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A characterization for $*$-isomorphisms in an indefinite inner product space

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Abstract

Let H_1 and H_2 be indefinite inner product spaces. Let $L(H_1)$ and $L(H_2)$ be the sets of all linear operators on H_1 and H_2, respectively. The following result is proved: If Φ is $[*]$-isomorphism from $L(H_1)$ onto $L(H_2)$ then there exists $U : H_1 \to H_2$ such that $\Phi(T) = cUTU^{[*]}$ for all $T \in L(H_1)$ with $UU^{[*]} = cI_2$, $U^{[*]}U = cI_1$ and $c = \pm 1$. Here I_1 and I_2 denote the identity maps on H_1 and H_2, respectively.

Keywords: Indefinite inner product; $*$-Isomorphism

1. Introduction

Recently, Kulkarni et al. [2] gave an elementary proof of characterizing onto $*$-isomorphisms of the algebra $BL(H)$ of all bounded linear operators on a Hilbert space H using simple and well-known properties of operators in a Hilbert space. The classical proof of this result utilizes the theory of irreducible representations of C^*-algebras [1, Corollary 2, p. 20]. In this paper we prove a similar characterization theorem for $*$-isomorphisms in an indefinite inner product space. One of the main results (Corollary 3.5) also demonstrates that there is no qualitative difference in the behaviour of a $*$-isomorphism of the algebra $BL(H)$, with H being a complex Hilbert space in one instance and an indefinite inner product space in the other. In other words, completeness

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of the norm and positive definiteness of the inner product are not of concern, in the representation of $*$-isomorphisms. The main results are Theorem 3.4 and Corollary 3.5.

2. Preliminary results

In this section, we prove some preliminary results which will be used in the sequel. Let $\langle .,. \rangle$ denote the conventional Hilbert space inner product on a Hilbert space H and N be an invertible Hermitian operator on H. An indefinite inner product on H is defined by the equation $[x, y] = \langle x, Ny \rangle$, where $x, y \in H$. Such a matrix N is called a weight. A space with an indefinite inner product is called an indefinite inner product space (IIPS). A vector x is called normalized vector if $[x, x] = \pm 1$. If $[x, y] = 0$ then the vectors x and y are called orthogonal vectors. Let T be an operator from H_1 into H_2. We define the adjoint $T^{[*]}$ (of the operator T) by $[T(x), y] = [x, T^{[*]}(y)]$ for all $x, y \in H_1$. T is called a projection iff $T = T^2$ and orthogonal projection iff T is a projection and $T = T^{[*]}$. Throughout this paper $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range space and the null space of T, respectively.

Lemma 2.1. *If P is an orthogonal projection then $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are orthogonal complementary subspaces. In this case $[x, x] \neq 0$ for all nonzero $x \in \mathcal{R}(P)$.*

Lemma 2.2. *If $P = P^2$ then $P = P^{[*]}$ iff $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are orthogonal complementary subspaces of H.*

Proof. Sufficiency follows from Lemma 2.1. We now prove the necessity part. Suppose $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are orthogonal subspaces. If $z \in \mathcal{N}(P)$, then $[P^{[*]}x, z] = [x, Pz] = 0$ for all x. Thus $\mathcal{N}(P) \subseteq \mathcal{N}(P^{[*]})$. Similarly, $\mathcal{N}(P^{[*]}) \subseteq \mathcal{N}(P)$. Thus $\mathcal{N}(P^{[*]}) = \mathcal{N}(P)$. Since $I - P$ is an orthogonal projection whenever P is so, it follows that $\mathcal{N}((I - P)^{[*]}) = \mathcal{N}(I - P)$. Equivalently, $\mathcal{R}(P^{[*]}) = \mathcal{R}(P)$. Thus $P = P^{[*]}$.

Lemma 2.3. *Let P, Q be orthogonal projections. Then*

(i) $\mathcal{R}(P) \subseteq \mathcal{R}(Q) \Leftrightarrow PQ = P = QP$;
(ii) $\mathcal{R}(P)[\perp]\mathcal{R}(Q) \Leftrightarrow PQ = \mathbf{0} = QP$.

Definition 2.4. Let H be a indefinite inner product space. For $x, y \in H$, define the operator $T_{x,y}$ on H by $T_{x,y}(u) = [u, y]x$, $u \in H$ and $P_x = \mathtt{sgn}(x)T_{x,x}$, where

$$\mathtt{sgn}(x) = \begin{cases} 1, & \text{if } [x, x] \geqslant 0, \\ -1, & \text{otherwise.} \end{cases}$$

Next we list some properties of $T_{x,y}$. Let $x, y, z \in H$.

Lemma 2.5. *$T_{x,y}$ is a linear and a rank one operator.*

Lemma 2.6. *$T_{\alpha x, \beta y} = \alpha \bar{\beta} T_{x,y}$, where α, β are scalars.*

Lemma 2.7. $T_{x,y}^{[*]} = T_{y,x}$.

Lemma 2.8. *If* $[x,x]=\pm 1$*, then* P_x *is an orthogonal projection.*

Proof. We have for all $u \in H$,

$$P_x(u) = \mathtt{sgn}(x)[u,x]x = \mathtt{sgn}(x)[u,x]P_x(x) = P_x^2(u).$$

It is easy to verify that $\mathcal{R}(P_x)$ and $\mathcal{N}(P_x)$ are orthogonal. Thus by Lemma 2.2, $P_x = P_x^{[*]}$. □

Lemma 2.9. *If* P *is an orthogonal projection of rank* 1*, then there exists* $x \in H$ *with* $[x,x]=\pm 1$ *such that* $P = P_x$.

Proof. If P is an orthogonal projection of rank 1, then there exists nonzero $x \in H$ such that $\mathcal{R}(P) = \operatorname{span}(\{x\})$. By Lemma 2.1, $[x,x] \neq 0$. So x can be normalized. Let $u \in H$. Then $P(u) = \alpha x$ where $\alpha = \mathtt{sgn}(x)[P(u),x]$. Thus

$$P(u) = \mathtt{sgn}(x)\big[P(u),x\big]x = \mathtt{sgn}(x)\big[u,P(x)\big]x = \mathtt{sgn}(x)[u,x]x = P_x(u).$$

Thus $P = P_x$. This completes the proof. □

Lemma 2.10. *Let* $[x,x]=\pm 1$ *and* $[y,y]=\pm 1$. *Then*

$$x \text{ and } y \text{ are orthogonal} \quad \Leftrightarrow \quad P_xP_y = P_yP_x = \mathbf{0}.$$

Proof. Follows from Lemmas 2.3 and 2.8. □

Lemma 2.11.

(i) *If* $[x,x]=\pm 1$ *then* $T_{x,z}T_{y,x} = \mathtt{sgn}(x)[y,z]P_x$.
(ii) *If* $[z,z]=\pm 1$ *then* $T_{x,z}T_{z,y} = \mathtt{sgn}(z)T_{x,y}$. *If in addition* $[x,x]=\pm 1$ *then* $T_{x,z}T_{z,x} = \mathtt{sgn}(x)\mathtt{sgn}(z)P_x$.

Proof. Let $u \in H$. Then

$$T_{x,z}T_{y,x}(u) = [u,x][y,z]x = [y,z]T_{x,x}(u) = \mathtt{sgn}(x)[y,z]P_x,$$

proving (i). Next, we have

$$\begin{aligned} T_{x,z}T_{z,y}(u) &= [u,y]T_{x,z}(z) \\ &= [u,y][z,z]x \\ &= \mathtt{sgn}(z)[u,y]x \\ &= \mathtt{sgn}(z)T_{x,y}(u). \end{aligned}$$

If in addition, $[x,x]=\pm 1$ then substituting $y=x$ in the above, we get $T_{x,z}T_{z,x} = \mathtt{sgn}(z)T_{x,x} = \mathtt{sgn}(x)\mathtt{sgn}(z)P_x$. □

Lemma 2.12. *Let* T *be a linear operator of rank* 1. *Then there exist nonzero* $x, y \in H$ *such that* $T = T_{x,y}$.

Proof. Since T is a linear operator of rank 1, then $\mathcal{R}(T) = \operatorname{span}(\{x\})$ for some nonzero x. If $[x,x]=\pm 1$, choose $y = \mathtt{sgn}(x)T^{[*]}(x)$. Then for any u, $T(u) = \mathtt{sgn}(x)[T(u),x]x = [u,y]x = T_{x,y}(u)$. Suppose $[x,x]=0$. For any u, $T(u)=\alpha x$ for some scalar α. Choose y such that $[u,y]=\alpha$. Then $T(u) = [u,y]x = T_{x,y}(u)$. □

3. Main results

In this section we define a $[*]$-isomorphism in an indefinite inner product space and prove the main result (Corollary 3.5) which characterizes all $[*]$-isomorphisms in an IIPS. Let $L(H)$ denote the space of all linear operators on the indefinite inner product space H.

Definition 3.1. Let H_1, H_2 be indefinite inner product spaces over $\mathbb{R}$ or $\mathbb{C}$. A linear map Φ between vector spaces $L(H_1)$ and $L(H_2)$ is called an isomorphism if it is one–one and $\Phi(TS) = \Phi(T)\Phi(S)$ for all $T, S \in L(H_1)$. An isomorphism Φ on $L(H_1)$ is called a $[*]$-isomorphism if $\Phi(T^{[*]}) = (\Phi(T))^{[*]}$ for all $T \in L(H_1)$.

Lemma 3.2. *Let Φ be a $[*]$-isomorphism from $L(H_1)$ onto $L(H_2)$ and $x \in H_1$. If $[x, x] = \pm 1$ then $\Phi(P_x)$ is an orthogonal projection of rank* 1.

Proof. Since $[x, x] = \pm 1$, P_x is an orthogonal projection, by Lemma 2.8. It is easy to prove that $\Phi(P_x)$ is an orthogonal projection. By Lemma 2.1, $[x, x] \neq 0$ for all nonzero $x \in \mathcal{R}(\Phi(P_x))$. The rest of the proof is similar to the Euclidean case [2, Step 1]. □

Lemma 3.3. *For each $x \in H_1$ with $[x, x] = \pm 1$, there exists $\tilde{x} \in H_2$ with $[\tilde{x}, \tilde{x}] = \pm 1$ such that $\Phi(P_x) = P_{\tilde{x}}$.*

Proof. By Lemma 3.2, $\Phi(P_x)$ is an orthogonal projection of rank 1. By Lemma 2.9, there exists $\tilde{x} \in H_2$ with $[\tilde{x}, \tilde{x}] = \pm 1$ such that $\Phi(P_x) = P_{\tilde{x}}$. □

Theorem 3.4. *Let H_1 and H_2 be indefinite inner product spaces. If Φ is a $[*]$-isomorphism from $L(H_1)$ onto $L(H_2)$ then there exists a linear operator $U : H_1 \to H_2$ such that $\Phi(T) = cUTU^{[*]}$ for all rank* 1 *operators $T \in L(H_1)$, with $UU^{[*]} = cI_2$ and $U^{[*]}U = cI_1$, where $c = \pm 1$. Here I_1, I_2 denote the identity maps on H_1, H_2, respectively.*

Proof. Fix $x_0 \in H_1$ with $[x_0, x_0] = \pm 1$. Then by Lemma 3.3, there exists $\tilde{x}_0 \in H_2$ such that $\Phi(P_{x_0}) = P_{\tilde{x}_0}$ and $[\tilde{x}_0, \tilde{x}_0] = \pm 1$. Define $U : H_1 \to H_2$ by

$$U(y) = \Phi(T_{y,x_0})(\tilde{x}_0).$$

It is easy to check that U is linear and

$$U(\alpha x_0) = \alpha\, \mathtt{sgn}(x_0)\tilde{x}_0, \tag{3.1}$$

for every scalar α. Also,

$$\begin{aligned}
\big[U(y), U(z)\big] &= \big[\Phi(T_{y,x_0})\tilde{x}_0, \Phi(T_{z,x_0})\tilde{x}_0\big] \\
&= \big[\big(\Phi(T_{z,x_0})\big)^{[*]}\Phi(T_{y,x_0})\tilde{x}_0, \tilde{x}_0\big] \\
&= \big[\Phi\big(T_{z,x_0}^{[*]}\big)\Phi(T_{y,x_0})\tilde{x}_0, \tilde{x}_0\big] \\
&= \big[\Phi(T_{x_0,z}T_{y,x_0})\tilde{x}_0, \tilde{x}_0\big] \\
&= \mathtt{sgn}(x_0)[y, z]\big[\Phi(P_{x_0})\tilde{x}_0, \tilde{x}_0\big] \\
&= \mathtt{sgn}(x_0)[y, z][\tilde{x}_0, \tilde{x}_0] \\
&= \mathtt{sgn}(x_0)\mathtt{sgn}(\tilde{x}_0)[y, z].
\end{aligned}$$

Thus $U^{[*]}U = cI_1$ and $UU^{[*]} = cI_2$, where $c = \mathtt{sgn}(x_0)\mathtt{sgn}(\tilde{x}_0)$. It follows that

$$U^{[*]}U\big(\alpha\,\mathtt{sgn}(x_0)x_0\big) = c\alpha\,\mathtt{sgn}(x_0)x_0.$$

Substituting the expression for $U(\alpha x_0)$ from Eq. (3.1), we obtain

$$U^{[*]}(\alpha x_0) = c\alpha\,\mathtt{sgn}(x_0)x_0. \tag{3.2}$$

Next we prove $\Phi(T) = cUTU^{[*]}$ for all rank 1 linear operators T on H_1. First we prove this for $T = T_{x_0,y}$ for $y \in H_1$. Let $u, y \in H_1$. Then

$$\begin{aligned}\Phi(T_{x_0,y})U(u) &= \Phi(T_{x_0,y})\Phi(T_{u,x_0})\tilde{x}_0\\ &= \Phi(T_{x_0,y}T_{u,x_0})\tilde{x}_0\\ &= \mathtt{sgn}(x_0)[u, y]\Phi(P_{x_0})(\tilde{x}_0)\\ &= \mathtt{sgn}(x_0)[u, y]\tilde{x}_0.\end{aligned}$$

Now, $T_{x_0,y}(u) = [u, y]x_0$ implies

$$\begin{aligned}U(T_{x_0,y})(u) &= \Phi\big([u, y]T_{x_0,x_0}\big)\tilde{x}_0\\ &= [u, y]\mathtt{sgn}(x_0)\Phi(P_{x_0})(\tilde{x}_0)\\ &= \mathtt{sgn}(x_0)[u, y]\tilde{x}_0.\end{aligned}$$

Thus $\Phi(T_{x_0,y})U = UT_{x_0,y}$. So we have

$$\Phi(T_{x_0,y}) = cUT_{x_0,y}U^{[*]}.$$

If T is a rank one linear operator, then by Lemma 2.12, there exist x and y such that $T = T_{x,y}$. Then

$$\begin{aligned}\Phi(T) &= \Phi(T_{x,y})\\ &= \Phi\big(\mathtt{sgn}(x_0)T_{x,x_0}T_{x_0,y}\big)\\ &= \mathtt{sgn}(x_0)\Phi\big(T^{[*]}_{x_0,x}\big)\Phi(T_{x_0,y})\\ &= \mathtt{sgn}(x_0)\big\{\Phi(T_{x_0,x})\big\}^{[*]}\Phi(T_{x_0,y})\\ &= c\,\mathtt{sgn}(x_0)\big\{UT_{x_0,x}U^{[*]}\big\}^{[*]}\big\{UcT_{x_0,y}U^{[*]}\big\}\\ &= c\,\mathtt{sgn}(x_0)UT^{[*]}_{x_0,x}T_{x_0,y}U^{[*]}\\ &= c\,\mathtt{sgn}(x_0)UT_{x,x_0}T_{x_0,y}U^{[*]}\\ &= cUT_{x,y}U^{[*]}\\ &= cUTU^{[*]}.\end{aligned}$$

In the above, the second equation follows from Lemma 2.11. Thus

$$\Phi(T) = cUTU^{[*]}$$

for all rank one operators T. This completes the proof. □

Corollary 3.5. *Let H_1 and H_2 be indefinite inner product spaces. If Φ is a $[*]$-isomorphism from $L(H_1)$ onto $L(H_2)$ then there exists a linear operator $U : H_1 \to H_2$ such that $\Phi(T) = cUTU^{[*]}$ for all $T \in L(H_1)$ with $UU^{[*]} = cI_2$ and $U^{[*]}U = cI_1$, where $c = \pm 1$. Moreover, U is unique up to a scalar multiple of absolute value* 1.

Proof. Let $x_0 \in H_1$ such that $[x_0, x_0] = \pm 1$. It is clear that TT_{v,x_0} is a rank 1 operator for any $v \in H_1$. Then by Theorem 3.4, there exists U such that $UU^{[*]} = cI_2$, $U^{[*]}U = cI_1$ and

$$\Phi(TT_{v,x_0}) = cUTT_{v,x_0}U^{[*]}.$$

Then

$$\begin{aligned}\Phi(T)U(v) &= \Phi(T)\Phi(T_{v,x_0})\tilde{x}_0\\ &= \Phi(TT_{v,x_0})(\tilde{x}_0)\\ &= cUTT_{v,x_0}U^{[*]}(\tilde{x}_0)\\ &= cUTT_{v,x_0}\big(c\,\mathtt{sgn}(x_0)x_0\big)\\ &= UT(v),\end{aligned}$$

where the fourth equation follows from Eq. (3.2). Thus $\Phi(T) = cUTU^{[*]}$ (for all $T \in L(H_1)$). Uniqueness is similar to the Euclidean case. □

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