A CLASS OF TORUS MANIFOLDS WITH NONCONVEX ORBIT SPACE

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ABSTRACT. We study a class of smooth torus manifolds whose orbit space has the combinatorial structure of a simple polytope with holes. We construct moment angle manifolds for such polytopes with holes and use them to prove that the associated torus manifolds admit stable almost complex structure. We give a combinatorial formula for the Hirzebruch χ_y genus of these torus manifolds. We show that they have (invariant) almost complex structure if they admit positive omniorientation. We give examples of almost complex manifolds that do not admit a complex structure. When the dimension is four, we calculate the homology groups and describe a method for computing the cohomology ring.

1. INTRODUCTION

The moment polytope of the Hamiltonian action of the real torus $\mathbb{T}^n$ on a smooth projective toric variety (toric manifold) may be identified with the orbit space of the action. The moment polytope (Delzant polytope) is rather rigid with severe integrality constraints, see Definition 2.1.1 of [Sil01]. In 1991 Davis and Januskiewicz [DJ91] introduced a generalization of toric manifolds, now known as quasitoric manifolds, which may be obtained as identification spaces of $\mathbb{T}^n \times P$ where P is a simple n-dimensional polytope. In general these spaces do not have algebraic or invariant symplectic structure, but they still have a lot of remarkable properties; see the survey [BP02]. In this article we study a class of even dimensional manifolds which may be obtained as identification space of $\mathbb{T}^n \times P$ where P is not convex, but a simple polytope with holes which are also simple polytopes. In [Mas99] and [HM03], Masuda and Hattori introduced the notion of torus manifold (see Definition 2.2). Our manifolds are a special class of torus manifolds. As in the case of quasitoric manifolds, the torus action on our manifolds is *locally standard*, i.e. locally equivalent to the natural action, up to automorphism, of $U(1)^n$ on $\mathbb{C}^n$.

We describe the combinatorial construction of these manifolds in section 2. However, these manifolds are also obtained by gluing quasitoric manifolds along deleted neighborhoods of principal torus orbits (Lemma 2.1). We refer to this as the fiber sum construction. It is a special case of a more general construction in [GK98]. This is used to endow the manifolds with smooth structure (Lemma 2.1), and in certain cases with almost complex structure (Theorem 5.1).

We realize each of our manifolds as the quotient of a submanifold of $\mathbb{C}^m$ by the free action of a compact torus in section 4. This may be viewed as a topological analogue of the construction of toric manifolds by symplectic reduction, or of the quotient construction of toric varieties. We use this to endow our manifolds with a stable complex structure (Lemma 5.1). Using it, we give a combinatorial formula for the χ_y genus of these manifolds

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(Theorem 5.3) following the work of Panov [Pan01] in quasitoric case. The formula also follows from Lemma 5.1 and a more general result in [HM03].

Our manifolds admit almost complex structure if they admit a positive omniorientation (Lemma 5.2 and Theorem 5.1). Positive omniorientation is also a necessary condition if we require the almost complex structure to be $\mathbb{T}^n$-invariant.

These manifolds cannot admit an invariant symplectic structure (Lemma 5.3) or an invariant integrable complex structure (see [IK12]) if the orbit space has at least one hole. It would be interesting to know if any of these torus manifolds admit a symplectic or complex structure. If the orbit space has one hole, then the manifold can not be Kahler (Lemma 5.4). We give examples of almost complex manifolds that do not admit a complex structure in section 5.2.

A lot is known about the topological invariants of these manifolds from the works [Mas99] and [HM03]. However as they have nontrivial homology in odd degrees (see Theorem 9.3 of [MP06]), the formula for the cohomology ring given in Corollary 7.8 of [MP06] does not hold when the orbit space has holes. Even explicit formulas for their (co)homology groups are not known in general. In section 3, we give a combinatorial formula for the homology groups when dimension is four. We also describe a method for computing the cohomology ring for the four dimensional manifolds.

2. Construction and smooth structure

2.1. Polytope with holes. A polytope is the convex hull of a finite set of points in $\mathbb{R}^n$. An n-dimensional polytope is said to be simple if every vertex is the intersection of exactly n codimension one faces. Let P_0 be an n-dimensional simple polytope in $\mathbb{R}^n$. Let $P_1, P_2, \ldots, P_s$ be a disjoint collection of simple polytopes belonging to the interior of P_0. Let

$$P = P_0 - \bigcup_{k=1}^{s} P_k^{\circ}. \tag{2.1}$$

We call P an n-dimensional *polytope with simple holes*. The polytopes $P_1, P_2, \ldots, P_s$ are called holes of P. The faces of P are the faces of P_k, $k = 0, \ldots, s$.

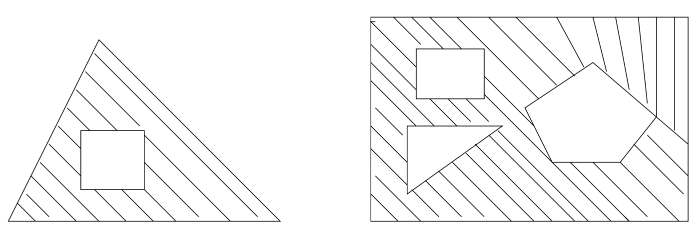

Figure 1. Polytopes with simple holes in $\mathbb{R}^2$.

2.2. Combinatorial construction. Let P be an n-dimensional simple polytope with s simple holes. Let $\mathcal{F}(P) = \{F_1, F_2, \ldots, F_m\}$ be the set of all codimension one faces (*facets*) of P. Note that $\mathcal{F}(P) = \bigcup_{k=0}^{s} \mathcal{F}(P_k)$. Also, if F is a nonempty face of P of codimension k then F is the intersection of a unique collection of k facets of P. The following definition is a straightforward generalization of the notion of characteristic function for a simple polytope, which is a crucial concept for studying quasitoric manifolds [DJ91, BP02].

Definition 2.1. *A function $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n$ is called a characteristic function if it satisfies the following condition: Whenever $F = \bigcap_{j=1}^{k} F_{i_j}$ is a $(n-k)$-dimensional face of P, the span of the vectors $\lambda(F_{i_1}), \lambda(F_{i_2}), \ldots, \lambda(F_{i_k})$ is a k-dimensional direct summand of $\mathbb{Z}^n$. We will denote $\lambda(F_i)$ by λ_i for simplicity and call it the characteristic vector of F_i.*

For any face $F = \bigcap_{j=1}^{k} F_{i_j}$ of P, let $N(F)$ be the submodule of $\mathbb{Z}^n$ generated by $\lambda_{i_1}, \ldots, \lambda_{i_k}$. The module $N(F)$ defines a sub-torus G_F of $\mathbb{T}^n = \mathbb{Z}^n \otimes \mathbb{R}/\mathbb{Z}^n = \mathbb{R}^n/\mathbb{Z}^n$ as follows.

$$G_F := (N(F) \otimes \mathbb{R})/N(F). \tag{2.2}$$

Define an equivalence relation $\sim$ on the product space $\mathbb{T}^n \times P$ by

$$(t, x) \sim (u, y) \text{ if } x = y \text{ and } u^{-1}t \in G_{F(x)} \tag{2.3}$$

where $F(x)$ is the unique face of P whose relative interior contains x.

We denote the quotient space as follows.

$$M = M(P, \lambda) := (\mathbb{T}^n \times P)/\sim. \tag{2.4}$$

The space M is a $2n$-dimensional manifold. The proof of this is analogous to the quasitoric case [DJ91]. The $\mathbb{T}^n$ action on $(\mathbb{T}^n \times P)$ induces a natural effective action of $\mathbb{T}^n$ on M, which is locally standard (see [DJ91]). Let $\pi\colon M \to P$ be the projection or orbit map defined by $\pi([(t, x)]) = x$.

Definition 2.2. [HM03] *A closed, connected, oriented, smooth manifold Y of dimension $2n$ with an effective smooth action of $\mathbb{T}^n$ with non-empty fixed point set is called a torus manifold if a preferred orientation is given for each characteristic submanifold. A characteristic submanifold is, by definition, any codimension two closed connected submanifold of Y, which is fixed by some circle subgroup of $\mathbb{T}^n$ and contains at least one $\mathbb{T}^n$-fixed point.*

In the case of M, the $\mathbb{T}^n$-fixed point set corresponds bijectively to the set of vertices of P. Observe that the spaces $X_i := \pi^{-1}(F_i)$, $i = 1, \ldots, m$, are the characteristic submanifolds of M. Each X_i is a $2(n-1)$-dimensional quasitoric manifold. We explain in section 2.4, how the characteristic function λ endows each X_i with a preferred orientation. We say that M is the torus manifold derived from the *characteristic pair* (P, λ).

2.3. **Fiber sum construction.**

Lemma 2.1. *The torus manifold $M(P, \lambda)$ is smooth and orientable.*

Proof. By induction it is sufficient to prove that $M(P, \lambda)$ has a smooth structure when P is a polytope with one hole, that is, $P = P_0 - P_1^0$. Let $\mathcal{F}(P_0)$ and $\mathcal{F}(P_1)$ be the set of facets of P_0 and P_1 respectively. The restrictions λ_0 and λ_1 of λ on $\mathcal{F}(P_0)$ and $\mathcal{F}(P_1)$ are characteristic functions on P_0 and P_1 respectively. Let M_0 and M_1 be the quasitoric manifolds associated to the characteristic pairs (P_0, λ_0) and (P_1, λ_1) respectively. These manifolds, being quasitoric, have smooth structure.

Let $\pi_k : M_k \to P_k,\ k = 0, 1$ be the orbit maps. Fix points $x_k \in P_k^{\circ}$. Let

$$L_k = \pi_k^{-1}(x_k). \tag{2.5}$$

Let $U_k \subset M_k$ be a $\mathbb{T}^n$ invariant neighborhood of L_k such that

$$B_k := \pi_k(U_k) \subset P_k \tag{2.6}$$

is diffeomorphic to an open ball in $\mathbb{R}^n$.

The quasitoric manifolds M_k are orientable. An orientation on M_k is determined by orientations on $\mathbb{R}^n \supset P_k$ and $\mathbb{T}^n$. Suppose $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be the standard Cartesian and angular coordinates on $\mathbb{R}^n$ and $\mathbb{T}^n$ respectively. Then the orientation on M_k corresponding to the ordering $\frac{\partial}{\partial p_1}, \frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial p_n}, \frac{\partial}{\partial q_n}$ will be assumed.

By (2.6) there exist equivariant orientation preserving diffeomorphisms

$$f_k : U_k \to \mathbb{T}^n \times B, \tag{2.7}$$

where B is the unit n-ball centered at the origin. Denote the punctured unit n-ball, $B - \{0\}$, by B^-. Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$. Define

$$r := |\mathbf{p}| \quad \text{and} \quad \Theta = (\theta_1, \dots, \theta_n) := \frac{\mathbf{p}}{r}. \tag{2.8}$$

The space $M(P, \lambda)$ can be obtained from $M_0 - L_0$ and $M_1 - L_1$ by identifying $U_0 - L_0$ and $U_1 - L_1$ as follows. Let $g : B^- \to B^-$ be the orientation preserving involution,

$$g(\mathbf{p}) = \frac{1-r}{r}(p_1, \dots, p_{n-1}, -p_n). \tag{2.9}$$

In other words, $g(r, \Theta) = (1 - r, \theta_1, \dots, \theta_{n-1}, -\theta_n)$.

Define

$$h = f_0^{-1} \circ (Id \times g) \circ f_1. \tag{2.10}$$

Identify $U_0 - L_0$ with $U_1 - L_1$ by the orientation preserving equivariant diffeomorphism h. □

REMARK 2.1. Note that the map g as used in (2.10), radially inverts a deleted neighborhood B_1^- of the point x_1 in P_1° and reflects it about the hyperplane $p_n = 0$. Then the map h identifies it to a deleted neighborhood of x_0 in P_0°. Up to homeomorphism, we can widen the puncture at x_0, and fit $P_1 - B_1$ into it and thus recover our picture of the orbit space of the glued manifold as the polytope with hole $P_0 - P_1^\circ$. A smooth embedding of this orbit space into Euclidean space is described in section 4. We do not know for sure if the smooth structure on the orbit space coincides with the smooth structure of P coming from its given embedding in $\mathbb{R}^n$.

REMARK 2.2. We refer to the above gluing construction as fiber sum construction because of its similarity to the symplectic fiber sum construction (see [Gro86], [Gom95]). A more general fiber sum construction for spaces with torus action was introduced in [GK98].

REMARK 2.3. As we may observe from section 4, the exact formula for the gluing map g is not important for the smooth structure.

REMARK 2.4. The sign of the characteristic vectors do not affect the equivariant diffeomorphism type of M. This follows from similar observation for quasitoric manifolds, see [DJ91, BR01].

2.4. **Omniorientation.** We fix an orientation for $M(P, \lambda)$ as above by choosing standard orientations on $\mathbb{T}^n$ and $\mathbb{R}^n$. Also each characteristic submanifold X_i is quasitoric and hence orientable.

Definition 2.3. *An omniorientation is an assignment of orientation for $M(P, \lambda)$ as well as for each X_i. Given such an assignment, we say that $M(P, \lambda)$ is omnioriented.*

Given the above choice of orientation for M, the characteristic function λ determines a natural omniorientation on M as follows: The characteristic vector λ_i determines a fiberwise S^1 action on the normal bundle of X_i, corresponding to the isotropy group G_{F_i}. This equips the normal bundle with a complex structure and therefore an orientation. This, together with the orientation on M, induces an orientation on X_i. We will refer to this omniorientation as the *characteristic omniorientation*.

Consider an omniorientation on M. Let $v \in M$ be a fixed point of the $\mathbb{T}^n$ action (or corresponding vertex of P). If the orientation of $T_v(M)$ determined by the orientation on M and the orientations of characteristic submanifolds containing v coincide then the *sign* $\sigma(v)$ is defined to be 1, otherwise $\sigma(v)$ is -1.

Definition 2.4. *An omniorientation is called positive if $\sigma(v) = 1$ for each fixed point v.*

For the characteristic omniorientation, the sign of a vertex v may be computed as follows [BP02]. Suppose $v = F_{i_1} \cap \ldots \cap F_{i_n}$. To each codimension one face F_{i_k} assign the unique edge E_k such that $E_k \cap F_{i_k} = v$. Let e_k be a vector along E_k with origin at v. Order (rename) the e_ks so that $e_1, \ldots, e_n$ is a positively oriented basis for $\mathbb{R}^n$. Consider the corresponding matrix $\Lambda_{(v)} = [\lambda_{i_1} \ldots \lambda_{i_n}]$. Then

$$\sigma(v) = \det \Lambda_{(v)}. \tag{2.11}$$

REMARK 2.5. It is also evident that the oriented intersection number of the submanifolds $X_{i_1}, \ldots, X_{i_n}$ is $\sigma(v)$.

3. Calculations in dimension four

Let $\pi : M(P, \lambda) \to P$ be a 4-dimensional torus manifold, where P is a polytope with s simple holes. We give a CW structure on $M(P, \lambda)$ and compute the homology groups.

First assume that P has only one hole. Then $P = P_0 - P_1^0$, where P_0 and P_1 are simple 2-dimensional polytopes with vertices $\{v_1, \ldots, v_{l_0}\}$ and $\{u_1, \ldots, u_{l_1}\}$ respectively. Assume that $dist(v_1 u_1) \leq dist(v_1 u_j)$ for all $j = 1, \ldots, l_1$. Let E_{v_i} and E_{u_j} be the edges of P joining the vertices $\{v_i, v_{i+1}\}$ and $\{u_j, u_{j+1}\}$ respectively for $i = 1, \ldots, l_0$; $j = 1, \ldots, l_1$. Here assume $v_{l_0+1} = v_1$ and $u_{l_1+1} = u_1$. Let $E_{v_1 u_1}$ be the line segment joining v_1 and u_1.

We construct the i-th skeleton X_i of $M(P, \lambda)$ as follows. Let $X_0 = \{v_1, \ldots, v_{l_0 - 1}, u_1, \ldots, u_{l_1}\}$. Define

$$\begin{array}{ll} e_i^1 = (\{(1,1)\} \times E_{v_i})/\sim & \text{for } i = 1, \ldots, l_0 - 2 \\ e_{l_0-1}^1 = (\{(1,1)\} \times E_{v_1 u_1})/\sim & \\ e_{l_0+j-1}^1 = (\{(1,1)\} \times E_{u_j})/\sim & \text{for } j = 1, \ldots, l_1 \\ X_1 = \cup_{i=1}^{l_0+l_1-1} \overline{e_i^1}. & \end{array} \tag{3.1}$$

A picture of the 1-skeleton for a polytope with one hole is given in figure 2 (a) on the next page. Define

$$\begin{array}{ll} e_i^2 = ((\mathbb{T}^2 \times E_{v_i})/\sim) - \overline{e_i^1} & \text{for } i = 1, \ldots, l_0 - 2 \\ e_{l_0-1}^2 = ((\{1\} \times S^1 \times E_{v_1 u_1})/\sim) - \overline{e_{l_0-1}^1} & \\ e_{l_0}^2 = ((S^1 \times \{1\} \times E_{v_1 u_1})/\sim) - \overline{e_{l_0-1}^1} & \\ e_{l_0+j}^2 = ((\mathbb{T}^2 \times E_{u_j})/\sim) - \overline{e_{l_0+j-1}^1} & \text{for } j = 1, \ldots, l_1 \\ X_2 = \cup_{i=1}^{l_0+l_1} \overline{e_i^2}. & \end{array} \tag{3.2}$$

Define

$$e^3 = ((\mathbb{T}^2 \times E_{v_1u_1})/\sim) - (\overline{e^2_{l_0-1}} \cup \overline{e^2_{l_0}}) \quad (3.3)$$
$$X_3 = \overline{e^3} \cup X_2.$$

Define

$$U^4 = P - \{E_{v_1} \cup \ldots \cup E_{v_{l_0-2}} \cup \partial P_1 \cup E_{v_1u_1}\}. \quad (3.4)$$

Clearly U^4 is homeomorphic to $\mathbb{R}^2_{\geq 0}$. So

$$(\mathbb{T}^2 \times U^4)/\sim \cong B^4 = \{x \in \mathbb{R}^4 : |x| < 1\}. \quad (3.5)$$

Define

$$e^4 = (\mathbb{T}^2 \times U^4)/\sim \text{ and } X_4 = \overline{e^4} \quad (3.6)$$

For the above CW structure, by reasons of either dimension or orientation, the cellular boundary maps d_2, d_3, d_4 are zero. Since X_1 is homotopic to a circle, we get the following result.

Theorem 3.1. *Suppose P is a 2-polytope with one hole. Then*

$$H_i(M(P,\lambda),\mathbb{Z}) = \begin{cases} \mathbb{Z}^{l_0+l_1} & \textit{if } i = 2 \\ \mathbb{Z} & \textit{if } i = 0,1,3,4 \\ 0 & \textit{if } i > 4. \end{cases}$$

We can give a similar CW structure on $M(P,\lambda)$ when P is a 2-polytope with multiple holes. The figure 2 (b) gives a representation of the 1-skeleton of such a structure when there are two holes.

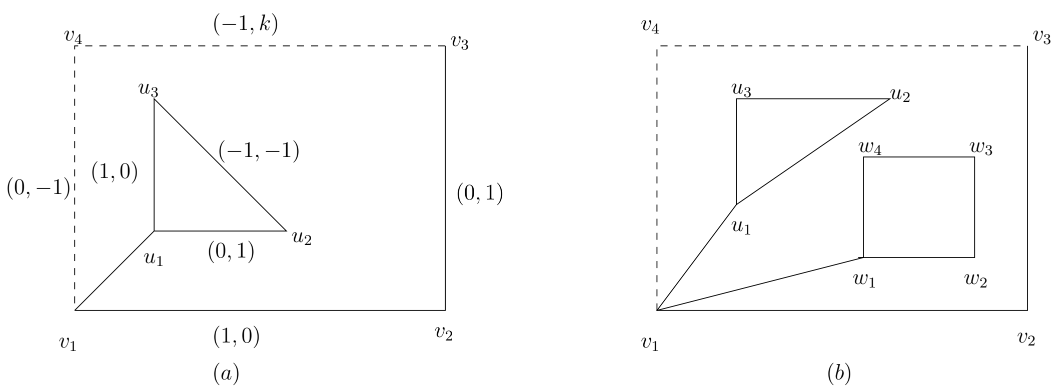

FIGURE 2. 1-skeleta for 2-polytopes with (a) one hole and (b) two holes.

Corollary 3.2. *Suppose P is a 2-polytope with m vertices and s simple holes. Then*

$$H_i(M(P,\lambda),\mathbb{Z}) = \begin{cases} \mathbb{Z}^{m+2s-2} & \textit{if } i = 2 \\ \mathbb{Z}^s & \textit{if } i = 1,3, \\ \mathbb{Z} & \textit{if } i = 0,4 \\ 0 & \textit{if } i > 4. \end{cases}$$

3.1. **Cohomology ring.** Assume that M has the characteristic omniorientation. In dimension four it is possible to compute the cohomology ring by using Poincaré duality and intersection product. To illustrate, we consider the case when there is one hole. Let $x_k \in H_2(M)$ denote the homology class of the sphere associated to the 2-cell e_k^2. Here characteristic orientation is chosen for the sphere if $k \neq l_0 - 1, l_0$. Otherwise orientation determined by the direction v_1u_1 and standard orientation of the associated S^1 is assumed.

The products of two classes x_i and x_j when i and j are both less than $l_0 - 1$, is the same as obtained by considering them as classes in $H_*(M_0)$. This is because the homotopies needed to achieve transversality can be done away from a neighborhood of any given principal torus fiber. Similar remarks apply when i and j both exceed l_0. If $i < l_0 - 1$ and $j > l_0$, or vice versa, then the product is obviously zero.

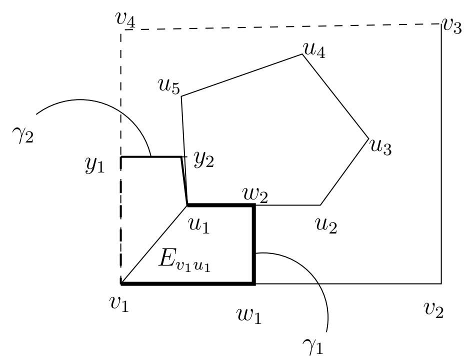

FIGURE 3. Homotopic copies of x_{l_0-1}, here $l_0 = 5$ and $l_1 = 4$.

Now consider the class x_{l_0-1}. To compute the self intersection $x_{l_0-1}^2$, we choose two different homotopy representatives, S_1^2 and S_2^2, of x_{l_0-1} which intersect only at v_1 and u_1. Let w_1, y_1 be points in the relative interior of the edges v_1v_2 and $v_{l_0}v_1$ respectively. Similarly let w_2, y_2 be points in the relative interior of the edges u_1u_2 and $u_{l_1}u_1$ respectively. Let γ_1, γ_2 be the piecewise linear paths $v_1w_1w_2u_1$ and $v_1y_1y_2u_1$ respectively. Let S_i^2 be the homotopy sphere $(\{1\} \times S^1) \times \gamma_i / \sim$. The circle subgroup $\{1\} \times S^1$ corresponds to the submodule of $\mathbb{Z}^2$ generated by $(0, 1)$. It is possible to express $(0, 1)$ uniquely as an integral linear combination $a_1\lambda_1 + a_2\lambda_{l_0}$. Let $d = \det[\lambda_{l_0}, \lambda_1] = \sigma(v_1)$. Near v_1, the sphere S_1^2 is homotopic to a_2d times the characteristic sphere over v_1v_2. Similarly S_2^2 is homotopic to a_1d times the characteristic sphere over $v_1v_{l_0}$. Therefore the contribution of v_1 to $x_{l_0-1}^2$ is $(d)(a_1d)(a_2d) = a_1a_2d$, see remark 2.5. The contribution from the point u_1 may be calculated similarly. Other intersection products of degree 2 classes may be calculated by using similar homotopies. For example, $x_1 \cdot x_{l_0-1} = (d)(a_1d) = a_1$. Finally the intersection of the generators degree one and degree three homology classes is 1 up to sign.

Example 3.3. Consider M to be the fiber sum of a Hirzebruch surface with $\mathbb{CP}^2$ corresponding to figure 2(a). Let $x_1, \ldots, x_7$ be the generators of $H_2(M)$ as defined above. Let y and z be the generators of $H_1(M)$ and $H_3(M)$ corresponding to the cells $\sum_{j=4}^{6} e_j^1$ and e^3 respectively. Then,

$$\begin{aligned}
&x_1^2 = x_3^2 = x_4^2 = 0, x_2^2 = -k, x_i^2 = 1 \text{ if } i \geq 5,\\
&x_1x_3 = x_2x_3 = x_2x_4 = x_3x_6 = x_3x_7 = x_4x_5 = x_4x_6 = 0,\\
&x_ix_j = 0 \text{ if } i = 1, 2 \text{ and } j = 5, 6, 7,\\
&x_1x_2 = x_1x_4 = x_5x_6 = x_5x_7 = x_6x_7 = 1, x_3x_5 = x_4x_7 = -1, yz = 1.
\end{aligned} \tag{3.7}$$

4. Moment angle manifold

Let M be the manifold obtained by fiber summing the smooth quasitoric manifolds $M_i(P_i, \lambda_i)$. We may assume that each P_i lies in a distinct copy of $\mathbb{R}^n$. Let S_i be the one point compactification of the copy of $\mathbb{R}^n$ that contains P_i, with standard smooth structure.

The orbit space O of M inherits a smooth structure from the gluing operations in Lemma 2.1. As noted in Remark 2.1, O is homeomorphic to P. Using the punctured balls B_k^- as tubes between different affine copies of $\mathbb{R}^n$, we may construct a smooth embedding of O into $\mathbb{R}^{n+s}$, where s is the number of holes of P. However, we need more. Consider the manifold with corners O^+, obtained by gluing a punctured copy of each S_i, $1 \leq i \leq s$, to P_0 punctured at s points, according to the gluing maps in Lemma 2.1. Then O^+ is homeomorphic to P_0. We may smoothly embed O^+ into $\mathbb{R}^{n+s}$.

For notational simplicity we will describe the embedding in terms of P and the P_i's. Induce smooth structures on P and P_0 using the homeomorphisms with O and O^+ respectively. Then there exists a smooth embedding ψ_0 of P_0 in $\mathbb{R}^{n+s} = \{(p_1, \ldots, p_{n+s})\}$ such that the following hold:

(1) The image of $P - \bigcup_{k=1}^{s} V_k$, where V_k is a small neighborhood of P_k in $\mathbb{R}^n$, lies in $\mathbb{R}^n = \{p_1, \ldots, p_n\}$.
(2) The image of V_k lies in the $(n+1)$-dimensional subspace $\{p_{n+j} = 0 | 1 \leq j \neq k \leq s\}$. The projection of $\psi_0(V_k)$ to $\mathbb{R}^n$ lies inside V_k.
(3) The embedding ψ_0 is affine when restricted to the boundary ∂P_k. The image of ∂P_k lies in the affine subspace $H_k := \{p_{n+k} = 1, p_{n+j} = 0 \,\forall\, j \text{ such that } 1 \leq j \neq k \leq s\}$.
(4) $\psi_0(P_0) \bigcap H_K = \partial P_k$.
(5) The image of P lies between the affine subspaces $p_{n+k} = 0$ and $p_{n+k} = 1$ for each $1 \leq k \leq s$.

Consider any facet F_i of P. Suppose $F_i \subset P_k$ where $k \geq 1$. Choose a linear polynomial A_i in the variables $p_1, \ldots, p_n, p_{n+k}$, other than $a_k := 1 - p_{n+k}$, which is zero on $\psi_0(F_i)$ and positive on $\psi_0(P) - \psi_0(F_i)$. Define $d_i = A_i + a_k + \sum_{1 \leq j \neq k} p_{n+j}$. If F_i is a facet of P_0, then let A_i be the defining linear polynomial of F_i in the variables $p_1, \ldots, p_n$ such that A_i is positive in the interior of P_0. In this case define $d_i = A_i + \sum_{1 \leq j} p_{n+j}$.

Then for a point x in $\psi_0(P)$, $d_i(x)$ can be thought of as an l_1-distance of x from the affine subspace of $\psi_0(F_i)$. We construct a smooth embedding ψ_1 of $\psi_0(P_0)$ into $\mathbb{R}^m$ by $\psi_1(x) = (d_1(x), \ldots, d_m(x))$ where $m = |\mathcal{F}(P)|$. The composition $\psi := \psi_1 \circ \psi_0$ defines an embedding of P_0 into $\mathbb{R}^m = \{(r_1, \ldots, r_m)\}$ such that the image of P lies in $\mathbb{R}^m_{\geq} = \{r_i \geq 0 \,\forall\, i\}$. Suppose $y \in \psi(P)$. Then $r_i(y) = 0$ if and only if $y \in \psi(F_i)$.

The space $\mathbb{C}^m$ can be regarded as a quotient of $\mathbb{T}^m \times \mathbb{R}^m_{\geq}$ by an equivalence relation $\sim_0$ as follows: Let $u_1, \ldots, u_m$ denote the standard basis of $\mathbb{Z}^m$. Let T_i denote the circle subgroup $(\mathbb{Z}u_i \otimes \mathbb{R})/\mathbb{Z}u_i$ of $\mathbb{T}^m$. For any face $F = \{r_j = 0 | j \in J\}$ of $\mathbb{R}^m_{\geq}$, we define the subgroup $T_F := \prod_{j \in J} T_j$. For any y in $\mathbb{R}^m_{\geq}$, let $F(y)$ denote the unique face of $\mathbb{R}^m_{\geq}$ whose relative interior contains y. Then define $\sim_0$ by

$$(t, x) \sim_0 (u, y) \text{ if } x = y \text{ and } u^{-1}t \in T_{F(y)}. \tag{4.1}$$

Definition 4.1. *Let $\pi_0 : \mathbb{C}^m \to \mathbb{R}^m$ denote the quotient map. Define the moment angle complex $Z(P)$ of P by*

$$Z(P) = \pi_0^{-1}(\psi(P)).$$

We may identify π_0 with the smooth map defined coordinate-wise by $z_i \mapsto |z_i|^2$. This shows that $Z(P)$ is smooth. The details are straightforward and left to the reader.

Given a characteristic function λ for P, let $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^n$ be the linear map defined by $\Lambda(u_i) = \lambda_i$. Let $K = \ker \Lambda$ and $T_K = (K \otimes \mathbb{R})/K$. Then it is easy to observe that topologically $Z(P)$ is a principal T_K bundle over $M(P, \lambda)$.

The leaf space $\mathcal{M}(P, \lambda)$ of the foliation corresponding to the smooth and free action of T_K on $Z(P)$ has a natural smooth structure. Since $\mathbb{T}^m \cong T_K \times \mathbb{T}^n$, it is not hard to check that $\mathcal{M}(P, \lambda)$ supports a smooth action of $\mathbb{T}^n$. Moreover $\mathcal{M}(P, \lambda)$ is equivariantly homeomorphic to $M(P, \lambda)$ with respect to this action. There is a one-to-one correspondence between normal orbit types and, in fact, an isomorphism of $\mathbb{T}^n$-normal systems (see [Dav78]) of $\mathcal{M}(P, \lambda)$ and $M(P, \lambda)$. (Here the smooth structures on the orbit spaces match that of O.) All of these may be ascertained by studying the local representations of $Z(P)$, up to equivariant diffeomorphism, by $T_K \times \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ near the faces of P. Therefore by Theorem 4.3 of [Dav78], $\mathcal{M}(P, \lambda)$ and $M(P, \lambda)$ are equivariantly diffeomorphic. We will henceforth identify $M(P, \lambda)$ with $\mathcal{M}(P, \lambda)$ without additional comments.

5. Almost complex structure

In this section we prove three results: i) That every omniorientation of M determines a stable almost complex structure on it, ii) that if M admits a positive omniorientation and $\dim(M) = 4$, then there exists an almost complex structure on M which is equivalent to the associated stable complex structure, and iii) that there exists a $\mathbb{T}^n$-invariant almost complex structure on M if and only if M has a positive omniorientation. It is not known to us if the invariant almost complex structure is equivalent to the associated stable almost complex structure.

Lemma 5.1. *Every omniorientation of the torus manifold $M(P, \lambda)$ determines a stable almost complex structure on it.*

Proof. Let $Q = \psi(P)$ and $Q_k = \psi(P_k)$ where ψ is the embedding of P_0 into $\mathbb{R}^m$ constructed in section 4. the normal bundle of Q_0 in $\mathbb{R}^m$ is trivial as Q_0 is contractible. Therefore the normal bundle N_Q of Q is also trivial. We may in fact identify N_Q with a tubular neighborhood of Q in $\mathbb{R}^m_{\geq}$ following an idea in [BR01]: Identify N_Q with $\{(x, v) | x \in Q,\, v \in (T_xQ)^{\perp} \subset \mathbb{R}^m\}$ where $\perp$ denotes orthogonal complement with respect to the dot product. Then define the map $f : N_Q \to \mathbb{R}^m_{\geq}$ by $f(x, v) = (e^{v_1}x_1, \ldots, e^{v_m}x_m)$ where the x_is and v_is denote the coordinates of x and v respectively. Then a careful analysis of the situation shows that $v \cdot Df_{(x,0)}(v) = \sum_{i=1}^m v_i^2 x_i$ is positive. This shows that $Df_x(N_Q)$ is transversal to T_xQ. We identify N_Q with $Df(N_Q)$.

Since a tubular neighborhood of Q in $\mathbb{R}^m_{\geq}$ pulls back to a tubular neighborhood of $Z(P)$ in $\mathbb{C}^m$ under π_0, we may identify the normal bundle N_Z of $Z(P)$ in $\mathbb{C}^m$ with $\pi_0^* N_Q$. Therefore N_Z is trivial. Let N_M denote the pullback of N_Q to M under $\psi \circ \pi$. Then by a slight generalization of the Atiyah sequence [At57], we obtain the following split exact sequence of bundles,

$$0 \to \mathfrak{t}_K \times M \to (TZ(P) \oplus N_Z)^{T_K} \to TM \oplus N_M \to 0. \tag{5.1}$$

Here $\mathfrak{t}_K$ denotes the Lie algebra of T_K. Since the action of T_K on $T\mathbb{C}^m$ is complex linear, therefore $(TZ(P) \oplus N_Z)^{T_K} = (T\mathbb{C}^m|_{Z(P)})^{T_K}$ inherits a complex structure. It follows that M admits a stable almost complex structure. □

As T_K acts diagonally on $\mathbb{C}^m$, the bundle $(T\mathbb{C}^m|_{Z(P)})^{T_K}$ splits naturally into a direct sum of m complex line bundles over M, namely $\nu_1, \ldots, \nu_m$, corresponding to the complex coordinate directions of $\mathbb{C}^m$. These directions correspond to (distance from) the facets of P.

Since the angular direction u_i maps to λ_i by Λ, the bundle ν_i restricts to the normal bundle of X_i on the characteristic submanifold X_i. The total Chern class of $M(P,\lambda)$ associated to the above stable complex structure admits the following product decomposition,

$$c(TM) = \prod_{i=1}^{m}(1 + c_1(\nu_i)). \tag{5.2}$$

Using standard localization formula or Theorem 5.3, we obtain

$$c_n(TM) = \sum \sigma(v) \tag{5.3}$$

where the sum is over all vertices of P.

Lemma 5.2. *If $M(P,\lambda)$ admits a positive orientation and $\dim(M) = 4$ then it admits an almost complex structure which is equivalent to the associated stable almost complex structure.*

Proof. By Theorem 1.7 of [Tho67], the lemma holds if $c_2(TM) = e(TM)$. This follows from (5.3) and Corollary 3.2. □

Theorem 5.1. *The torus manifold $M(P,\lambda)$ admits a $\mathbb{T}^n$-invariant almost complex structure if and only if it has a positive omniorientation.*

Proof. The necessity of positive omniorientation for existence of $\mathbb{T}^n$-invariant almost complex structure follows from similar argument as in quasitoric case, see [BP02].

To prove sufficiency, first assume that the number of holes is one. Note that a positive omniorientation of $M(P,\lambda)$ induces positive omniorientation on M_0 and M_1. Then by the work of Kustarev [Kus09], there exist $\mathbb{T}^n$-invariant orthogonal almost complex structures J_k on M_k, $k = 0, 1$. In particular, these structures are orientation preserving. We may assume that the complex structure J_k is locally constant in the normal direction near L_k, as explained below.

Recall the orientation preserving diffeomorphisms f_k in (2.7). Since $T(\mathbb{T}^n \times B)$ is trivial, df_k defines an isomorphism

$$df_k : TU_k \to \mathbb{T}^n \times B \times \mathbb{R}^{2n}. \tag{5.4}$$

Consider the almost complex structures

$$\widehat{J}_k = df_k \circ J_k \circ df_k^{-1} \tag{5.5}$$

on $\mathbb{T}^n \times B \times \mathbb{R}^{2n}$. Choose a smooth non-decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma(t) = \begin{cases} 0 & \text{if } t \leq \epsilon_1 \\ t & \text{if } t \geq \epsilon_2 \end{cases} \tag{5.6}$$

where $0 < \epsilon_1 < \epsilon_2 < 1$ are small real numbers. Define

$$J'_k(\mathbf{q}, r, \Theta) = \widehat{J}_k(\mathbf{q}, \gamma(r), \Theta). \tag{5.7}$$

Replace J_k by $df_k^{-1} J'_k df_k$ on U_k. Denote the resulting almost complex structure on M_k by J_k without confusion. Note that these new almost complex structures are orientation preserving and $\mathbb{T}^n$-invariant.

Recall the orientation preserving diffeomorphism g in (2.9). Define

$$\phi_0 := f_0, \quad \phi_1 := (Id \times g) \circ f_1 : U_1 - L_1 \to \mathbb{T}^n \times B^-. \tag{5.8}$$

We have orientation preserving isomorphisms,

$$d\phi_k : T(U_k - L_k) \to \mathbb{T}^n \times B^- \times \mathbb{R}^{2n}. \tag{5.9}$$

Consider the almost complex structures

$$\widetilde{J}_k = d\phi_k \circ J_k \circ d\phi_k^{-1} \tag{5.10}$$

on $\mathbb{T}^n \times B^- \times \mathbb{R}^{2n}$. The space of orientation preserving almost complex structures on $\mathbb{R}^{2n}$ may be identified with $GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$. Since ϕ_k is orientation preserving, we can regard $\widetilde{J}_k$ as a map

$$\widetilde{J}_k : \mathbb{T}^n \times B^- \to GL^+(2n,\mathbb{R})/GL(n,\mathbb{C}). \tag{5.11}$$

Since J_k is locally constant in the normal direction near L_k, we may define

$$\widetilde{J}_0(\mathbf{q}, 0, \Theta) = \widetilde{J}_0(\mathbf{q}, \epsilon_1/2, \Theta), \quad \widetilde{J}_1(\mathbf{q}, 1, \Theta) := \widetilde{J}(\mathbf{q}, 1-\epsilon_1/2, \Theta). \tag{5.12}$$

The space $GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$ is path connected. Hence there exists a smooth path

$$F(t) : [0.4, 0.6] \to GL^+(2n,\mathbb{R})/GL(n,\mathbb{C}),\ F(0.4) = \widetilde{J}_1(\mathbf{1}, 1, \Theta),\ F(0.6) = \widetilde{J}_0(\mathbf{1}, 0, \Theta). \tag{5.13}$$

By $\mathbb{T}^n$-invariance, we construct a smooth family of paths $F(\mathbf{q}, t) : \mathbb{T}^n \times [0.4, 0.6] \to GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$,

$$F(\mathbf{q}, t) := d\mathbf{q}F(t)d\mathbf{q}^{-1}, \tag{5.14}$$

satisfying $F(\mathbf{q}, 0.4) = \widetilde{J}_1(\mathbf{q}, 1, \Theta),\ F(\mathbf{q}, 0.6) = \widetilde{J}_0(\mathbf{q}, 0, \Theta)$.
Choose a smooth non-decreasing function $\alpha : (0,1) \to [0,1)$ such that

$$\alpha(t) = \begin{cases} t & \text{if } t \geq 0.8 \\ 0 & \text{if } t \leq 0.6. \end{cases} \tag{5.15}$$

Choose another smooth non-decreasing function $\beta : (0,1) \to (0,1]$ such that

$$\beta(t) = \begin{cases} t & \text{if } t \leq 0.2 \\ 1 & \text{if } t \geq 0.4. \end{cases} \tag{5.16}$$

Define a map $\widetilde{J} : \mathbb{T}^n \times B^- \to GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$ by

$$\widetilde{J}(\mathbf{q}, r, \Theta) = \begin{cases} \widetilde{J}_0(\mathbf{q}, \alpha(r), \Theta) & \text{if } r > 0.6 \\ F(\mathbf{q}, r) & 0.6 \geq r \geq 0.4 \\ \widetilde{J}_1(\mathbf{q}, \beta(r), \Theta) & \text{if } r < 0.4. \end{cases} \tag{5.17}$$

Note that

$$\widetilde{J}(\mathbf{q}, r, \Theta) = \begin{cases} \widetilde{J}_0(\mathbf{q}, r, \Theta) & \text{if } r > 0.8 \\ \widetilde{J}_1(\mathbf{q}, r, \Theta) & \text{if } r < 0.2. \end{cases} \tag{5.18}$$

Define a $\mathbb{T}^n$-invariant almost complex structure $\overline{J}_k$ on $T(U_k - L_k)$ by

$$\overline{J}_k = d\phi_k^{-1} \circ \widetilde{J} \circ d\phi_k. \tag{5.19}$$

By construction, $\overline{J}_k$ agrees with J_k in a neighborhood of the outer boundary of $U_k - L_k$. Therefore $\overline{J}_k$ extends to a $\mathbb{T}^n$-invariant almost complex structure on $M_k - L_k$. Moreover $\overline{J}_0 \circ dh = dh \circ \overline{J}_1$ on $U_1 - L_1$ since $h = \phi_0^{-1} \circ \phi_1$, see (2.10) and (5.8). Therefore $\overline{J}_0$ and $\overline{J}_1$ glue to produce a $\mathbb{T}^n$-invariant almost complex structure $\overline{J}$ on M. Finally, note that we may apply induction when the number of holes is greater than one. □

REMARK 5.2. In dimension four, the sufficiency part of Theorem 5.1 also follows from section 13 of [GK98] together with the main theorem of [Kus09].

5.1. **The χ_y genus.** The Hirzebruch χ_y genus is an invariant of the complex cobordism class of the manifold and thus depends on the stable almost complex structure. We give a combinatorial formula of the χ_y genus of M, following Panov's work on quasitoric manifolds. The proofs are the same as in [Pan01].

Let E be an edge of P^n. The isotropy subgroup of $\pi^{-1}(E)$ is an $(n-1)$-dimensional torus generated by a submodule K of rank $(n-1)$ in $\mathbb{Z}^n$. A primitive vector μ in $(\mathbb{Z}^n)^*$ is called an edge vector corresponding to E if $\mu(\alpha)=0$ for each $\alpha \in K$. The edge vector of E is therefore unique up to sign.

Let ν be a primitive vector in $\mathbb{Z}^n$ such that

$$\mu(\nu) \neq 0 \text{ for any edge vector } \mu. \tag{5.20}$$

Then the circle $S^1_\nu = (\mathbb{Z} < \nu > \otimes \mathbb{R})/\mathbb{Z} < \nu >$ acts smoothly on M with only isolated fixed points corresponding to the vertices of P.

We choose signs for each edge vector at a vertex v according to the characteristic omniorientation as follows. Order the codimension one faces meeting at v and corresponding edges E_ks as in subsection 2.4. Let μ_k be an edge vector corresponding to E_k. Let $M_{(v)}$ be the matrix, $M_{(v)} = [\mu_1, \ldots, \mu_k]$. Then choose sign for each μ_k such that $M^t_{(v)}\Lambda_{(v)} = I_n$. Under this choice of signs the action of S^1_ν induces a representation of S^1 on the tangent space T_vM with weights $\mu_1(\nu), \ldots, \mu_n(\nu)$.

Definition 5.1. *Define the index of a vertex $v \in P$ as the number of negative weights of the S^1 representation on $T_v(M)$,*

$$\mathrm{ind}_\nu(v) = |\{k : \mu_k(\nu) < 0\}|.$$

Theorem 5.3. *For any vector ν satisfying (5.20),*

$$\chi_y(M) = \sum_v (-y)^{\mathrm{ind}_\nu(v)} \sigma(v).$$

Note that the Theorem 5.3 also follows from Lemma 5.1 together with Theorem 10.1 of [HM03]. Specializing the formula in Theorem 5.3 to $y=-1$ and $y=1$, respectively yield formulas for the top Chern number and the signature. Moreover following Theorem 3.4 of [Pan01] or Theorem 4.2 of [Mas99] we obtain the following formula for Todd genus of M,

$$\mathrm{td}(M) = \sum_{\mathrm{ind}_\nu(v)=0} \sigma(v). \tag{5.21}$$

5.2. **Integrability questions.**

Lemma 5.3. *If the polytope P has at least one hole, then the torus manifold $M(P,\lambda)$ does not support any symplectic form for which the torus action is symplectic.*

Proof. When the dimension $2n > 4$, $M(P,\lambda)$ is simply connected. So any symplectic circle action is Hamiltonian. Therefore if $M(P,\lambda)$ supports a T^n-invariant symplectic form, then the action of T^n must be Hamiltonian. Then $M(P,\lambda)$ would be a symplectic toric manifold with a moment map whose image is a Delzant polytope. Then the orbit space of the T^n-action on $M(P,\lambda)$ would be a Delzant polytope, see Theorem 2.6.2 of [Sil01]. Therefore, as the orbit space of $M(P,\lambda)$ is not convex it cannot support an invariant symplectic form.

When $2n = 4$, a result of McDuff [McD88] states that a symplectic circle action on a compact four dimensional manifold is Hamiltonian if and only if it has fixed points. Therefore, again, if $M(P, \lambda)$ supports a T^n-invariant symplectic form, then the action of T^n must be Hamiltonian. We get a contradiction as above. □

It follows from the main result of [IK12] that $M(P, \lambda)$ cannot admit a complex structure with respect to which the torus action is holomorphic if it is not a toric variety, for instance when P has at least one hole.

More generally, we may ask whether $M(P, \lambda)$ admits any symplectic or complex structure. We do not know of any example that does so in case P has at least one hole.

Lemma 5.4. *If P has one hole, then $M(P, \lambda)$ can not be Kahler.*

Proof. If P has one hole, by Theorem 3.1, the first Betti number $b_1(M) = 1$. As Gompf points out in p. 560 of [Gom95], it is known from the Enriques-Kodaira classification of surfaces (see [BPV84]) that a symplectic manifold with $b_1 = 1$ can not admit a complex structure. The result follows. □

It is not hard to produce examples of M that admit almost complex structure but do not admit an integrable complex structure. For an almost complex 4-manifold, c_1^2 and c_2 are determined by the Euler characteristic and signature, and are therefore independent of the choice of almost complex structure. Consider the equivariant connected sum Y of three copies of $\mathbb{C}P^2$. This is a quasitoric manifold with a pentagon as P. The characteristic vectors may be chosen to be $(1, 0)$, $(-1, 1)$, $(1, -2)$, $(0, 1)$ and $(-1, -1)$, thus endowing Y with a positive omniorientation and an almost complex structure. However, Y has $c_1^2 = 19$ and $c_2 = 5$. Therefore the Bogomolov-Miyaoka-Yau inequality, $c_1^2 \leq 3c_2$, is not satisfied and Y does not admit a complex structure.

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