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A CLASS OF SINGULAR R_0-MATRICES AND EXTENSIONS TO SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEMS

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Abstract: For $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the linear complementarity problem $LCP(A, q)$ is to determine if there is $x \in \mathbb{R}^n$ such that $x \geq 0,\ y = Ax + q \geq 0$ and $x^T y = 0$. Such an x is called a solution of $LCP(A, q)$. A is called an R_0-matrix if $LCP(A, 0)$ has zero as the only solution. In this article, the class of R_0-matrices is extended to include typically singular matrices, by requiring in addition that the solution x above belongs to a subspace of $\mathbb{R}^n$. This idea is then extended to semidefinite linear complementarity problems, where a characterization is presented for the multplicative transformation.

Keywords: R_0-matrix; semidefinite linear complementarity problems; Moore-Penrose inverse; group inverse.

MSC: 90C33; 15A09

1. INTRODUCTION

Let $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. The linear complementarity problem $LCP(A, q)$ is to determine if there exists $x \in \mathbb{R}^n$ such that $x \geq 0,\ y = Ax + q \geq 0$ and $\langle x, y \rangle = 0$, where for $u, v \in \mathbb{R}^n$, we have $\langle u, v \rangle = u^T v$. Throughout this section, for $u \in \mathbb{R}^n$, the notation $u \geq 0$ signifies that all the coordinates of u are nonnegative. If $B \in \mathbb{R}^{m\times n}$, then $B \geq 0$ denotes that all the entries of B are nonnegative. Motivated by questions concerning the existence and uniqueness of $LCP(A, q)$, many matrix classes have been considered in the literature. Let us recall two such classes. A matrix $A \in \mathbb{R}^{n\times n}$ is called a Q-matrix if $LCP(A, q)$ has a solution for all $q \in \mathbb{R}^n$. $A \in \mathbb{R}^{n\times n}$ is called an R_0-matrix if $LCP(A, 0)$ has zero as the only solution. Just to recall one of the well known results for an R_0-matrix, we point out that A is an R_0-matrix if and only if for every $q \in \mathbb{R}^n$, the problem $LCP(A, q)$ has a bounded solution set. For more details and relationships with other classes of matrices, we refer the reader to [5].

In this article, we propose two generalizations of this class of matrices in the classical linear complementarity theory and study extensions to the semidefinite linear complementarity problem. Briefly, we restrict the solution x of $LCP(A, 0)$ to lie in a subspace. Specifically, we consider the subspaces $R(A)$ (the range space of A) and $R(A^T)$, leading to what we refer to as $R_{\#}$-matrices and $R_{\dagger}$-matrices, respectively. The main results in connection with the classical problem are presented in Theorem 3.7 and Theorem 3.8. Extensions of these notions to semidefinite linear complementarity problems are referred to as $R_{\#}$-operators and $R_{\dagger}$-operators, respectively. In Theorem 4.14, we present sufficient conditions under which the Lyapunov operator is an $R_{\#}$-operator and in Theorem 4.15, we prove a similar result for the Stein operator. The case of the multiplication operator is taken up next and we prove a characterization in Theorem 4.16. Necessary conditions for the three operators mentioned above to be $R_{\#}$-operators are derived in Theorem 4.17. The notion of $R_{\dagger}$-operator is presented briefly at the end.

2. PRELIMINARIES

Let $\mathbb{R}^n$ denote the n dimensional real Euclidean space and $\mathbb{R}^n_+$ denote the nonnegative orthant in $\mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{m\times n}$, the set of all $m \times n$ matrices of reals, we denote the null space and the transpose of A by $N(A)$ and A^T, respectively. Let K, L be complementary subspaces of $\mathbb{R}^n$, i.e., $K \oplus L = \mathbb{R}^n$. Then $P_{K,L}$ denotes the (not necessarily orthogonal) projection of $\mathbb{R}^n$ onto K along L. So we have $P^2_{K,L} = P_{K,L}$, $R(P_{K,L}) = K$ and $N(P_{K,L}) = L$. If, in addition, $K \perp L$, then $P_{K,L}$ will be denoted by P_K. In such a case, we also have $P_K^T = P_K$. The spectral radius of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$ where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A.

The Moore-Penrose inverse of matrix $A \in \mathbb{R}^{m\times n}$, denoted by $A^{\dagger}$ is the unique solution $X \in \mathbb{R}^{n\times m}$ of the equations: $A = AXA$, $X = XAX$, $(AX)^T = AX$ and $(XA)^T = XA$. The group inverse of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $A^{\#}$ (if it exists), is the unique matrix X satisfying $A = AXA$, $X = XAX$ and $AX = XA$. Let us reiterate the interesting fact that while the Moore-Penrose inverse exists for all matrices, the group inverse does not exist for some matrices. One of the well known equivalent conditions for the existence of $A^{\#}$ is that $N(A^2) = N(A)$ (equivalently, $R(A^2) = R(A)$). We also mention another equivalent condition: $A^{\#}$

exists if and only if $R(A)$ and $N(A)$ are complementary subspaces of $\mathbb{R}^n$. If A is nonsingular, then of course, we have $A^{-1} = A^{\dagger} = A^{\#}$. Recall that $A \in \mathbb{R}^{n\times n}$ is called range-symmetric (or an EP matrix) if $R(A^T) = R(A)$. For this class of matrices, the group inverse coincides with the Moore-Penrose inverse. For details, we refer to [1]. Next, we collect some well known properties of $A^{\dagger}$ and $A^{\#}$ ([1]) that will be frequently used in this paper: $R(A^T) = R(A^{\dagger})$; $N(A^T) = N(A^{\dagger})$; $AA^{\dagger} = P_{R(A)}$; $A^{\dagger}A = P_{R(A^T)}$; $R(A) = R(A^{\#})$; $N(A) = N(A^{\#})$; $AA^{\#} = P_{R(A),N(A)}$. In particular, if $x \in R(A^T)$ then $x = A^{\dagger}Ax$ and if $x \in R(A)$ then $x = A^{\#}Ax$. We also use the fact that $(A^T)^{\#} = (A^{\#})^T$.

Next, we list certain results to be used in the sequel. The first result is well known, for instance one could refer to [1].

Lemma 2.1. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $Ax = b$ has a solution if and only if $AA^{\dagger}b = b$. In this case, the general solution is given by $x = A^{\dagger}b + z$ for some $z \in N(A)$.*

Recall that, for $A \in \mathbb{R}^{m\times n}$ a decomposition $A = U - V$ of A is called a proper splitting of A, if $R(U) = R(A)$ and $N(U) = N(A)$ [2].

The following properties of a proper splitting will be used in the sequel.

Theorem 2.2. *(Theorem 1, [2])*
Let $A = U - V$ be a proper splitting. Then
(a) $AA^{\dagger} = UU^{\dagger}$; $A^{\dagger}A = U^{\dagger}U; VU^{\dagger}U = V$
(b) $A = U(I - U^{\dagger}V)$.
(c) $I - U^{\dagger}V$ is invertible.
(d) $A^{\dagger} = (I - U^{\dagger}V)^{-1}U^{\dagger}$.

The case of group inverses has also been studied in the literature. The next result is in this direction.

Theorem 2.3. *(Theorem 3.1, [8])*
Let $A = U - V$ be a proper splitting. Suppose that $A^{\#}$ exists. Then $U^{\#}$ exists and we have the following:
(a) $AA^{\#} = UU^{\#}$
(b) $A = U(I - U^{\#}V)$.
(c) $I - U^{\#}V$ is invertible.
(d) $A^{\#} = (I - U^{\#}V)^{-1}U^{\#}$.

3. A GENERALIZATION OF THE R_0-PROPERTY

In this section, we propose an extension of the R_0-property by requiring the solution x of $LCP(A, 0)$ to lie in certain subspaces related to the matrix A. This leads to two generalizations, as we will discuss next.

Definition 3.4. *Let $A \in \mathbb{R}^{n\times n}$ and M be a subspace of $\mathbb{R}^n$. Then A is called an R_0-matrix relative to M, if the only solution for $LCP(A, 0)$ in M is the zero* *solution. In other words, A is an R_0-matrix relative to M if $x = 0$ is the only vector $x \in M$ such that $x \geq 0, y = Ax \geq 0$ and $\langle x, y\rangle = 0$.*

Next, we assume that the subspace M has some relationship with the matrix A. Specifically, we consider the following two particular cases.

Definition 3.5. *$A \in \mathbb{R}^n$ is called an $R_\dagger$-matrix, if A is an R_0-matrix relative to $R(A^T)$.*

Definition 3.6. *$A \in \mathbb{R}^n$ is called an $R_\#$-matrix, if A is an R_0-matrix relative to $R(A)$.*

First, we present a result which provides sufficient conditions under which a matrix is an $R_\dagger$-matrix. Recall that $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if the off-diagonal entries of A are nonpositive. If A is a Z-matrix, then $A = sI - B$, where $B \geq 0$ and $s \geq 0$. If $s > \rho(B)$, then A is invertible and in this case A is well known as an (invertible) M-matrix. It is also quite well known that if A is a Z-matrix, then A is an M-matrix if and only if A is a Q-matrix. For details, we refer to Chapter 6 in [3]. Observe that a Z-matrix A could be written as $A = U - V$, where $U = sI$ and $V \geq 0$. Then, $U^{-1} \geq 0, U^{-1}V \geq 0$ and U^{-1} is strictly copositive. In the result under discussion for a typically singular matrix A, we assume that $A = U - V$ where U and V satisfy more general conditions than the above. Let us point out that the proof is an adaptation of the first part of the proof of Theorem 9 in [6], where the authors study matrices of the form $A = I - S$, where S is a certain nonnegative matrix satisfying $\rho(S) < 1$. Finally, let us recall that $B \in \mathbb{R}^{n\times n}$ is *copositive* if $\langle x, Bx\rangle \geq 0$ for all $x \geq 0$ and *strictly copositive* if $\langle x, Bx\rangle > 0$ for all $x \geq 0$.

Theorem 3.7. *For $A \in \mathbb{R}^{n\times n}$, let $A = U - V$ be a proper splitting where $U^\dagger \geq 0$, $U^\dagger V \geq 0$ and $U^\dagger$ is strictly copositive. If $\rho(U^\dagger V) < 1$, then A is an $R_\dagger$-matrix.*

Proof. Suppose that there exists $x \in R(A^T)$ such that $x \geq 0, y = Ax \geq 0$ and $\langle x, y\rangle = 0$. Then $x = A^\dagger y + z$, for some $z \in N(A)$, by Lemma 2.1. Since $x \in R(A^T)$ and $A^\dagger y \in R(A^\dagger) = R(A^T)$, it follows that $z = 0$. Hence, $x = A^\dagger y = (I - U^\dagger V)^{-1}U^\dagger y$, where we have utilized the expression for $A^\dagger$ from Theorem 2.2. Since $\rho(U^\dagger V) < 1$, it follows that $(I - U^\dagger V)^{-1} = \sum_{j=0}^{\infty}(U^\dagger V)^j \geq 0$. So, $x = \sum_{j=0}^{\infty}(U^\dagger V)^j U^\dagger y = U^\dagger y + Wy$, where $W = \sum_{j=1}^{\infty}(U^\dagger V)^j U^\dagger \geq 0$. Since $y \geq 0$, it follows that $Wy \geq 0$. Thus $0 = \langle x, y\rangle = \langle U^\dagger y, y\rangle + \langle Wy, y\rangle$. Since both terms on the right hand side are nonnegative, and since $U^\dagger$ is strictly copositive, it follows that $y = 0$. Hence, $x = 0$ and so A is an $R_\dagger$-matrix. □

The version for $R_\#$-matrices follows. The proof is similar to the proof of the theorem above and hence, it is omitted. Let us point out that an expression for $A^\#$ similar to the expression for $A^\dagger$ in the above holds, by Theorem 2.3.

Theorem 3.8. *For $A \in \mathbb{R}^{n\times n}$ suppose that $A^\#$ exists. Let $A = U - V$ be a proper splitting where $U^\# \geq 0$, $U^\# V \geq 0$ and $U^\#$ is strictly copositive. If $\rho(U^\# V) < 1$, then A is an $R_\#$-matrix.*

Lemma 3.9. *Let $U \in \mathbb{R}^{n\times n}$ be strictly copositive, range symmetric and $U^\dagger \geq 0$. Suppose further that each column of U has at least one nonzero entry. Then, $U^\dagger$ is strictly copositive.*

Proof. We must show that for all $0 \neq y \geq 0$, it holds that $\langle U^\dagger y, y\rangle > 0$. Let $x = U^\dagger y \in R(U^T)$. If possible, let $x = 0$. Set $y = y_1e^1 + y_2e^2 + \ldots + y_ne^n$, where e^i denotes the ith standard basis vector of $\mathbb{R}^n$. Then $y_i \geq 0$ for all i. Also, $0 = U^\dagger y = y_1U^\dagger e^1 + y_2U^\dagger e^2 + \ldots + y_nU^\dagger e^n$. Since each term is a nonnegative vector, we conclude that $y_iU^\dagger e^i = 0$ for all i. Since $y \neq 0$ there is some i for which $y_i \neq 0$. For such an i, we have $U^\dagger e^i = 0$ so that $e^i \in N(U^T)$. Since U is range symmetric, it then follows that $e^i \in N(U^T) = N(U)$, so that $Ue^i = 0$, a contradiction, since this would mean that U has a zero column. Hence $0 \neq x \geq 0$. By Lemma 2.1, we have $y = Ux + z$, for some $z \in N(U^T)$. Then $\langle U^\dagger y, y\rangle = \langle x, Ux + z\rangle = \langle x, Ux\rangle + \langle x, z\rangle > 0$, where we have used the strict copositivity of U and the fact that $R(U) = R(U^T)$ which guarantees that $\langle x, z\rangle = 0$. □

We close this section with the observation that if U is strictly copositive, range symmetric and $U^\# \geq 0$, then $U^\#$ is strictly copositive, since in this case, $U^\dagger = U^\#$.

4. EXTENSIONS TO THE SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEM

In this section, we study generalizations of the matrix classes studied in the previous section, specifically in the case of three rather well studied operators in the theory of semidefinite linear complementarity problems. Let S^n denote the space of all real symmetric $n \times n$ matrices. Recall that for $T : S^n \to S^n$, and $Q \in S^n$, the semidefinite linear complementarity problem denoted by $SDLCP(T, Q)$ is to determine if there exists $X \in S^n$ such that $X \geq 0$, $Y = T(X) + Q \geq 0$ and $tr(XY) = 0$. Here, for $W \in S^n$, $W \geq 0$ denotes that W is positive semidefinite (meaning that $\langle x, Wx\rangle \geq 0$ for all $x \in \mathbb{R}^n$) and for $Z \in S^n$, the notation $tr(WZ)$ refers to the trace inner product. It follows that for $X, Y \in S^n$ and $X, Y \geq 0$, if $tr(XY) = 0$ then $XY = 0$.

Next, let us consider the following notions, extending Definition 3.4 and Definition 3.6.

Definition 4.10. *Let M be a subspace of S^n. A linear operator $T : S^n \to S^n$ is called an R_0-operator relative to M, if $SDLCP(T, 0)$ has zero as the only solution in M. Further, T is called an $R_\#$-operator, if $X = 0$ is the only matrix $X \in R(T)$ such that $X \geq 0$, $Y = T(X) \geq 0$ and $XY = 0$.*

Next, we consider the three most widely studied maps on S^n.

Definition 4.11. *For a fixed $A \in \mathbb{R}^{n\times n}$, let $M_A, L_A, S_A : S^n \to S^n$ be defined by $M_A(X) = AXA^T$, $L_A(X) = AX + XA^T$ and $S_A(X) = X - AXA^T, X \in S^n$. Then M_A is called the multiplicative transformation, L_A is called the Lyapunov transformation, and S_A is called the Stein transformation.*

Example 4.12. *Let* $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. *If* $X \in S^2$ *is given by* $X = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, *then* $M_A(X) = \begin{pmatrix} \alpha + 2\beta + \gamma & 0 \\ 0 & 0 \end{pmatrix}$, $L_A(X) = \begin{pmatrix} 2(\alpha+\beta) & \beta+\gamma \\ \beta+\gamma & 0 \end{pmatrix}$ *and* $S_A(X) = \begin{pmatrix} -(2\beta+\gamma) & \beta \\ \beta & \gamma \end{pmatrix}$.

Let us consider M_A *first. If* $X \in R(M_A)$, *then* $\beta = \gamma = 0$. *Also* $0 = XM_A(X) = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 0 \end{pmatrix}$ *implies that* $\alpha = 0$. *Hence* $X = 0$, *showing that* M_A *is an* $R_{\#}$*-operator.*

Next, we take L_A. *If* $X \in R(L_A)$, *then* $\gamma = 0$. *Also* $0 = XL_A(X) = \begin{pmatrix} 2\alpha(\alpha+\beta)+\beta^2 & \alpha\beta \\ 2\beta(\alpha+\beta) & \beta^2 \end{pmatrix}$ *implies that* $\beta = 0$ *and hence* $\alpha = 0$. *Hence* $X = 0$, *showing that* L_A *is an* $R_{\#}$*-operator.*

Finally, $0 = XS_A(X) = \begin{pmatrix} -\alpha(2\beta+\gamma)+\beta^2 & \beta(\alpha+\gamma) \\ -2\beta^2 & \beta^2+\gamma^2 \end{pmatrix}$ *implies that* $\beta = \gamma = 0$. *If, in addition,* $X \in R(S_A)$, *then it follows that* $\alpha = 0$. *Hence* $X = 0$, *and so* S_A *is an* $R_{\#}$*-operator.*

In what follows, first we present a class of matrices A for which L_A is an $R_{\#}$-operator, in Theorem 4.14 and a class of matrices A for which S_A is an $R_{\#}$-operator, in Theorem 4.15. The circumstance under which M_A has the $R_{\#}$-property is completely characterized in Theorem 4.16.

Let us recall that $B \in \mathbb{R}^{k\times k}$ is called positive stable if all the eigenvalues of B have positive real parts. It is known that (Theorem 5, [7]) if B is a positive stable matrix, then the only symmetric matrix X that satisfies $X \geq 0, L_B(X) \geq 0$ and $XL_B(X) = 0$ is the zero matrix.

The following notation is used in the next Theorem: For $B \in \mathbb{R}^{(n-1)\times(n-1)}$ define $A \in \mathbb{R}^{n\times n}$ by $A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$, where the zero in the top right corner is the zero (column) vector in $\mathbb{R}^{n-1}$, the zero in the bottom left is the transpose of the top right zero vector, and the bottom right zero is scalar. We shall make use of the following result:

Lemma 4.13. *Let* $B \in \mathbb{R}^{m\times m}$ *be (possibly non-symmetric and) positive definite (meaning that for all* $0 \neq x \in \mathbb{R}^m$, *we have* $x^TBx > 0$*). Then* B *is positive stable.*

Proof. Let $Bx = \lambda x$, where $\lambda = \alpha + i\beta$ and $0 \neq x = u + iv$. Then $Bu = \alpha u - \beta v$ and $Bv = \alpha v + \beta u$. We then have $0 \leq u^TBu = \alpha u^Tu - \beta u^Tv$ and $0 \leq v^TBv = \alpha v^Tv + \beta v^Tu$. Since at least one of the vectors u, v is non-zero, it follows by adding the last two inequalities that $\alpha(u^Tu + v^Tv) > 0$. Thus $\alpha > 0$. □

Theorem 4.14. *Let* $B \in \mathbb{R}^{(n-1)\times(n-1)}$ *be positive definite. Let* $A \in \mathbb{R}^{n\times n}$ *be defined as above. Then* L_A *is an* $R_{\#}$*-operator.*

Proof. It follows that if $Y = \begin{pmatrix} X & x \\ x^T & \alpha \end{pmatrix} \in S^n$, where $X \in S^{n-1}$, $x \in \mathbb{R}^{n-1}$

is a column vector and $\alpha \in \mathbb{R}$, then $L_A(Y) = \begin{pmatrix} BX + XB^T & Bx \\ (Bx)^T & 0 \end{pmatrix}$. Thus, if $Y \in R(L_A)$, then $\alpha = 0$. Also, $Y \geq 0, L_A(Y) \geq 0$ get transformed to $X \geq 0, BX + XB^T \geq 0$, respectively. Finally, if $YL_A(Y) = 0$, then

$$\begin{pmatrix} X(BX + XB^T) + x(Bx)^T & XBx \\ x^T(BX + XB^T) & x^T Bx \end{pmatrix} = 0.$$

Since B is positive definite, the equation $x^T Bx = 0$ implies that $x = 0$. The top left corner entry is $X(BX + XB^T)$, and this is the zero matrix. However, as mentioned earlier, since B is positive stable (by Lemma 4.13), and it is required that $X \geq 0$ (and X is symmetric), it follows that $X = 0$ and hence, $Y = 0$. This proves that L_A is an $R_\#$-operator. $\square$

Next, we consider the Stein transformation. Here, we recall that (Theorem 11, [6]) if B is a matrix with $\rho(B) < 1$, (i.e., all the eigenvalues of B lie in the open unit disk), then the only symmetric matrix X that satisfies $X \geq 0, S_B(X) \geq 0$ and $XS_B(X) = 0$ is the zero matrix.

Theorem 4.15. *Let $B \in \mathbb{R}^{(n-1)\times(n-1)}$ be such that $\rho(B) < 1$. Let $A \in \mathbb{R}^{n\times n}$ be defined as above. Then S_A is an $R_\#$-operator.*

Proof. It follows that if $Y = \begin{pmatrix} X & x \\ x^T & \alpha \end{pmatrix} \in S^n$, where $X \in S^{n-1}$, $x \in \mathbb{R}^{n-1}$ is a column vector and $\alpha \in \mathbb{R}$, then $S_A(Y) = Y - AYA^T = \begin{pmatrix} X - BXB^T & x \\ x^T & \alpha \end{pmatrix}$. Thus, $Y \geq 0, S_A(Y) \geq 0$ imply $X \geq 0, X - BXB^T \geq 0$, respectively. If $YS_A(Y) = 0$, then $\begin{pmatrix} X(X - BXB^T) + xx^T & Xx + \alpha x \\ x^T(X - BXB^T) + \alpha x^T & \alpha^2 + x^T x \end{pmatrix}$ equals the zero matrix. The bottom right entry being equal to zero implys that $x = 0$ and $\alpha = 0$. We then have $X(X - BXB^T) = 0$. However, as mentioned just before the theorem, it follows that $X = 0$ and hence, $Y = 0$. This proves that S_A is an $R_\#$-operator. $\square$

Now, we turn our attention to the multiplicative transformation. We observe that the matrix A in Example 4.12 is positive definite on $R(A)$. That this is not a coincidence is proved in the next result, where, we provide two necessary and sufficient conditions for M_A to be an $R_\#$-operator. This theorem generalizes a part of Theorem 17 of [4].

Theorem 4.16. *Let $A \in \mathbb{R}^{n\times n}$ be given. Then following statements are equivalent:*
(a) M_A is an $R_\#$-operator.
(b) X is symmetric, $XAX = 0,\ R(X) \subseteq R(A) \Rightarrow X = 0$.
(c) A is either positive definite or negative definite on $R(A)$.

Proof. $(a) \Rightarrow (b)$: Suppose that X is symmetric, $R(X) \subseteq R(A)$ and $XAX = 0$. For $u \in \mathbb{R}^n$, set $v = Xu$ and $V = vv^T \geq 0$. Further, since $R(X) \subseteq R(A)$, we have $v = Xu = Ap, p \in \mathbb{R}^n$. So, $V = vv^T = Ap(Ap)^T = APA^T$, where $P = pp^T$ is symmetric. Thus $V = M_A(P)$ so that $V \in R(M_A)$. Also, $M_A(V) = AVA^T = Avv^TA^T = Av(Av)^T \geq 0$. We also have $v^TAv = u^TX^TAXu = u^TXAXu = 0$. Thus $VAVA^T = vv^TAv(Av)^T = 0$. Since M_A is an $R_\#$-operator, it follows that $V = 0$ and so $X = 0$.

$(b) \Rightarrow (c)$: Suppose that A is not negative definite on $R(A)$. We show that A is positive definite on $R(A)$. Let us suppose that this is not the case. Then there exist $x, y \in R(A)$ such that $x^TAx < 0$ and $y^TAy > 0$. It then follows that for a specific value of $\lambda \in (0,1)$, the vector $z = \lambda x + (1-\lambda)y \in R(A)$ satisfies the equation $z^TAz = 0$. Suppose that $z = 0$. Then $x = \alpha y$ for some $0 \neq \alpha \in \mathbb{R}$. Then $0 > x^TAx = \alpha^2 y^TAy$, contradicting $y^TAy > 0$. Hence $z \neq 0$. Define $Z = zz^T$. Then $R(Z) \subseteq R(A)$. Also, $ZAZ = zz^TAzz^T = 0$. This means that $Z = 0$ and so $z = 0$, which is a contradiction.

$(c) \Rightarrow (b)$: Let X be symmetric, satisfying $XAX = 0$ and $R(X) \subseteq R(A)$. Then for all $y \in \mathbb{R}^n$, $0 = y^TXAXy = (Xy)^TA(Xy) = z^TAz$, $z = Xy \in R(X) \subseteq R(A)$. Since (c) holds, we have $Xy = 0$ for all $y \in \mathbb{R}^n$, showing that $X = 0$.

$(b) \Rightarrow (a)$: Suppose that $X \in R(M_A)$, $X \geq 0$, $M_A(X) \geq 0$ and $XM_A(X) = 0$. Then $X = AYA^T$ for some Y, and so $R(X) \subseteq R(A)$. Also, $XAXA^T = 0$ so that y^TXAXA^Ty for all $y \in \mathbb{R}^n$. By setting $z = XA^Ty \in R(X) \subseteq R(A)$, we then have $z^TAz = 0$. By (c) (which is now equivalent to (b)), it follows that $0 = z = XA^Ty$ for all $y \in \mathbb{R}^n$. Hence $XA^T = 0$, so that $AX = 0$ and so $XAX = 0$. It now follows that $X = 0$. □

Let us recall that $T : S^n \to S^n$ is called an R_0-operator if $SDLCP(T, 0)$ has zero as the only solution. In this connection, it is known that (Theorem 3, [7]), if L_A is an R_0-operator, then A is nonsingular. It can be shown that if M_A is an R_0-operator, then A is nonsingular. In the next result we prove analogues for $R_\#$-operators, including a result for the Stein transformation.

Theorem 4.17. *Let $A \in \mathbb{R}^{n\times n}$. We have the following:*
(a) If M_A or L_A is an $R_\#$-operator, then $A^\#$ exists.
(b) If S_A is an $R_\#$-operator, then $(I - A)^\#$ exists.

Proof. (a): Let $y \in R(A) \cap N(A)$. Then $Ay = 0$ and $y = Ax$ for some $x \in \mathbb{R}^n$. We must show that $y = 0$. Set $Y = yy^T$. Then $Y \geq 0$ and $M_A(Y) = Ayy^TA^T = 0$. Set $U = xx^T$. Then $U \in \mathcal{S}^n$. Also, $M_A(U) = Axx^TA^T = yy^T = Y$. Thus, $Y \in R(M_A)$. Since M_A is an $R_\#$-operator, it now follows that $Y = 0$ so that $y = 0$. Hence $N(A^2) = N(A)$, proving that $A^\#$ exists. This proves the first part.

Next, let $y \in R(A) \cap N(A)$. Then $Ay = 0$ and $y = Ax$ for some $x \in \mathbb{R}^n$. Set $Y = yy^T \in S^n$. Then $Y \geq 0$, $AY = Ayy^T = 0$ and $YA^T = yy^TA^T = y(Ay)^T = 0$, so that $L_A(Y) = 0$. Set $U = \frac{1}{2}(xy^T + yx^T)$. Then $U \in \mathcal{S}^n$. Also, $AU = \frac{1}{2}(Axy^T + Ayx^T) = \frac{1}{2}yy^T = \frac{1}{2}Y$. Further, $UA^T = \frac{1}{2}(xy^TA^T + y(Ax)^T) = \frac{1}{2}(x(Ay)^T + yy^T) = \frac{1}{2}Y$. Thus $Y = L_A(U)$ so that $Y \in R(L_A)$. It now follows that $Y = 0$ so that $y = 0$, proving the second part.

(b): We prove the existence of $(I - A)^\#$. Let $y \in R(I - A) \cap N(I - A)$. Then $(I - A)y = 0$ and $y = (I - A)x$ for some $x \in \mathbb{R}^n$. We must show that

$y = 0$. Set $Y = yy^T \geq 0$. Then $S_A(Y) = yy^T - Ayy^TA^T = yy^T - Ay(Ay)^T = 0$, since $Ay = y$. Set $U = \frac{1}{2}(xy^T + yx^T)$. Then $U \in \mathcal{S}^n$. Also, $S_A(U) = \frac{1}{2}(xy^T + yx^T - A(xy^T + yx^T)A^T) = \frac{1}{2}(xy^T + yx^T - (Ax)(Ay)^T - (Ay)(Ax)^T) = \frac{1}{2}(xy^T + yx^T - (Ax)y^T - y(Ax)^T) = \frac{1}{2}(y(x^T - (Ax)^T) + (x - (Ax))y^T) = yy^T$, where the last equation holds due to the fact that $(I - A)x = y$. Thus $Y = S_A(U)$ so that $Y \in R(S_A)$. It now follows that $Y = 0$, so that $y = 0$. □

Finally, let us turn our attention to a notion, extending Definition 3.5.

Definition 4.18. *A linear operator $T : S^n \rightarrow S^n$ is called an $R_\dagger$-operator, if $SDLCP(T, 0)$ has zero as the only solution in $R(T^T)$.*

When one attempts to derive necessary conditions for the three operators to be $R_\dagger$-operators, *a la* Theorem 4.17, rather interestingly, again one is led to the existence of the corresponding group inverses, as we show next.

Theorem 4.19. *Let $A \in \mathbb{R}^{n \times n}$. We have the following:*
(a) If M_A or L_A is an $R_\dagger$-operator, then $A^\#$ exists.
(b) If S_A is an $R_\dagger$-operator, then $(I - A)^\#$ exists.

Proof. The proof follows by starting with $y \in R(A^T) \cap N(A^T)$ and proceeding in an entirely similar manner to the proof of Theorem 4.17. The other facts that we may make use of are: $M_A^T = M_{A^T}$, $L_A^T = L_{A^T}$, $S_A^T = S_{A^T}$ and $(A^T)^\# = (A^\#)^T$. □

We close this paper with a version of Theorem 4.16 for $R_\dagger$-operators. The proof is entirely similar and hence, omitted.

Theorem 4.20. *Let $A \in \mathbb{R}^{n \times n}$ be given. The following statements are equivalent:*
(a) M_A is an $R_\dagger$-operator.
(b) X is symmetric, $XAX = 0$, $R(X) \subseteq R(A^T) \Rightarrow X = 0$.
(c) A is either positive definite or negative definite on $R(A^T)$.

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