIC) decoders at receivers.

Encoding at S (State S_1): Source intends to send a message to Node 2 in the presence of interfering signals from Nodes 3 and 4. Since source is the originator of all messages flowing through the network, the messages from Nodes 3 and 4 are assumed to be known to S. We propose that the source does dirty paper coding (DPC) [9] to cancel the known interference at receiver Node 2, assuming further that h_{23} and h_{24} are also known at S. Under this coding, reliable transmission along link $(S, 2)$ requires that the rate R_{S2} must satisfy:

$$R_{S2} \leq C\left(h_{S2}^2 P/\sigma^2\right), \tag{1}$$

where $C(x) = \frac{1}{2}\log_2(1+x)$.

Encoding at Node 3 (State S_1): Transmitter 3 can reach receivers 2 and 5. Since we use DPC at the source, we set $R_{32} = 0$. For reliable transmission along link $(3, 5)$, rate R_{35} must satisfy:

$$R_{35} \leq C\left(h_{35}^2 P/\sigma^2\right). \tag{2}$$

Encoding at Node 4 (State S_1): Transmitter 4 can reach receivers 2, 5 and D. Since we use DPC at source, we set $R_{42} = 0$. We propose that Node 4 uses superposition coding (SC) to send codewords $\mathbf{x}_{45}$ and $\mathbf{x}_{4D}$ to receivers 5 and D with power sharing variables α_{45}, α_{4D} such that $\alpha_{45}+\alpha_{4D} = 1$. For real a, b, indicator function $\mathbb{I}_{a>b} = 1$ if $a > b$ else $\mathbb{I}_{a>b} = 0$. The achievable rates R_{45} and R_{4D} satisfy [10]:

$$R_{45} \leq C\left(\frac{h_{45}^2\alpha_{45}P}{\sigma^2 + \mathbb{I}_{|h_{45}|<|h_{4D}|}\alpha_{4D}h_{45}^2 P}\right), \tag{3}$$

$$R_{4D} \leq C\left(\frac{h_{4D}^2\alpha_{4D}P}{\sigma^2 + \mathbb{I}_{|h_{45}|>|h_{4D}|}\alpha_{45}h_{4D}^2 P}\right). \tag{4}$$

The indicator function is used to compactly express the bound on the rates to the strong and weak receivers under SC. In summary, State S_1 is a 3×3 interference network with 4 messages, which is different from the standard 3×3 interference channel with 3 messages [11].

Decoding at Nodes 2 and D (State S_1): The DPC coded message from S is decoded at 2, while the superposition coded message from Node 4 is decoded at D.

Decoding at Node 5 (State S_1): The received signal at receiver 5 is

$$\mathbf{y}_5 = h_{35}\mathbf{x}_3 + h_{45}(\mathbf{x}_{45} + \mathbf{x}_{4D}) + \mathbf{w}_5,$$

where $\mathbf{x}_3$ is the signal from Node 3 and $\mathbf{w}_5$ is the noise. We propose the following decoding depending on channel gains h_{45} and h_{4D}: when $|h_{45}| \geq |h_{4D}|$, Node 5 jointly decodes codewords $\hat{\mathbf{x}}_3, \hat{\mathbf{x}}_{45}, \hat{\mathbf{x}}_{4D}$. When $|h_{45}| < |h_{4D}|$, it decodes only codewords $\hat{\mathbf{x}}_3, \hat{\mathbf{x}}_{45}$ treating $\mathbf{x}_{4D}$ as noise. In either case, decoding is same as SIC decoding in Gaussian multiple access [10]. So, we have

$$\sum_{(p,q)\in A} R_{pq} \leq C\left(\frac{\sum_{(p,q)\in A}\alpha_{pq}h_{p5}^2 P}{\sigma^2 + \mathbb{I}_{|h_{45}|<|h_{4D}|}\alpha_{4D}h_{45}^2 P}\right), \tag{5}$$

$\forall\ A \subseteq \mathcal{A}$. Here $\alpha_{35} = 1$, and

$$\mathcal{A} = \begin{cases} \{(3,5),(4,5)\} & \text{if} \quad |h_{45}| < |h_{4D}|, \\ \{(3,5),(4,5),(4,D)\} & \text{otherwise.} \end{cases}$$

Rate region: The achievable rate region in State S_1 under the coding schemes described is

$$\mathcal{R}_1 = \{(R_{S2}, R_{35}, R_{45}, R_{4D}) : \text{ satisfying } (1)-(5)\}. \tag{6}$$

We call this scheme as DPC-SC coding. The coding scheme for state S_2 is similar to that of state S_1 with the links $(S, 2), (3, 5), (4, 5), (4, D)$ replaced by $(S, 3), (2, 4), (5, 4), (5, D)$, respectively, with corresponding channel gains and rates. The rate region $\mathcal{R}_2$ in State S_2 is:

$$\mathcal{R}_2 = \{(R_{S3}, R_{24}, R_{54}, R_{5D}) : \text{ satisfying } (1)-(5) \text{ with respective variable changes}\}. \tag{7}$$

C. Information flow and achievable S-D rate

Information flow from S to D happens by a time-sharing of states S_1 and S_2 which are active for λ_1 and λ_2 fraction of the time with $\lambda_1 + \lambda_2 = 1$. Fig. 2 illustrates the entire information flow under 2P2S schedule and DPC-SC coding. Let z_1, z_2 be the flow (in bits per unit time) along links $(S, 2)$ and $(S, 3)$, respectively. To conserve flow in 2P2S schedule with DPC at S, the flows out of Nodes 2 and 3 are also equal to z_1 and z_2, respectively. SC at Node 4 splits the flow from Node 2 into az_1 units for receiver 5 and $(1-a)z_1$ units for receiver D. Node 5 also does SC to split the flow from Node 3 into bz_2 for receiver 4 and $(1-b)z_2$ units for receiver D. Therefore, Node

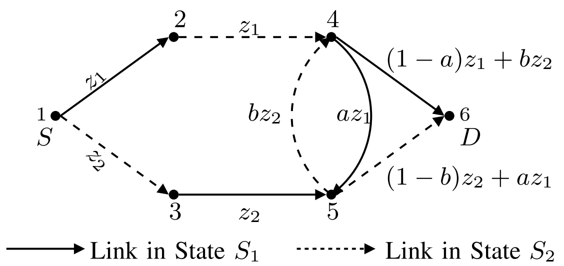

Fig. 2. Information flow graph along with average flow rates.

4 receives a total flow of z_1+bz_2 from links $(2,4)$ and $(5,4)$. It forwards a flow of az_1 and $(1-a)z_1+bz_2$ along links $(4,5)$ and $(4,D)$, respectively, and conserves flow. Similarly, Node 5 also conserves the flow. This leads to an achievable rate of z_1+z_2. The achievable rate R_{SD} from S to D is maximized by solving the constrained flow problem described below:

$$\max_{0\leq\lambda_1,\lambda_2,a,b\leq 1} R_{SD}=z_1+z_2, \tag{8}$$

subject to:

$$\begin{gathered} z_1\leq\lambda_1 R_{S2},\quad z_1\leq\lambda_2 R_{24},\quad z_2\leq\lambda_2 R_{S3},\\ z_2\leq\lambda_1 R_{35},(1-a)z_1+bz_2\leq\lambda_1 R_{4D},\\ 0\leq\lambda_1+\lambda_2\leq 1,(1-b)z_2+az_2\leq\lambda_2 R_{5D},\\ az_1\leq\lambda_1 R_{45},\quad bz_2\leq\lambda_2 R_{54},\\ (R_{S2},R_{35},R_{45},R_{4D})\in\mathcal{R}_1,\\ (R_{S3},R_{24},R_{54},R_{5D})\in\mathcal{R}_2. \end{gathered}$$

In the above optimization, the transmit powers have been set to be equal at all nodes. However, the constraints can be readily altered to allow for unequal transmit powers, if necessary.

Though the information flow graph of Fig. 2 is shown for the specific two-stage network of Fig. 1(a), an extension to any other network with two non-overlapping paths from the source to the sink is readily possible. The 2P2S schedule and the optimization framework can be extended to such relay networks as well.

III. Approaching the Cut-set Bound

A. Upper Bounds on Relaying Rate

In a relay network with source S and destination D, a subset of the nodes Ω such that $S\in\Omega$ and $D\in\Omega^c$ defines a cut with the edges $\{(u,v): u\in\Omega, v\in\Omega^c\}$ being the cut edges. The cut edges define a Multiple-Input Multiple-Output (MIMO) channel, whose sum capacity denoted $C_{\text{MIMO}}(\Omega;\Omega^c)$ is a full-duplex cut-set upper bound on the rate R_{SD} from S to D [10].

1) Half-duplex cut-set bound [12]: Suppose a half-duplex relay network operates in M states, $S_k=(I_k,J_k), 1\leq k\leq M$, where I_k and J_k denote the nodes in transmit and receive mode in state k, respectively. Assuming state S_k is active for a fraction of time λ_k, the rate R_{SD} is bounded as follows [12]:

$$R_{SD}\leq\sup_{\lambda_k,\sum\lambda_k=1}\ \min_{\Omega}\sum_{k=1}^{M}\lambda_k C_{\text{MIMO}}(\Omega\cap I_k;\Omega^c\cap J_k). \tag{9}$$

This upper bound on the half-duplex cut-set bound is computed by solving a linear program [4]. In computations, we use the following upper bound for $C_{\text{MIMO}}(I;J)$ as in [7], [8], [13]:

$$C_{\text{MIMO}}(I;J)\leq\frac{1}{2}\log_2(\det(\mathbf{I}_n+mP\mathbb{H}\mathbb{H}^H)), \tag{10}$$

where $m=|I|, n=|J|$, $\mathbf{I}_n$ is an $n\times n$ identity matrix, matrix $\mathbb{H}=[h_{ij}], i\in J, j\in I$ and receiver noise variance is normalized to 1.

2) A closed-form half-duplex cut-set bound: For the network of Fig. 1(a), we consider the channel condition: $h_{S2}=h_{S3}=h_{4D}=h_{5D}=\alpha$, $h_{24}=h_{35}=\beta$, $h_{23}=h_{25}=h_{34}=h_{45}=\gamma$. We determine a closed form upper bound of (9) by considering the three cuts: $\Omega_1=\{S\}$, $\Omega_2=\{S,2,3\}$ and $\Omega_3=\{S,2,3,4,5\}$ representing the three stages in the network of Fig. 1(a). Note that reducing the number of cuts in the minimization in (9) still provides an upper bound. For the maximization of (9), it turns out that the six states shown in Table I are sufficient. In Table I, $CUT_i=C_{\text{MIMO}}(\Omega_i\cap I_k;\Omega_i^c\cap J_k)$ and $C_0=C\left(4P(\gamma^2+\beta^2)+4P^2((\gamma^2-\beta^2)^2)\right)$, which is an upper bound on $C_{\text{MIMO}}(\Omega_2;\Omega_2^c)$ obtained by using (10). The six states S_2 to S_7 in Table I are sufficient because, for any other state, the cut capacities $[CUT_1, CUT_2, CUT_3]$ are smaller or equal (coordinate-wise) to those for one of the states S_2 to S_7. For example, for state S_1 the cut capacities are the same as for state S_2. In this scenario, the half-duplex cut-set bound is computed by the linear program (LP):

$$\max\mathbf{c}^T\mathbf{x}=R,\quad s.t.\quad A\mathbf{x}\leq\mathbf{b},\mathbf{x}\geq 0, \tag{11}$$

where A is the coefficient matrix defined in (12), $\mathbf{x}=[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,R]$, $\mathbf{c}=[0,0,0,0,0,0,1]$, and $\mathbf{b}=[0,0,0,1]$. To obtain a closed-form upper bound on the optimal cut-set bound, we consider the dual

$$\min\quad\mathbf{b}^T\mathbf{y}=\tilde{R},\quad s.t.\quad A^T\mathbf{y}\geq\mathbf{c},\mathbf{y}\geq 0, \tag{13}$$

where $\mathbf{y}=[\tau_1,\tau_2,\tau_3,\tilde{R}]$. Note that any feasible point in the dual (13) gives an upper bound to the optimal cut-set bound. To find a feasible point in (13), we let $\tau_3=0$ and $\tau_1+\tau_2=1$. With these choices for $[\tau_1,\tau_2,\tau_3]$ and using $C_0\geq C((\beta^2+\gamma^2)P)$, $A^T\mathbf{y}\geq\mathbf{c}$ simplifies to:

$$\begin{aligned}\tilde{R}\ \geq\ &\max\{\tau_2C_0,\tau_1C(2\alpha^2P),\\ &\tau_1C(\alpha^2P)+\tau_2C((\beta^2+\gamma^2)P)\}.\end{aligned} \tag{14}$$

The lowest value of $\tilde{R}$ satisfying (14) can now be computed to be the expression in (15). The $\tilde{R}$ in (15) is a closed-form upper bound to the half-duplex cut-set bound for the network of Fig. 1(a) under the chosen channel conditions.

B. Relaying rates of proposed MDF protocol

The optimal rates in (6) and (7) can be expressed in closed form under suitable assumptions on flow in certain channel regimes. For the analysis, we assume that in the MDF protocol information flows only through the edges in Paths P_1 and P_2 and compute the rate achieved by it. All other edges have zero flow and are processed as interference at the receivers. This sets $a=b=0$ in Fig. 2.

TABLE I
STATES AND CUT CAPACITIES.

State	I_k	J_k	CUT_1	CUT_2	CUT_3
S_2	$\{S,2,5\}$	$\{3,4,D\}$	$C(\alpha^2 P)$	$C(\beta^2 P)$	$C(\alpha^2 P)$
S_3	$\{S,2,3,5\}$	$\{4,D\}$	0	$C((\beta+\gamma)^2 P)$	$C(\alpha^2 P)$
S_4	$\{S,2,3\}$	$\{4,5D\}$	0	C_0	0
S_5	$\{S,2,4\}$	$\{3,5,D\}$	$C(\alpha^2 P)$	$C(\gamma^2 P)$	$C(\alpha^2 P)$
S_6	$\{S,2\}$	$\{3,4,5,D\}$	$C(\alpha^2 P)$	$C((\beta^2+\gamma^2)P)$	0
S_7	$\{S,4,5\}$	$\{2,3,D\}$	$C(2\alpha^2 P)$	0	$C(4\alpha^2 P)$

$$A = \begin{bmatrix} -C(\alpha^2P) & 0 & 0 & -C(\alpha^2P) & -C(\alpha^2P) & -C(2\alpha^2P) & 1 \\ -C(\beta^2P) & -C((\beta+\gamma)^2P) & -C_0 & -C(\gamma^2P) & -C((\beta^2+\gamma^2)P) & 0 & 1 \\ -C(\alpha^2P) & -C(\alpha^2P) & 0 & -C(\alpha^2P) & 0 & -C(4\alpha^2P) & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{12}$$

$$\tilde{R} = \begin{cases} \frac{C(\alpha^2P)C_0}{C(\alpha^2P)+C_0-C((\beta^2+\gamma^2)P)} & \text{if } \; 1-\frac{C(\alpha^2P)}{C(2\alpha^2P)} < \frac{C((\beta^2+\gamma^2)P)}{C_0} \; \text{ and } \; \alpha^2 > \beta^2+\gamma^2, \\ \frac{C(2\alpha^2P)C((\beta^2+\gamma^2)P)}{C(2\alpha^2P)-C(\alpha^2P)+C((\beta^2+\gamma^2)P)} & \text{if } \; 1-\frac{C(\alpha^2P)}{C(2\alpha^2P)} < \frac{C((\beta^2+\gamma^2)P)}{C_0} \; \text{ and } \; \alpha^2 \le \beta^2+\gamma^2, \\ \frac{C_0C(2\alpha^2P)}{C_0+C(2\alpha^2P)} & \text{if } \; 1-\frac{C(\alpha^2P)}{C(2\alpha^2P)} \ge \frac{C((\beta^2+\gamma^2)P)}{C_0}. \end{cases} \tag{15}$$

1) Strong interference condition: [1] We suppose that all nodes in states S_1 and S_2 transmit at a common rate $R_1 \le C(h_{S2}^2 P)$ and $R_2 \le C(h_{S3}^2 P)$, respectively. Further, information received by a node at rate R_1 when state S_1 is operational is forwarded in state S_2 by the same node at rate R_2. For flow conservation, we require that $R_1\lambda_1 = R_2\lambda_2$. Using $\lambda_1 + \lambda_2 = 1$, we have $\lambda_1 = R_2/(R_1+R_2)$ and $\lambda_2 = R_1/(R_1+R_2)$ and a total rate of $2R_1R_2/(R_1+R_2)$. The question to be addressed is the condition for successful decoding by receivers in each state. Receiver 5 in state S_1 sees a two-user Gaussian MAC channel from transmitters 3 and 4 with respective channel gains h_{35} and h_{45} under a transmit power constraint P. The rate pair (R_1, R_1) is feasible at receiver 5 in State S_1, if $R_1 \le C(h_{35}^2 P)$, $R_1 \le C(h_{45}^2 P)$ and the sum rate of this two-user MAC channel satisfies: $2R_1 \le 2C(h_{S2}^2 P) \le C((h_{35}^2 + h_{45}^2)P)$. These conditions simplify to $\min(|h_{35}|, |h_{45}|) \ge |h_{S2}|$, and

$$|h_{45}| \ge \sqrt{\frac{(1+h_{S2}^2P)^2 - 1 - h_{35}^2P}{P}} \triangleq h_1$$

Similarly to achieve (R_2, R_2) at receiver 4 in State S_2, the channel gains should satisfy $\min(|h_{24}|, |h_{54}|) \ge |h_{S3}|$, and

$$|h_{54}| \ge \sqrt{\frac{(1+h_{S3}^2P)^2 - 1 - h_{24}^2P}{P}} \triangleq h_2.$$

Sink Node D is interference free in both states. Hence $|h_{4D}| \ge |h_{S2}|$ and $|h_{5D}| \ge |h_{S3}|$ are sufficient to forward information to D at rates R_1 and R_2 in states S_1 and S_2, respectively.

Remark 1: When the channel gains satisfy the following *strong interference conditions*: $\min(|h_{4D}|, |h_{35}|, |h_{45}|) \ge |h_{S2}|$, $\min(|h_{5D}|, |h_{24}|, |h_{54}|) \ge |h_{S3}|$, and $|h_{54}| = |h_{45}| \ge \max(h_1, h_2)$, the achievable rate under the proposed MDF protocol in the two-stage relay network is

$$R(h_{S2}, h_{S3}) \triangleq \frac{2C(h_{S2}^2P)C(h_{S3}^2P)}{C(h_{S2}^2P)+C(h_{S3}^2P)}. \tag{16}$$

(a) When $|h_{S2}| = |h_{S3}|$ the achievable rate of the MDF protocol in the strong interference regime is $R(h_{S2}, h_{S3}) = C(h_{S2}^2 P)$ with the full-duplex source cut bound being $C(2h_{S2}^2 P)$. The gap to capacity is at most $C(2h_{S2}^2P) - C(h_{S2}^2P) = C(\frac{h_{S2}^2 P}{1+h_{S2}^2 P}) \le 0.5$ bits, $\forall h_{S2}$.

(b) In the *strong interference regime* when $h_{S2} = h_{S3} = h_{4D} = h_{5D} = \alpha$, $h_{24} = h_{35} = \beta \ge \alpha$, $h_{23} = h_{25} = h_{34} = h_{45} = \gamma \ge \alpha$ and $\gamma \ge \sqrt{\frac{(1+\alpha^2P)^2-1-\beta^2P}{P}} \ge 0$, the gap Δ_s from the closed-form half-duplex bound (15) is given by (17).

2) Weak interference condition: Suppose that receiver 5 in state S_1 decodes the data along link $(3,5)$ and treats interference along link $(4,5)$ as noise. Since we assume Gaussian codebooks at all transmitters, a rate R_1 is achievable whenever $R_1 = C(h_{S2}^2P) \le C\left(\frac{h_{24}^2P}{1+h_{45}^2P}\right)$. The above condition reduces to $|h_{45}| \le \left(\sqrt{\frac{h_{24}^2}{h_{S2}^2P} - \frac{1}{P}}\right)^+ \triangleq h_3$, where $x^+ = \max(x, 0)$. Similarly, rate R_2 is achievable at receiver 4 in state S_2 whenever $|h_{54}| \le \left(\sqrt{\frac{h_{35}^2}{h_{S3}^2P} - \frac{1}{P}}\right)^+ \triangleq h_4$.

Remark 2: When the channel gains satisfy the following *weak interference conditions*: $\min(|h_{24}|, |h_{4D}|) \ge |h_{S2}|$, $\min(|h_{35}|, |h_{5D}|) \ge |h_{S3}|$, $|h_{54}| = |h_{45}| \le \min(h_3, h_4)$, the achievable rate under the MDF protocol is $R(h_{S2}, h_{S3})$.

(a) In the weak interference regime, the achievable rate of the MDF protocol is $R(h_{S2}, h_{S3}) = C(h_{S2}^2 P)$, when $|h_{S2}| = |h_{S3}|$. So, the gap to capacity is at most 0.5 bits as seen from the comparison with the full-duplex source cut-set bound $C(2h_{S2}^2P)$.

(b) Consider the *weak interference regime* with $h_{S2} = h_{S3} = h_{5D} = h_{4D} = \alpha, h_{24} = h_{35} = \beta, h_{23} = h_{25} = h_{45} =$

[1]The terms "strong" and "weak" are used to merely describe conditions satisfied by the relative strengths of network links. We do not imply that capacity of the relay network is known in these regions.

$$\Delta_s = \lim_{\alpha\to\infty}\left[\frac{C(2\alpha^2P)C((\beta^2+\gamma^2)P)}{C(2\alpha^2P)-C(\alpha^2P)+C((\beta^2+\gamma^2)P)} - C(\alpha^2P)\right] = 0.25 \text{ bits.} \tag{17}$$

$$\Delta_w = \begin{cases} \frac{C(2\alpha^2P)C(\beta^2P)}{C(2\alpha^2P)-C(\alpha^2P)+C(\beta^2P)} - C(\alpha^2P) & \text{if} \quad 1-\frac{C(\alpha^2P)}{C(2\alpha^2P)} < \frac{C(\beta^2P)}{2C(2\beta^2P)}, \\ \frac{2C(2\beta^2P)C(2\alpha^2P)}{2C(2\beta^2P)+C(2\alpha^2P)} - C(\alpha^2P) & \text{if} \quad 1-\frac{C(\alpha^2P)}{C(2\alpha^2P)} \geq \frac{C(\beta^2P)}{2C(2\beta^2P)}. \end{cases} \tag{18}$$

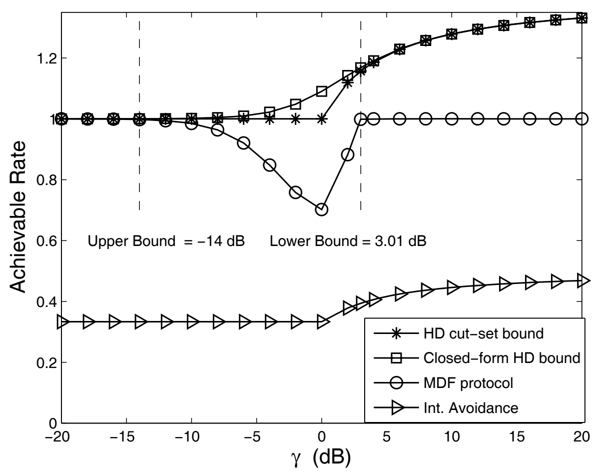

Fig. 3. Performance of the MDF protocol; Channel condition A: $\alpha = 1, \beta = 1$, vary γ.

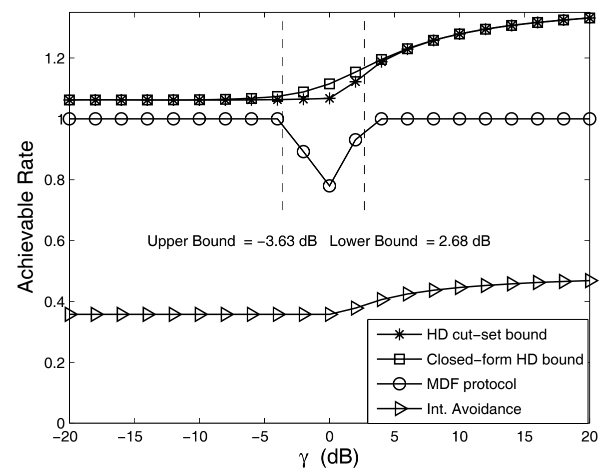

Fig. 4. Performance of the MDF protocol; Channel condition B: $\alpha = 1, \beta = 1.25$, vary γ.

$h_{34} = \gamma$, with $\alpha \leq \beta$, and $\gamma \to 0$. For these channel conditions, we have $C_0 \to 2C(2\beta^2P)$ and the gap Δ_w to the closed-form half-duplex cut-set bound is given by (18).

When $\alpha = \beta$, further simplification shows that the gap Δ_w in (18) reduces to zero when $\alpha^2P > \frac{1+\sqrt{5}}{2}$, and otherwise to $\frac{2}{3}C(2\alpha^2P) - C(\alpha^2P) \leq 0.07$ bits.

Though Remarks 1 and 2 are made for the specific two-stage relay network of Fig. 1(a), extensions to any network with two non-overlapping paths is possible as long as the on-path gains are either strong or weak, when compared to the inter-path gains. The coding ideas remain the same, but computing the half-duplex cut-set bound will become more complicated. However, the gap to the full-duplex cut-set bound will still remain small in suitably defined strong and weak interference channel gain regimes.

IV. Numerical Evaluation

In this section, we numerically evaluate the performance of the proposed MDF protocol for the two-stage relay network and verify the results of Section III. The achievable S-D rate is found by solving the optimization (8) in Section II-C using standard optimization routines. We consider half-duplex cut-set bound, the closed-form half-duplex bound described in Section III-A and the interference avoidance (IA) scheme for comparison. In the IA scheme, all states with only non-interfering links are considered. We set $P = 3, \sigma^2 = 1$ and $h_{S2} = h_{S3} = h_{5D} = h_{4D} = \alpha, h_{24} = h_{35} = \beta, h_{23} = h_{25} = h_{45} = h_{34} = \gamma$ for illustration.

In Fig. 3, $\alpha = \beta = 1$ and γ is varied. In the strong ($\gamma \geq 3.01$ dB) and weak ($\gamma \leq -14$ dB) interference regimes, the rate achieved by MDF protocol is $C(\alpha^2P) = 1$ as determined in Remark 1(a). In the weak interference regime, capacity is achieved following Remark 2(b). In the strong interference regime, the gap from the half-duplex cut-set bound is at most 0.33 bits, as per (9) and (15).

In Fig. 4, $\alpha = 1, \beta = 1.25$, and γ is varied. The MDF protocol achieves a rate of $C(\alpha^2P) = 1$ for a larger range of γ, *i.e.,* strong interference regime ($\gamma \geq 2.68$ dB) and weak interference regime ($\gamma \leq -3.63$ dB) according to Remarks 1 and 2. The gap from the HD cut-set bound in the weak and strong interference regimes are 0.06 and 0.33 bits respectively, as per (9) and (15).

Fig. 5 shows the performance of the MDF protocol with varying α, with $\beta = \alpha$ and $\gamma = 2\sqrt{\frac{(1+\alpha^2P)^2-1-\beta^2P}{P}}$ in the strong interference regime. We notice that the gap to the full-duplex bound is at most 0.5 bits verifying Remark 1(a). The gap between the achievable rate and the derived half-duplex cut-set bound is only 0.25 bits as determined in Remark 1(b) even when the rate achieved is large (for large α).

In Figs. 3, 4 and 5, we notice the proposed MDF protocol performs significantly better than the interference avoidance scheme in all three channel conditions. Overall, the numerical results agree with the analytical results for strong and weak interference regimes and for the half-duplex cut-set bound. They show that the closed-form half-duplex bound is close to the computed one and illustrate the good performance of the proposed protocol in various channel conditions. Based on Figs. 3, 4 and 5, we can conclude that more complicated

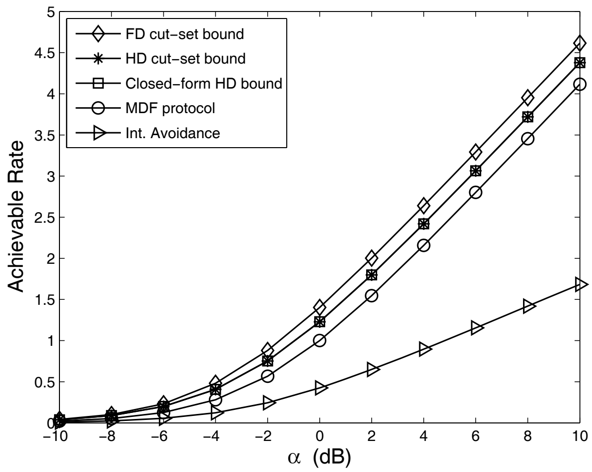

Fig. 5. Performance gap of the MDF protocol; Channel condition C: $\beta = \alpha$, $\gamma = 2\sqrt{\frac{(1+\alpha^2 P)^2 - 1 - \beta^2 P}{P}}$, vary α.

coding schemes that exploit significant cooperation among the nodes will only provide marginal or no gains in the strong and weak interference regimes.

V. Conclusion

We have proposed and analyzed a multi-hopping decode and forward (MDF) protocol for a two-stage Gaussian relay network. The protocol is shown to perform well under some practical assumptions such as half-duplex nodes, non-cooperative decoding among relay nodes and finite SNR. Through analysis, we show that the MDF protocol used with a simple schedule and suitable coding can approach the cut-set bound under strong and weak interference regimes of channel gains. Extensions to use of finite constellations at transmitters[14][15] and inclusion of fading in the channel model[16][17] are possible considerations for future work in the study of the proposed MDF protocol.

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