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Two moonshines for $L_2(11)$ but none for M_{12}

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Abstract

In this paper, we revisit an earlier conjecture by one of us that related conjugacy classes of M_{12} to Jacobi forms of weight zero and index one. We construct Jacobi forms for all conjugacy classes of M_{12} that are consistent with constraints from group theory as well as modularity. However, we obtain 1427 solutions that satisfy these constraints (to the order that we checked) and are unable to provide a unique Jacobi form. Nevertheless, as a consequence, we are able to provide a group theoretic proof of the evenness of the coefficients of all EOT Jacobi forms associated with conjugacy classes of $M_{12}:2 \subset M_{24}$. We show that there exists no solution where the Jacobi forms (for order 4/8 elements of M_{12}) transform with phases under the appropriate level. In the absence of a moonshine for M_{12}, we show that there exist moonshines for two distinct $L_2(11)$ sub-groups of the M_{12}. We construct Siegel modular forms for all $L_2(11)$ conjugacy classes and show that each of them arises as the denominator formula for a distinct Borcherds–Kac–Moody Lie superalgebra.

1. Introduction

Following the discovery of monstrous moonshine, came a moonshine for the largest sporadic Mathieu group, M_{24}. This related ρ, a conjugacy class of M_{24}, to a multiplicative eta product that we denote by η_ρ via the map [1,2]:

$$\rho = 1^{a_1} 2^{a_2} \cdots N^{a_N} \longrightarrow \eta_\rho(\tau) := \prod_{m=1}^{N} \eta(m\tau)^{a_m} \quad . \tag{1.1}$$

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The same multiplicative eta products appeared as the generating function of $\frac{1}{2}$-BPS states twisted by a symmetry element (in the conjugacy class ρ) in type II string theory compactified on $K3 \times T^2$. This was further extended to the generating function of $\frac{1}{4}$-BPS states in the same theory. The generating function in this case was a genus-two Siegel modular form that we denote by $\Phi^\rho(\mathbf{Z})$ [3].

Renewed interest in this moonshine (now called *Mathieu moonshine*) appeared following the work of Eguchi–Ooguri–Tachikawa (EOT) who observed the appearance of the dimensions of irreps of M_{24} in the elliptic genus of $K3$ when expanded in terms of characters of the $\mathcal{N}=4$ superconformal algebra [4]. The Siegel modular form $\Phi^\rho(\mathbf{Z})$, when it exists, unifies the two Mathieu moonshines, the one related to multiplicative eta products as well as the one related to the elliptic genus [5,6].

It has now been established that there is a moonshine that relates conjugacy classes of M_{24} to (the EOT) Jacobi forms of weight zero and index one that we denote by $Z^\rho(\tau,z)$ [7–9]. These Jacobi forms arise as twistings (also called 'twinings') of the elliptic genus of $K3$. Given a conjugacy class ρ of M_{24}, the associated Jacobi form expressed in terms of $\mathcal{N}=4$ characters: the massless character (with zero isospin) $\mathcal{C}(\tau,z)$ and the massive characters $q^{h-\frac{1}{8}}\,\mathcal{B}(\tau,z)$ (with $h\geq 0$) takes the form [10,11]

$$Z^\rho(\tau,z) = \alpha^\rho\, \mathcal{C}(\tau,z) + q^{-\frac{1}{8}}\, \Sigma^\rho(\tau)\, \mathcal{B}(\tau,z)\ , \tag{1.2}$$

where $\alpha^\rho = 1+\chi_{23}(\rho)$ and the character expansion of the function $\Sigma^\rho(\tau)$ is as follows:

$$\begin{aligned}\Sigma^\rho(\tau) = -2 &+ [\chi_{45}(\rho)+\chi_{\overline{45}}(\rho)]\,q + [\chi_{231}(\rho)+\chi_{\overline{231}}(\rho)]\,q^2 \\ &+ [\chi_{770}(\rho)+\chi_{\overline{770}}(\rho)]\,q^3 + 2\chi_{2277}(\rho)\,q^4 + 2\chi_{5796}(\rho)\,q^5+\cdots\end{aligned} \tag{1.3}$$

where the subscript denotes the dimension of the irrep of M_{24} and $q=e^{2\pi i\tau}$. An all-orders proof of the existence of such an expansion has been given by Gannon [12]. In particular, $\Phi^\rho(\mathbf{Z})$ can be constructed in two ways: an additive lift which uses the eta product $\eta_\rho(\tau)$ as input and a multiplicative lift where $Z^\rho(\tau,z)$ is the input. However, the additive lift is not known for all conjugacy classes.

In some cases, the square-root of $\Phi^\rho(\mathbf{Z})$ is related to a Borcherds–Kac–Moody (BKM) Lie superalgebra with the additive and multiplicative lifts providing the sum and product side of the Weyl denominator formula. An attempt at understanding this square-root was done in [13] where it was argued that there might be a moonshine involving the Mathieu group M_{12} relating its conjugacy classes to BKM Lie superalgebras. In this paper, we revisit that proposal from several viewpoints. Our results may be summarised as follows:

1. We study a Conjecture 2.1 (due to one of us) that implies a moonshine for M_{12} that provides Jacobi forms of weight zero and index 1 for every conjugacy class of M_{12}. We find 1427 families of Jacobi forms that have a positive definite character expansion. This result, albeit non-unique, is sufficient to show that all the Fourier coefficients of the EOT Jacobi forms for M_{24} conjugacy classes that reduce to conjugacy classes of $M_{12}{:}2$ (a maximal sub-group of M_{24}) are even. This provides an alternate group-theoretic proof of a result due to Creutzig et al. [14].
2. In an attempt to obtain a unique solution, we introduce a stronger form of the conjecture (Proposition 2.4) – this imposes a condition that the Jacobi forms transform with suitable phases under an appropriate level. For conjugacy classes $4a/4b/8a/8b$, we find **no** solutions, thereby concluding that there is no moonshine for M_{12}.

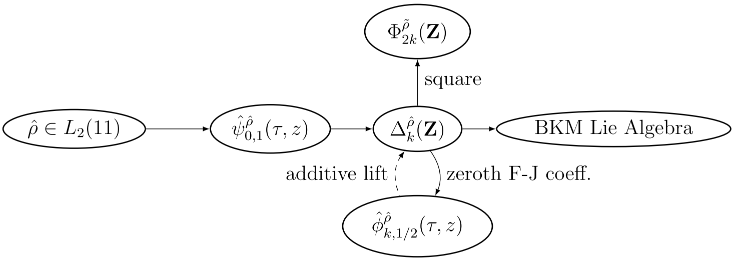

Fig. 1. Moonshines for $L_2(11)$: $\hat{\rho}$ is an $L_2(11)_{A/B}$ conjugacy class.

3. We address the non-existence of a moonshine for M_{12} by providing two moonshines for $L_2(11)$ that arise as two distinct sub-groups of M_{12}. Using this, we construct Siegel modular forms for all conjugacy classes using a product formula that arises naturally as a consequence of $L_2(11)$ moonshine. The modularity of this product is proven in two ways: (i) as an additive lift determined by the eta product, $\eta_{\hat{\rho}}(\tau)$ – this works for all but three conjugacy classes ($1^1 11^1$, 3^4 and 6^2), and (ii) as a product of rescaled Borcherds products – this works for all conjugacy classes.
4. For all conjugacy classes of $L_2(11)$, we show the existence of Borcherds–Kac–Moody (BKM) Lie superalgebras for conjugacy classes of both the $L_2(11)$ by showing that the Siegel modular forms, $\Delta_k^{\hat{\rho}}(\mathbf{Z})$, arise as their Weyl–Kac–Borcherds denominator identity. Fig. 1 pictorially summarises the moonshines for $L_2(11)$.

The plan of the manuscript is as follows. Following the introductory section, in section 2, we describe the Conjecture 2.1 for M_{12} moonshine. We find multiple solutions that satisfy all the constraints that are imposed. In Proposition 2.4, we show that a stronger form of M_{12} moonshine does not hold. We then show that there are unique solutions for two distinct $L_2(11)$ subgroups of M_{12}. In section 3, we construct genus-two Siegel modular forms for all conjugacy classes for both $L_2(11)$ moonshines using a multiplicative lift. In section 4, we show that each of these Siegel modular forms arises as the Weyl–Kac–Borcherds denominator formula for a BKM Lie superalgebra. We conclude with brief remarks in section 5. Appendix A contains various definitions and details of modular forms, Jacobi forms and Siegel modular forms. Appendix B provides some details of the computations that prove modularity of the multiplicative lift. Finally, Appendix C contains details of (finite) group theory that are relevant for this paper.

Notation

ρ	Conjugacy class of M_{24}
$\tilde{\rho}$	Conjugacy class of $M_{12}:2$
$\hat{\rho}$	Conjugacy class of M_{12} and $L_2(11)$
ϱ	Weyl vector for a BKM Lie superalgebra
$\hat{\rho}_m$	Conjugacy class of the m-th power of an element of conjugacy class $\hat{\rho}$
$\eta_\rho(\tau)$	Eta product for cycle shape ρ
$Z^\rho(\tau, z)$	EOT Jacobi form for conjugacy class ρ of M_{24}
$\widehat{\psi}^{\hat{\rho}}(\tau, z)$	Jacobi form of weight zero and index one for conjugacy class $\hat{\rho}$ of M_{12}
$\phi_{k,1/2}^{\hat{\rho}}(\tau, z)$	Additive seed corresponding to M_{12} conjugacy class $\hat{\rho}$
$\Phi_k^{\rho}(\mathbf{Z})$	Siegel modular form of weight k for conjugacy class ρ of M_{24}
$\Delta_k^{\hat{\rho}}(\mathbf{Z})$	Siegel modular form of weight k for conjugacy class $\hat{\rho}$ of M_{12} and $L_2(11)$

2. The M_{12} conjecture

In some cases, the square-root of the Siegel modular form that unifies the additive and multiplicative moonshines for M_{24} is related to the denominator formula of a Borcherds–Kac–Moody (BKM) Lie superalgebra. A necessary condition is that the multiplicative seed have even coefficients. A group-theoretic answer to this question comes via $M_{12}{:}2$, a maximal subgroup of M_{24} that can be constructed from M_{12} and its outer automorphism. We denote characters and conjugacy classes of $M_{12}{:}2$ by $\widetilde{\chi}_i$ and $\widetilde{\rho}$ throughout this paper. Similarly, we will use a hat for M_{12} characters and conjugacy classes.

$M_{12}{:}2$ is a maximal subgroup of M_{24}. The construction of $M_{12}{:}2$ has its origin in the order two outer automorphism of M_{12} that is generated by an element that we denote by φ. There are two classes of elements, $(\hat{g}, e)$ and $(\hat{g}, \varphi)$, where $\hat{g} \in M_{12}$. The composition rule is given by ($\hat{g}_1, \hat{g}_2 \in M_{12}$ and $h \in (e, \varphi)$)

$$(\hat{g}_1, e) \cdot (\hat{g}_2, h) = (\hat{g}_1 \cdot \hat{g}_2, h) \quad , \quad (\hat{g}_1, \varphi) \cdot (\hat{g}_2, h) = (\hat{g}_1 \cdot \varphi(\hat{g}_2), \varphi \cdot h)$$

The existence of the decomposition of $Z^{\rho}(\tau, z)$ in terms of characters of M_{24} as in Eq. (1.2) immediately implies the following decomposition for all the 21 conjugacy classes of $M_{12}{:}2$.

$$Z^{\widetilde{\rho}}_{0,1}(\tau, z) = \alpha^{\widetilde{\rho}}\, \mathcal{C}(\tau, z) + q^{-\frac{1}{8}}\, \Sigma^{\widetilde{\rho}}(\tau)\, \mathcal{B}(\tau, z)\ , \tag{2.1}$$

where $\alpha^{\widetilde{\rho}} = 2 + \widetilde{\chi}_2(\widetilde{\rho}) + \widetilde{\chi}_3(\widetilde{\rho})$ and the function $\Sigma^{\widetilde{\rho}}(\tau)$ can be expanded in terms of characters of $M_{12}{:}2$ as follows[1] :

$$\Sigma^{\widetilde{\rho}}(\tau) = -2 + \sum_{n=1}^{\infty} \left(\sum_{a=1}^{21} \widetilde{N}_a(n) \widetilde{\chi}_a(\widetilde{\rho}) \right) q^n\ , \tag{2.2}$$

where the multiplicities $\widetilde{N}_a(n)$ are non-negative integers.

Characters of representations of $M_{12}{:}2$ arise in two ways from characters of M_{12}. For representations of *splitting* type, a pair of $M_{12}{:}2$ characters are determined by a single M_{12} character and for representations of *fusion* type, a $M_{12}{:}2$ character is determined by a couple of M_{12} characters.

Rep. type	Conjugacy class	
	(g, e)	(g, φ)
Splitting	$\tilde{\chi}_a = \tilde{\chi}_{a'} = \hat{\chi}_m$	$\tilde{\chi}_a + \tilde{\chi}_{a'} = 0$
Fusion	$\tilde{\chi}_a = \hat{\chi}_m + \hat{\chi}_{m'}$	$\tilde{\chi}_a = 0$

Let $\hat{\rho}$ denote a conjugacy class of M_{12} associated with $\hat{g} \in M_{12}$. Then, the pair of conjugacy classes $(\hat{\rho}, \varphi(\hat{\rho}))$ become a conjugacy class $\tilde{\rho}$ associated with the element $(\hat{g}, e)$ of $M_{12}{:}2$ and hence of M_{24} as well. 12 of the 21 conjugacy classes of $M_{12}{:}2$ arise in this fashion.

Conjecture 2.1 *(Govindarajan [13]). There exists a moonshine for M_{12} that associates a unique weight zero, index one real Jacobi form $\widehat{\psi}^{\hat{\rho}}_{0,1}$ to every conjugacy class $\hat{\rho}$ of M_{12} such that*

[1] We indicate conjugacy classes and other objects related to $M_{12}{:}2$ with a tilde and a hat for M_{12}. In addition, characters of $M_{12}{:}2$ are labelled with the beginning letters of the alphabet while letters beginning from m are used for characters of M_{12}.

1. *$Z^{\tilde{\rho}}(\tau,z)=\widehat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)+\widehat{\psi}^{\varphi(\hat{\rho})}_{0,1}(\tau,z)$, where $\tilde{\rho}$ is the conjugacy class of the element $(g,e)\in M_{12}{:}2$, $g\in M_{12}$ is in the conjugacy class $\hat{\rho}$ and $Z^{\tilde{\rho}}_{0,1}(\tau,z)$ is the EOT Jacobi form that appears in the moonshine for M_{24}.*
2. *The Jacobi form written in terms of $\mathcal{N}=4$ characters*

$$\widehat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)=\hat{\alpha}^{\hat{\rho}}\ \mathcal{C}(\tau,z)+q^{-\frac{1}{8}}\ \hat{\Sigma}^{\hat{\rho}}(\tau)\ \mathcal{B}(\tau,z)\ ,$$

where $\hat{\alpha}^{\hat{\rho}}$ and $\hat{\Sigma}^{\hat{\rho}}(\tau)$ can be expressed in terms of M_{12} characters. One has $\hat{\alpha}^{\hat{\rho}}=1+\hat{\chi}_2(\hat{\rho})$ and

$$\hat{\Sigma}^{\hat{\rho}}(\tau)=-1+\sum_{n=1}^{\infty}\left(\sum_{m=1}^{15}\hat{N}_m(n)\hat{\chi}_m(\hat{\rho})\right)q^n\ , \tag{2.3}$$

with $\hat{N}_m(n)$ being non-negative integers for all $m\geq 1$ and $n\geq 1$.

2.1. Implications of Conjecture 2.1

We will later see that Conjecture 2.1 holds only after one relaxes the uniqueness condition on the Jacobi forms. We will now study its implications.

Proposition 2.2. *For M_{12} conjugacy classes $\hat{\rho}\neq 4a/4b/8a/8b$*

$$\widehat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)=\frac{1}{2}\ Z^{\tilde{\rho}}(\tau,z)\ .$$

Remark. It is easy to verify that $\alpha^{\hat{\rho}}=\alpha^{\varphi(\hat{\rho})}$ for these conjugacy classes. Thus it suffices to show that $\hat{\Sigma}^{\hat{\rho}}=\hat{\Sigma}^{\varphi(\hat{\rho})}$. A complete list of EOT Jacobi forms that are related to conjugacy classes of $M_{12}{:}2$ is given in Table 2.

Proof. For M_{12} conjugacy classes $\hat{\rho}$ such that $\varphi(\hat{\rho})=\hat{\rho}$, Conjecture 2.1 implies that $\widehat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)=\widehat{\psi}^{\varphi(\hat{\rho})}_{0,1}(\tau,z)$ and hence they are given by half the corresponding $M_{12}{:}2$ Jacobi form. This excludes the conjugacy classes $4a/4b/8a/8b/11a/11b$. The M_{12} characters $\hat{\chi}_4$ and $\hat{\chi}_5$ take complex values for the conjugacy classes $11a/b$. The reality condition on the Jacobi forms (in Conjecture 2.1) requires that the multiplicities that appear in Eq. (2.3) must be such that

$$\hat{N}_4(n)=\hat{N}_5(n)\text{ for all }n\geq 1\ .$$

Further, it follows that $\widehat{\psi}^{11a}_{0,1}(\tau,z)=\widehat{\psi}^{11b}_{0,1}(\tau,z)$. □

Remark. There is a proposal due to Eguchi and Hikami where they propose a moonshine called 'Enriques' moonshine [15]. For the four classes that are undetermined, they propose to take one half of the M_{24} Jacobi form as the required Jacobi form. That is not consistent with the decomposition in the expression for $\hat{\alpha}^{\hat{\rho}}$ as it implies that $\hat{\alpha}^{\hat{\rho}}=1+\frac{1}{2}(\hat{\chi}_2+\hat{\chi}_3)$ – the coefficients are half integral and do not make sense from group theory. The outer automorphism of M_{12} that we make extensive use of also does not make an appearance in their considerations. We believe that their proposal in not related to our work.

Proposition 2.3. *For all conjugacy classes of $M_{12}{:}2$, the Fourier–Jacobi coefficients of the associated Jacobi form are even i.e.,*

$$Z^{\widetilde{\rho}}(\tau) = 0 \mod 2\ .$$

Remark. This follows from a theorem of Creutzig, Höhn and Miezaki [14] where they show that the $Z^{\rho}(\tau, z)$ for M_{24} conjugacy classes ($7a/b$, $14a/b$, $15a/b$, $23a/b$) have some odd coefficients. These are precisely the conjugacy classes of M_{24} that do not reduce to conjugacy classes of $M_{12}{:}2$. We provide an alternate proof assuming that the M_{12} conjecture holds.

Proof. It is useful to rewrite the above character decomposition taking into account the outer automorphism φ of M_{12}. With this in mind, we define

$$\hat{N}_2^{\pm}(n) = (\hat{N}_2(n) \pm \hat{N}_3(n)) \quad , \quad \hat{N}_9^{\pm}(n) = (\hat{N}_9(n) \pm \hat{N}_{10}(n))\ ,$$
$$\hat{\chi}_2^{\pm}(\hat{\rho}) = (\hat{\chi}_2(\hat{\rho})) \pm \hat{\chi}_3(\hat{\rho}))) \quad , \quad \hat{\chi}_9^{\pm}(\hat{\rho})) = (\hat{\chi}_9(\hat{\rho})) \pm \hat{\chi}_{10}(\hat{\rho}))\ .$$

Note that $\hat{\chi}_2^{\pm}(\varphi(\hat{\rho})) = \pm\hat{\chi}_2^{\pm}(\hat{\rho})$ and $\hat{\chi}_9^{\pm}(\varphi(\hat{\rho})) = \pm\hat{\chi}_9^{\pm}(\hat{\rho})$ by construction. We can then write the character decomposition as follows

$$\hat{\Sigma}^{\hat{\rho}} = -1 + \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{m=1\\ m\neq 2,3,9,10}}^{15} \hat{N}_m(n)\hat{\chi}_m(\hat{\rho}) + \tfrac{1}{2}\hat{N}_2^{+}(n)\hat{\chi}_2^{+}(\hat{\rho}) + \tfrac{1}{2}\hat{N}_9^{+}(n)\hat{\chi}_9^{+}(\hat{\rho})$$
$$+ \tfrac{1}{2}\hat{N}_2^{-}(n)\hat{\chi}_2^{-}(\hat{\rho}) + \tfrac{1}{2}\hat{N}_9^{-}(n)\hat{\chi}_9^{-}(\hat{\rho})\Bigg)q^n\ . \quad (2.4)$$

Since the $\mathcal{N} = 4$ characters have integer coefficients, it suffices to show that $\alpha^{\widetilde{\rho}} = 0 \mod 2$ and $\Sigma^{\widetilde{\rho}} = 0 \mod 2$ for all conjugacy classes.

Part 1: $\alpha^{\widetilde{\rho}} = 1 + \widetilde{\chi}_2(\widetilde{\rho}) + \widetilde{\chi}_3(\widetilde{\rho}) = 0 \mod 2$ for all conjugacy classes as can be explicitly checked.

Part 2: Consider the $M_{12}{:}2$ conjugacy classes that arise from elements of type (g, e). For such cases,

$$\Sigma^{\widetilde{\rho}}(\tau) = \hat{\Sigma}^{\hat{\rho}} + \hat{\Sigma}^{\varphi(\hat{\rho})}\ ,$$
$$= -2 + \sum_{n=1}^{\infty}\Bigg(\Bigg[\sum_{\substack{m=1\\ m\neq 2,3,9,10}}^{15} 2\hat{N}_m(n)\hat{\chi}_m(\hat{\rho})\Bigg] + \hat{N}_2^{+}(n)\hat{\chi}_2^{+}(\hat{\rho}) + \hat{N}_9^{+}(n)\hat{\chi}_9^{+}(\hat{\rho})\Bigg)q^n.$$

Comparing the above equation with Eq. (2.2), we obtain

$$2\hat{N}_m(n) = \tilde{N}_a(n) + \tilde{N}_{a'}(n) \text{ for splitting characters}\ , \quad (2.5)$$

and for the fusion type,

$$\hat{N}_2^{+}(n) = \tilde{N}_3(n) \quad , \quad \hat{N}_9^{+}(n) = \tilde{N}_{11}(n) \quad , \quad 2\hat{N}_4(n) = 2\hat{N}_5(n) = \tilde{N}_4(n)\ .$$

Next, considering the expression for $\Sigma^{\widetilde{\rho}}(\tau)$ above modulo 2, we obtain

$$\Sigma^{\widetilde{\rho}}(\tau) = \hat{N}_2^{+}(n)\hat{\chi}_2^{+}(\hat{\rho}) + \hat{N}_9^{+}(n)\hat{\chi}_9^{+}(\hat{\rho}) \mod 2\ . \quad (2.6)$$

Further, one has $\hat{\chi}_2^{+}(\hat{\rho}) = \hat{\chi}_9^{+}(\hat{\rho}) = 0 \mod 2$ as one can explicitly check. Thus, for conjugacy classes for elements of type (g, e), one has $\Sigma^{\widetilde{\rho}}(\tau) = 0 \mod 2$.

Part 3: Now consider conjugacy classes of $M_{12}{:}2$ associated with elements of type (g,φ). For these conjugacy classes, the character for all fusion representations vanish. Thus,

$$\begin{aligned}\Sigma^{\widetilde{\rho}}(\tau) &= -2+\sum_{n=1}^{\infty}\left(\sum_{a\in\text{splitting}}\widetilde{N}_a(n)\widetilde{\chi}_a(\widetilde{\rho})\right)q^n \\ &= -2+\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{\text{pairs}\\(a,a')\in\text{splitting}}}\big(\widetilde{N}_a(n)\widetilde{\chi}_a(\widetilde{\rho})+\widetilde{N}_{a'}(n)\widetilde{\chi}_{a'}(\widetilde{\rho})\big)\Bigg)q^n \\ &= -2+\sum_{n=1}^{\infty}\left(\sum_{\substack{\text{pairs}\\(a,a')\in\text{splitting}}}(\widetilde{N}_a(n)-\widetilde{N}_{a'}(n))\widetilde{\chi}_a(\widetilde{\rho})\right)q^n\ ,\end{aligned}$$

where we have used the relation $\widetilde{\chi}_a+\widetilde{\chi}_{a'}=0$ for all pairs of splitting representations. The characters for the pairs (7, 8) and (16, 17) are irrational for these conjugacy classes. For such pairs, rationality (and hence integrality) of the Fourier–Jacobi coefficients of the EOT Jacobi forms implies that $(\widetilde{N}_a(n)-\widetilde{N}_{a'}(n))=0$ for $(a,a')=(7,8),(16,17)$. For all other pairs, equation (2.5) implies the weaker condition, $(\widetilde{N}_a(n)-\widetilde{N}_{a'}(n))=0 \mod 2$. Thus, for conjugacy classes for elements of type (g,φ), one has $\Sigma^{\widetilde{\rho}}(\tau)=0 \mod 2$. □

2.2. *Checking the M_{12} conjecture*

We have seen in Proposition 2.2 that for M_{12} conjugacy classes $\widehat{\rho}\neq 4a/4b/8a/8b$, one has

$$\widehat{\psi}^{\widehat{\rho}}(\tau,z)=\frac{1}{2}Z^{\rho}(\tau,z)\ , \tag{2.7}$$

where $Z^{\rho}(\tau,z)$ is the EOT Jacobi form for the M_{24} conjugacy class $\rho=(\widehat{\rho})^2$ i.e., $\rho=1^{2a_1}2^{2a_2}\cdots N^{2a_N}$ if $\widehat{\rho}=1^{a_1}2^{a_2}\cdots N^{a_N}$. That leaves four undetermined Jacobi forms associated with the classes $4a/4b$ and $8a/8b$. In these cases, we have the relation that relates the sums of Jacobi form of the two M_{12} conjugacy classes to EOT Jacobi forms.

$$Z^{4b}(\tau,z)=\widehat{\psi}^{\widehat{4a}}(\tau,z)+\widehat{\psi}^{\widehat{4b}}(\tau,z) \tag{2.8}$$

$$Z^{8b}(\tau,z)=\widehat{\psi}^{\widehat{8a}}(\tau,z)+\widehat{\psi}^{\widehat{8b}}(\tau,z)\ . \tag{2.9}$$

Thus it suffices to determine $\widehat{\psi}^{\widehat{4a}}(\tau,z)$ and $\widehat{\psi}^{\widehat{8a}}(\tau,z)$ to obtain Jacobi forms for all conjugacy classes. These two examples have vanishing twisted Witten index as the corresponding cycle shapes have no one-cycles. Thus, for $\widehat{\rho}=4a$ and $8a$, one has

$$\widehat{\psi}^{\widehat{\rho}}(\tau,z)=\gamma^{\widehat{\rho}}(\tau)\,\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\ , \tag{2.10}$$

where $\gamma^{\widehat{\rho}}(\tau)$ is a weight two modular form of suitable subgroup of $SL(2,\mathbb{Z})$.

Constraints

There are two kinds of constraints that we impose on the weight two modular forms.

1. **Non-negativity of coefficients in the character expansion:** We anticipate that all the Jacobi forms associated with the fifteen M_{12}-conjugacy classes admit a decomposition in terms of the fifteen M_{12} characters. The $\mathcal{N}=4$ decomposition (as carried out by Eguchi–Hikami [10,11]) implies

$$\widehat{\psi}^{\widehat{\rho}}_{0,1}(\tau,z) = \widehat{\alpha}^{\,\widehat{\rho}}\ \mathcal{C}(\tau,z) + q^{-\frac{1}{8}}\ \widehat{\Sigma}^{\,\widehat{\rho}}(\tau)\ \mathcal{B}(\tau,z)\ , \tag{2.11}$$

where $\widehat{\alpha}^{\,\hat{\rho}} = 1 + \widehat{\chi}_2(\hat{\rho})$ and

$$\widehat{\Sigma}^{\widehat{\rho}}(\tau) = -1 + \widehat{\chi}_6\, q + [\widehat{\chi}_8 + \widehat{\chi}_{15}]\, q^2 + [\widehat{\chi}_{11} + 2\,\widehat{\chi}_{13} + 2\,\widehat{\chi}_{14} + \widehat{\chi}_{15})]\, q^3 + \cdots$$

We have written out the first four terms in the character expansion of $\widehat{\Sigma}^{\widehat{\rho}}$ as there is no ambiguity arising from the undetermined conjugacy classes. The ambiguity arises from irreps that get exchanged by the outer automorphism of M_{12} – these correspond to the four characters $\widehat{\chi}_2 \leftrightarrow \widehat{\chi}_3$ and $\widehat{\chi}_9 \leftrightarrow \widehat{\chi}_{10}$. In particular, this implies that we know the first four terms in the q-series for $\gamma^{\widehat{4a}}$ and $\gamma^{\widehat{8a}}$. One has

$$\gamma^{\widehat{4a}} = 1 - 4q + 4q^2 + 4q^3 + \cdots \quad , \quad \gamma^{\widehat{8a}} = 1 - 2q - 2q^2 + 2q^3 + \cdots \tag{2.12}$$

The constraint from group theory is that coefficients in the character expansion of $\widehat{\Sigma}^{\widehat{\rho}}$ are all *non-negative* integers.

2. **Modularity:** The multiplicative eta products for the two classes $4a/8a$ are modular forms at level 16 and 64 respectively. From our experience with determining such examples for the M_{24} Jacobi forms for conjugacy classes with no one-cycles, these do provide a rough guide to determining the levels for our Jacobi forms. For the class $4a$, the first four terms already determined imply that the level must be larger than 8.

We find solutions where both conjugacy classes are determined by weight two modular forms at level 32.

2.3. *The conjugacy classes* $4a/8a$

Our non-unique proposal for the Jacobi forms for the conjugacy classes $4a/8a$ are as follows:

$$\begin{aligned}\widehat{\psi}^{\widehat{4a}}(\tau,z) &= \Big[\gamma_{4a}(\tau) + 4\alpha(\tau)\Big]\ \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\ ,\\ \widehat{\psi}^{\widehat{8a}}(\tau,z) &= \Big[\gamma_{8a}(\tau) - 2\alpha(\tau)\Big]\ \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\ ,\end{aligned} \tag{2.13}$$

where $\gamma_{4a}(\tau)$ and $\gamma_{8a}(\tau)$ are the following weight two modular forms of $\Gamma_0(32)$:

$$\begin{aligned}\gamma_{4a}(\tau) &:= -\tfrac{31}{192}E_2^{(2)}(\tau) + \tfrac{37}{64}E_2^{(4)}(\tau) - \tfrac{77}{48}E_2^{(8)}(\tau) + \tfrac{35}{16}E_2^{(16)}(\tau) - \tfrac{9}{4}f(\tau)\\ &\qquad - \tfrac{1}{2}\eta_{4^2 8^2}(\tau)\\ &= 1 - 4q + 4q^2 + 4q^3 - 4q^4 - 20q^5 + 16q^6 + 8q^7 + 24q^8 + \cdots\end{aligned} \tag{2.14}$$

$$\begin{aligned}\gamma_{8a}(\tau) &:= \tfrac{41}{384}E_2^{(2)}(\tau) - \tfrac{45}{128}E_2^{(4)}(\tau) + \tfrac{301}{192}E_2^{(8)}(\tau) - \tfrac{155}{32}E_2^{(16)}(\tau)\\ &\qquad + \tfrac{217}{48}E_2^{(32)}(\tau) + \tfrac{3}{8}f(\tau) + \tfrac{1}{2}f(2\tau) - \tfrac{13}{4}\eta_{4^2 8^2}(\tau)\\ &= 1 - 2q - 2q^2 + 2q^3 + 2q^4 + 14q^5 - 12q^6 + 4q^7 - 32q^8 + \cdots\ .\end{aligned} \tag{2.15}$$

The ambiguity in our solution is given by a modular form $\alpha(\tau)$ with integral coefficients and is parametrised by four integers (d_4, d_5, d_6, d_8).

$$\begin{aligned}\alpha(\tau) = \ & d_4(-\tfrac{1}{32}E_2^{(4)}(\tau) + \tfrac{7}{64}E_2^{(8)}(\tau) - \tfrac{5}{64}E_2^{(16)}(\tau)) \\ & + d_5(-\tfrac{1}{256}E_2^{(2)}(\tau) + \tfrac{1}{256}E_2^{(4)}(\tau) - \tfrac{1}{16}f(\tau) + \tfrac{1}{8}\eta_{4^2 8^2}(\tau)) \\ & + d_6(-\tfrac{1}{384}E_2^{(2)}(\tau) + \tfrac{3}{256}E_2^{(4)}(\tau) - \tfrac{7}{768}E_2^{(8)}(\tau) - \tfrac{1}{8}f(2\tau)) \\ & + d_8(-\tfrac{7}{192}E_2^{(8)}(\tau) + \tfrac{15}{128}E_2^{(16)}(\tau) - \tfrac{31}{384}E_2^{(32)}(\tau)) \\ = \ & d_4\, q^4 - d_5\, q^5 + d_6\, q^6 + d_8\, q^8 + \cdots .\end{aligned}$$

We can see that these four integers would be determined if we could fix either $\gamma^{\widehat{4a}}$ or $\gamma^{\widehat{8a}}$ to order q^8. Non-negativity of the multiplicities defined in Eq. (2.3) for the first few values of n leads to the following inequalities:

$$\begin{aligned}& -4 \le d_4 \le 0 \quad , \quad -3 \le (d_5 - 3d_4) \le 3 \, , \\ & -10 \le d_6 + 3(3d_4 - d_5) \le 8 \quad , \quad -32 \le d_8 + 9d_6 - 22d_5 + 51d_4 \le 46 \, .\end{aligned}$$

There exists no solution with $d_5 = -1$ and $d_6 = d_8 = 0$. Thus there is no solution for which $\gamma^{\widehat{4a}}$ is a modular form of $\Gamma_0(16)$. To order q^{128}, we find 1427 solutions that include $d_4 = d_5 = d_6 = d_8 = 0$. We have checked that all these solutions continue to satisfy the positivity constraints to order q^{512} and possibly to all orders.[2]

The result can also be understood in terms of the expansion of $\hat{\Sigma}^{\hat{\rho}}$ in terms of representations of M_{12}. The above solution completely determines the multiplicities $\hat{N}_m(n)$ defined in Eq. (2.3) for all representations except $m = 9, 10$. In particular, it uniquely fixes $\hat{N}_2(n)$ and $\hat{N}_3(n)$. Table 3 lists out the multiplicities of the various M_{12} characters, i.e., $\hat{N}_m(n)$, in the character decomposition for the M_{12} Jacobi form for $n \in [0, 32]$ when $d_4 = d_5 = d_6 = d_8 = 0$.

2.4. *There is no moonshine for M_{12}*

The EOT Jacobi forms associated with elements of order N transform as Jacobi form at level N with phases that are (powers of) twelfth-roots of unity [16]. Further, it is known that $H_3(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$. The non-uniqueness of our solution for M_{12} moonshine suggests we look for further constraints beyond the ones that we have imposed. It is known that like M_{24}, M_{12} has a non-trivial 3-cocycle [17]. One has $H_3(M_{12}, \mathbb{Z}) = \mathbb{Z}_8 \oplus \mathbb{Z}_6$. With this in mind we looked to see if whether suitable powers of the 1427 solutions for $\gamma^{\widehat{4a}}$ and $\gamma^{\widehat{8a}}$ are modular forms of $\Gamma_0(4)$ and $\Gamma_0(8)$ respectively. We did not find any solution. A more detailed analysis where we go beyond looking at just the 1427 solutions leads to a stronger negative result.

Proposition 2.4. *There exist no modular forms $\gamma^{\widehat{4a}}$ and $\gamma^{\widehat{8a}}$, whose initial terms are as given in Eq. (2.12), that transform with phases given by (powers of) eighth-roots of unity at levels 4 and 8 respectively such that the non-negativity conditions on the multiplicities $\hat{N}_2(n)$, $\hat{N}_3(n)$, $\hat{N}_9(n)$ and $\hat{N}_{10}(n)$ defined in Eq. (2.3) holds for all n.*

[2] All 1427 solutions can be accessed by downloading the LaTeX source file from arXiv.org. They have been listed in the source file after the `\end{document}` command.

Proof. Let $\tilde{N}_a(n)$ denote multiplicity of the representation with $M_{12}{:}2$ character $\tilde{\chi}_a$ in the expansion of $\Sigma^{\tilde{\rho}}$ and similarly $\hat{N}_m(n)$ is defined for M_{12}. Since $\hat{N}_2(n)$, $\hat{N}_3(n)$, $\hat{N}_9(n)$ and $\hat{N}_{10}(n)$ remain unfixed, we write them as follows:

$$\hat{N}_2(n) = \frac{\tilde{N}_3(n)}{2} + c_2(n),\ \hat{N}_3(n) = \frac{\tilde{N}_3(n)}{2} - c_2(n)\ ,$$

and

$$\hat{N}_9(n) = \frac{\tilde{N}_{11}(n)}{2} + c_9(n),\ \hat{N}_{10}(n) = \frac{\tilde{N}_{11}(n)}{2} - c_9(n)\ .$$

We can now write the Fourier expansion of weight two modular form for $\gamma^{\widehat{4a}}$ and $\gamma^{\widehat{8a}}$ in terms of multiplicities with $c_2(n)$ and $c_9(n)$ as unknowns. From the positive definiteness of the multiplicities we have following constraints.

$$|c_2(n)| \leq \frac{\tilde{N}_3(n)}{2} \text{ and } |c_9(n)| \leq \frac{\tilde{N}_{11}(n)}{2}\ .$$

Eq. (2.12) determines $c_2(n)$ and $c_9(n)$ for $n \leq 4$ with the others remaining unfixed.

We look for them to be modular forms with unknown phases that are powers of an eighth-root of unity. We take the eighth powers of $\gamma^{\widehat{4a}}$ and $\gamma^{\widehat{8a}}$ – these are expected to be modular forms of weight 16 at level 4 (of dimension 9) and 8 (of dimension 17) respectively. The remaining 16 coefficients can be expressed in terms of $c_2(n)$ ($n \in [5, 16]$) and $c_9(n)$ ($n \in [5, 8]$) by matching up to order q^8 for $\gamma^{\widehat{4a}}$ and order q^{16} for $\gamma^{\widehat{8a}}$. Modularity determines all other $c_2(n)$ and $c_9(n)$. Let us focus on $c_9(9)$ which can be expressed in terms of the 16 unknowns. We obtain

$$\begin{aligned} c_9(9) = &-1078695 + 2359c_2(6) - 396c_2(7) + 31c_2(8) - c_2(9) - 65930c_9(4) \\ &- 420c_9(4)^2 - 1830c_9(5) + 28c_9(4)c_9(5) + 2359c_9(6) - 396c_9(7) + 31c_9(8)\ . \end{aligned}$$

Using the constraints from positivity on the coefficients appearing on the right hand side of the above equation, we get

$$-1254891 \leq c_9(9) \leq -905523\ .$$

Further, positive definiteness of $\hat{N}_9(9)$ implies $-75 \leq c_9(9) \leq 75$. Clearly the two constraints are not compatible with each other. Thus, there exists no solution that is compatible with the positive definiteness of $\hat{N}_9(9)$. □

2.5. Moonshines for $L_2(11)$

We have seen that there is no moonshine for M_{12}. With this in mind, we look for subgroups of M_{12} for which the characters $\hat{\chi}_2^-(\hat{\rho})$ and $\hat{\chi}_9^-(\hat{\rho})$ vanish on restriction to the sub-group. There are two such sub-groups, both isomorphic to $L_2(11)$. The first is a maximal subgroup of M_{12} and the second is a maximal subgroup of $M_{11} \subset M_{12}$. As sub-groups of M_{12}, these two groups are *not* conjugate to each other and thus lead to distinct moonshines [18].

2.5.1. $L_2(11)$

$L_2(11)$ is Artin's notation for the finite simple group $PSL(2, \mathbb{F}_{11}) = SL(2, \mathbb{F}_{11})/\mathbb{F}_{11}^\times$, where $\mathbb{F}_{11}$ is the prime field of integers modulo 11. It has a natural action on the projective line, $PL(11)$, via projective linear transformations:

$$x \to \frac{ax+b}{cx+d} \quad , \quad x \in PL(11) \ .$$

The projective line $PL(11)$ consists of 12 points whose inhomogeneous coordinates are given by the set $\Omega = (0, 1, 2, 3, \ldots, 9, X = 10, \infty)$. This provides a 12-dimensional permutation representation of $L_2(11)$. In this representation, $L_2(11)$ is generated as $\langle \alpha, \beta, \gamma \rangle =: L_2(11)_A$, where

$$\alpha : x \to x+1 \quad , \quad \beta : x \to 3 \cdot x \quad , \quad \gamma : x \to -1/x \quad . \tag{2.16}$$

Explicitly, one has

$$\alpha = (\infty)(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X)$$
$$\beta = (\infty)(0)(1, 3, 9, 5, 4)(2, 6, 7, X, 8)$$
$$\gamma = (\infty, 0)(1, X), (2, 5), (3, 7)(4, 8)(6, 9)$$

One has $\alpha^{11} = \beta^5 = \gamma^2 = 1$. The eight conjugacy classes of $L_2(11)$ are given by the following cycle shapes in $L_2(11)_A$:

ρ	$1a$	$2a$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
cycle shape	1^{12}	2^6	3^4	$1^2 5^2$	$1^2 5^2$	6^2	$1^1 11^1$	$1^1 11^1$
element	1	γ	$\alpha\gamma$	β	β^{-1}	$\alpha\gamma\beta$	α	α^{-1}

Let δ represent the permutation (with cycle shape $1^4 2^4$) acting on $PL(11)$:

$$\delta = (\infty)(0)(1)(2, X)(3, 4)(5, 9)(6, 7)(8) \ .$$

A second construction of $L_2(11)$, that we call $L_2(11)_B$, is generated by $\langle \alpha, \beta, \delta \rangle$. All three generators fix ∞ and thus $L_2(11)_B$ permutes points in $\Omega \backslash \infty$. The cycle shapes for the conjugacy classes for $L_2(11)_B$ are

ρ	$1a$	$2a$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
cycle shape	1^{12}	$1^4 2^4$	$1^3 3^3$	$1^2 5^2$	$1^2 5^2$	$1^1 2^1 3^1 6^1$	$1^1 11^1$	$1^1 11^1$
element	1	δ	$\alpha\delta$	β	β^{-1}	$\alpha\delta\beta$	α	α^{-1}

The important observation here is that both $L_2(11)_A$ and $L_2(11)_B$ do not have any elements of order 4 and 8. Thus the conjugacy classes $4a/4b$ and $8a/8b$ do not reduce to conjugacy classes of these sub-groups. The conjugacy class $10a$ of M_{12}, for which we do know the Jacobi form, also does not appear.

Below we provide first few terms that appear in the character expansion which is the analog of Eq. (2.3) for the two $L_2(11)$ subgroups. For $L_2(11)_A \subset M_{12}$, one has[3]

$$\begin{aligned}\Sigma = &-\chi_1 + (\chi_1 + 2\chi_5 + \chi_7 + \chi_8)\, q + (\chi_2 + \chi_3 + 5\chi_4 + 2\chi_5 + 5\chi_6 + 4\chi_7 + 4\chi_8)\, q^2 \\ &+ (\chi_1 + 8\chi_2 + 8\chi_3 + 9\chi_4 + 12\chi_5 + 13\chi_6 + 14\chi_7 + 14\chi_8)\, q^3 \\ &+ (2\chi_1 + 15\chi_2 + 15\chi_3 + 39\chi_4 + 32\chi_5 + 37\chi_6 + 42\chi_7 + 42\chi_8)\, q^4 + O\left(q^5\right)\end{aligned} \tag{2.17}$$

For $L_1(11)_B \subset M_{11} \subset M_{12}$

[3] For the two equations that follow, the characters that appear are those for $L_2(11)$.

$$\Sigma = -\chi_1 + (\chi_4+\chi_6+\chi_7+\chi_8)\,q + (\chi_1+3\chi_2+3\chi_3+2\chi_4+4\chi_5+4\chi_6+4\chi_7+4\chi_8)\,q^2 + (\chi_1+4\chi_2+4\chi_3+15\chi_4+10\chi_5+13\chi_6+14\chi_7+14\chi_8)\,q^3 + (4\chi_1+19\chi_2+19\chi_3+31\chi_4+38\chi_5+35\chi_6+42\chi_7+42\chi_8)\,q^4 + O\left(q^5\right) \quad (2.18)$$

3. Siegel modular forms for $L_2(11)_A$ and $L_2(11)_B$

The construction of Borcherds–Kac–Moody Lie algebras is intimately connected to modular forms that appear as the Weyl denominator formula for the BKM Lie algebra. In this section, we shall pursue this approach by constructing genus-two Siegel modular forms. First, we show that the two distinct $L_2(11)$ moonshines naturally lead to a product formula given by Eq. (3.4). Modularity of this formula is not manifest in the construction and we prove this in two ways – (i) by constructing the sum side as an additive lift and (ii) by showing that the product formula is equivalent to Borcherds products. The second method always works while the additive lift works in most cases.

3.1. The multiplicative lift

The connection with $L_2(11)$ moonshine leads to a Siegel modular form that is given by the following formula (on repeating arguments given in [19]):

$$\Delta_k^{\hat\rho}(\mathbf{Z}) = s^{1/2}\,\hat\phi_{k,1/2}^{\hat\rho}(\tau,z)\times\exp\Big(\sum_{m=1}^{\infty} s^m\,\hat\psi^{\hat\rho}\big|_0 T(m)(\tau,z)\Big)\,, \quad (3.1)$$

where the twisted Hecke operator (first defined in [19]) is given by

$$s^m\;\hat\psi^{\hat\rho}\big|_0 T(m)(\tau,z) = s^m\;\tfrac{1}{m}\sum_{\substack{ad=m\\ b \bmod d}}\hat\psi^{\hat\rho_a}\left(\tfrac{a\tau+b}{d},az\right)$$

where ρ_a is the conjugacy class of the a-th power of an element in the conjugacy class ρ. Further, $\hat\phi_{k,1/2}^{\hat\rho}(\tau,z)$ is defined as follows:

$$\hat\phi_{k,1/2}^{\hat\rho}(\tau,z) = \frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\times\eta_{\hat\rho}(\tau)\,, \quad (3.2)$$

where $\eta_{\hat\rho}(\tau)$ is an eta product, $k_{\hat\rho}$ its weight as given in Table 1 and $k=(k_{\hat\rho}-1)$.

Again, as in [19] with $L_2(11)$ playing the role of M_{24}, we will show that Eq. (3.1) implies a product formula for $\Delta_k^{\hat\rho}(\mathbf{Z})$. Let g be an element of order N and $\hat\rho_a$ denote the conjugacy class of the element g^a. Further, set $\hat\rho_0=1^{12}$. Define the Fourier coefficients, $c^a(n,\ell)$, of the Jacobi form $\hat\psi_{0,1}^{\hat\rho_a}(\tau,z)$ as follows

$$\hat\psi_{0,1}^{\hat\rho_a}(\tau,z) = \sum_{n=0}^{\infty}\sum_{\ell\in\mathbb{Z}} c^a(n,\ell)\;q^n r^\ell\,. \quad (3.3)$$

Then, define $f_\alpha(n,\ell)$ via the discrete Fourier transform ($\omega_N=\exp(2\pi i/N)$):

$$c^a(n,\ell) = \sum_{\alpha=0}^{N-1}\omega_N^{\alpha a}\;f_\alpha(n,\ell)\,.$$

Table 1
M_{12} conjugacy classes and the corresponding cycle shapes. The associated eta products are modular forms of $\Gamma_0(2N_{\hat{\rho}}\, p_{\hat{\rho}}, 2)$ with weight $k_{\hat{\rho}}$ and Dirichlet character χ with $\Gamma_1(2N_{\hat{\rho}}\, p_{\hat{\rho}}, 2) \subset \ker(\chi)$. The values are given in columns 3–5.

M_{12} Conj. class	Cycle shape $\hat{\rho}$	$N_{\hat{\rho}}$	$p_{\hat{\rho}}$	$k_{\hat{\rho}}$	$\widehat{\chi}_{\hat{\rho}}(d)$
1a	1^{12}	1	1	6	
2a	2^6	2	2	3	$\left(\frac{-1}{d}\right)$
2b	$1^4 2^4$	2	1	2	
3a	$1^3 3^3$	3	3	3	$\left(\frac{-3}{d}\right)$
3b	3^4	3	3	2	
4a	$2^2 4^2$	4	2	2	
4b	$1^4 2^2 4^4/2^2 4^2$	4	2	3	$\left(\frac{-1}{d}\right)$
5a	$1^2 5^2$	5	1	2	
6a	6^2	6	6	1	$\left(\frac{-1}{d}\right)$
6b	$1^1 2^1 3^1 6^1$	6	1	2	
8a	$4^1 8^1$	8	4	1	$\left(\frac{-2}{d}\right)$
8b	$1^2 2^1 4^1 8^2/4^1 8^1$	8	4	2	
10a	$2^1 10^1$	10	2	1	$\left(\frac{-20}{d}\right)$
11a/b	$1^1 11^1$	11	1	1	$\left(\frac{-11}{d}\right)$

One then can rewrite the formula for the multiplicative lift as follows:

$$\Delta_k^{\hat{\rho}}(\mathbf{Z}) = s^{\frac{1}{2}}\, \hat{\phi}_{k,1/2}^{\hat{\rho}}(\tau, z) \times \prod_{\alpha=0}^{N-1} \prod_{m=1}^{\infty} \prod_{n=0}^{\infty} \prod_{\substack{\ell\in\mathbb{Z} \\ 4nm-\ell^2\geq 0}} \left(1 - \omega_N^{\alpha}\, q^n r^\ell s^m\right)^{f_\alpha(nm,\ell)} . \tag{3.4}$$

For the cases when the order of $g \in L_2(11)$ is prime (i.e., $N = 2, 3, 5, 11$), one has the conjugacy class of g^a for $a \neq 0 \bmod N$ is the same as that of g. Thus, one has $\rho_a = \rho$ for $a \neq 0 \bmod N$. For these cases, on using the product representation for the theta and eta functions that appear in $\hat{\phi}_{k,1/2}^{\hat{\rho}}(\tau, z)$, the above formula simplifies to

$$\Delta_k^{\hat{\rho}}(\mathbf{Z}) = q^{\frac{1}{2}} r^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{c^1(nm,\ell)} \left(1 - q^{nN} r^{\ell N} s^{mN}\right)^{\frac{c^0(nm,\ell)-c^1(nm,\ell)}{N}} , \tag{3.5}$$

where $(n, \ell, m) > 0$ implies $n > 0$, or $n = 0$ and $m > 0$, or $n = m = 0$ and $\ell < 0$.

For $N = 6$, the product formula takes the form

$$\Delta_k^{\hat{\rho}}(\mathbf{Z}) = q^{\frac{1}{2}} r^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{c^1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^6\right)^{f_1(nm,\ell)}$$
$$\left(1 - (q^n r^\ell s^m)^2\right)^{\frac{1}{2}(c^2(nm,\ell)-c^1(nm,\ell))} \left(1 - (q^n r^\ell s^m)^3\right)^{\frac{1}{3}(c^3(nm,\ell)-c^1(nm,\ell))} , \tag{3.6}$$

with $f_1(n, \ell) = \frac{1}{6}(c^0(n, \ell) + c^1(n, \ell) - c^2(n, \ell) - c^3(n, \ell))$.

Remark. All the terms in the product formula that appear with $m=0$ arise from the product representation of $\hat{\phi}^{\hat{\rho}}_{k,1/2}(\tau,z)$. Further, the formula is manifestly symmetric under the exchange $q \leftrightarrow s$ and odd under $r \to r^{-1}$.

3.2. Modularity of the multiplicative lift

The product formulae that we obtained starting from Eq. (3.1) is not standard in the context of Siegel modular forms. Thus, we need to establish the modular properties of the product formula. We establish modularity in the next couple of sub-sections.

3.2.1. Modularity by comparing with the additive lift

The additive 'seed' for the Siegel modular form is a Jacobi form of weight $k=(k_{\hat{\rho}}-1)$ and index 1/2 and is defined in Eq. (3.2). For $k>0$, the additive lift is given by

$$\mathcal{A}\Big(\hat{\phi}^{\hat{\rho}}_{k,1/2}\Big)(\mathbf{Z}) := \sum_{m=1}^{\infty} s^{(2m-1)/2}\, \hat{\phi}^{\hat{\rho}}\big|_{k,1/2} T_-^M(2m-1)(\tau,z)\ , \tag{3.7}$$

where $T_-^M(m)$ is the Hecke operator defined by Clery–Gritsenko [20]. Let $\phi(\tau,z)$ be a Jacobi form of weight k of $\Gamma_0(M)$ with character χ and index which is integral or half-integral. Then

$$\phi^\rho\big|_k T_-^M(m)(\tau,z) = m^{k-1} \sum_{\substack{ad=m\\(a,Mq)=1\\ b \bmod d}} d^{-k}\, \chi(a)\, \phi^\rho\left(\tfrac{a\tau+qb}{d}, az\right)\ ,$$

where q is chosen such that $\Gamma_1(Mq,q)\subset \ker(\chi)$.[4] For all the cases of interest, one has $q=2$ and $M=p_{\hat{\rho}}N_{\hat{\rho}}$ as given in Table 1. The additive lift $\mathcal{A}\Big(\hat{\phi}^{\hat{\rho}}_{k,1/2}\Big)(\mathbf{Z})$ is a genus-two Siegel modular form of a level N subgroup of $Sp(2,\mathbb{Z})$ with weight k.

A necessary condition for the compatibility of the additive lift with the multiplicative lift is:

$$\left[\tfrac{\theta_1(\tau,z)}{\eta(\tau)^3}\ \eta_{\hat{\rho}}(\tau)\right]\Big|_k T_-^M(3)(\tau,z) \overset{?}{=} \hat{\psi}^{\hat{\rho}}(\tau,z)\left[\tfrac{\theta_1(\tau,z)}{\eta(\tau)^3}\ \eta_{\hat{\rho}}(\tau)\right]$$

This is the coefficient of $s^{3/2}$ in both the lifts i.e., the ones given in Eq. (3.7) and Eq. (3.1). This condition holds for all the cycle shapes appearing in $L_2(11)_{A/B}$ except for the following three cycle shapes where we observe that:

$$\begin{aligned}
\frac{T_3\hat{\phi}^{3^4}}{\hat{\phi}^{3^4}} - \hat{\psi}^{3^4} &= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\Big[9\ \eta_{1^3 3^{-2} 9^3}(\tau)\Big]\ ,\\
\frac{T_3\hat{\phi}^{1^1 11^1}}{\hat{\phi}^{1^1 11^1}} - \hat{\psi}^{1^1 11^1} &= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\Big[\tfrac{11}{3}\ \eta_{1^2 11^2}(\tau)\Big]\ ,\\
\frac{T_3\hat{\phi}^{6^2}}{\hat{\phi}^{6^2}} - \hat{\psi}^{6^2} &= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\Big[\tfrac{2}{3}\eta_{6^4}(\tau) + \tfrac{1}{3}\eta_{1^3 3^{-3} 9^3}(\tau) + 2\eta_{1^3 3^{-3} 9^3}(2\tau) + \tfrac{8}{3}\eta_{1^3 3^{-3} 9^3}(4\tau)\Big],
\end{aligned} \tag{3.8}$$

[4] The group $\Gamma_1(Mq,q)$ is defined as follows.

$$\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})\ \Big|\ c=0 \bmod Mq,\ b=0 \bmod q, a=1 \bmod Mq, d=1 \bmod Mq\right\}\ .$$

.

where $T_3\hat{\phi}^{\hat{\rho}}$ is short for $\hat{\phi}^{\hat{\rho}}|_k T_-^M(3)$. The last two examples potentially correspond to Siegel modular forms with weight $k = 0$ and we have naïvely applied a formula which assumes $k > 0$. So we need to prove modularity in another way for these three examples.

3.2.2. Modularity by comparing with a Borcherds formula

We begin with a theorem due to Clery–Gritsenko (see also [21–23]) that leads to a Borcherds product formula for a meromorphic Siegel modular form starting from a nearly holomorphic Jacobi form of weight zero and index t specialised to the case for the case $t = 1$.

Theorem 3.1 *(Clery–Gritsenko [20]). Let ψ be a nearly holomorphic Jacobi form of weight 0 and index 1 of $\Gamma_0(N)$. Assume that for all cusps of $\Gamma_0(N)$ one has $\frac{h_e}{N_e} c_{f/e}(n, \ell) \in \mathbb{Z}$ if $4n - \ell^2 \leq 0$. Then the product*

$$B_\psi(\mathbf{Z}) = q^A r^B s^C \prod_{f/e \in \mathcal{P}} \prod_{\substack{n,\ell,m \in \mathbb{Z} \\ (n,\ell,m)>0}} \left(1 - (q^n r^\ell s^m)^{N_e}\right)^{\frac{h_e}{N_e} c_{f/e}(nm,\ell)} ,$$

with

$$A = \frac{1}{24} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}}} h_e\, c_{f/e}(0, \ell),\ B = \frac{1}{2} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}_{>0}}} \ell h_e\, c_{f/e}(0, \ell),\ C = \frac{1}{4} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}}} \ell^2 h_e\, c_{f/e}(0, \ell) ,$$

defines a meromorphic Siegel modular form of weight

$$k = \frac{1}{2} \sum_{\substack{f/e \in \mathcal{P} \\ \ell \in \mathbb{Z}}} \frac{h_e}{N_e} c_{f/e}(0, 0)$$

with respect to $\Gamma_1(N)^+$ possibly with character. The character is determined by the zeroth Fourier–Jacobi coefficient of $B_\psi(\mathbf{Z})$ which is a Jacobi form of weight k and index C of the Jacobi subgroup of $\Gamma_1(N)^+$.

Remark. As discussed by Clery and Gritsenko, the poles and zeros of B_ψ lie on rational quadratic divisors defined by the Fourier coefficients $c_{f/e}(n, \ell)$ for $4n - \ell^2 \leq 0$. The condition $\frac{h_e}{N_e} c_{f/e}(n, \ell) \in \mathbb{Z}$ ensures that one has only poles or zeros at these divisors. In our case, multiple cusps contribute to the same term in the product formula and hence we relax the condition. For all cusps that have identical values of (N_e, h_e), we require that the sum of $\frac{h_e}{N_e} c_{f/e}(n, \ell)$ (with $4n - \ell^2 \leq 0$) for all such cusps be integral. This suffices to ensure that one has only zeros or poles at all divisors.

Since only the coefficients $c_{f/e}(n, \ell)$, with $n \in \mathbb{Z}$ appear in the product formula, we define the projection (also defined in [24]), π_{FE}, as follows

$$\pi_{FE}\left(\phi | M_{f/e}\right)(\tau, z) := \frac{1}{h_e} \sum_{b=0}^{h_e - 1} \phi | M_{f/e}(\tau + b, z) , \tag{3.9}$$

where $M_{f/e} = \begin{pmatrix} f & * \\ e & * \end{pmatrix} \in SL(2, \mathbb{Z})$ maps the cusp at $i\infty$ to f/e. It is the Fourier coefficients of the projected Jacobi form that appears in the product formula.

Following Raum [24], we look to prove modularity of the product given in Eq. (3.4) by considering products of rescaled Borcherds products. Our considerations not only extend his results but also provide a systematic method of obtaining the precise rescaled Borcherds products that are needed. For all conjugacy classes of $L_2(11)_B$, we find that the product formula is equivalent to a single Borcherds formula. More generally, we find that the following holds.

Proposition 3.2. *For $g \in L_2(11)_{A/B}$, let $\hat{\rho}_m = [g^m]$, $\hat{\rho} = \hat{\rho}_1$, $\psi^{\hat{\rho}_m} = \hat{\psi}^{\hat{\rho}_m}_{0,1}(\tau, z)$ and $\Delta^{\hat{\rho}}_k(\mathbf{Z})$ be the modular form defined by the multiplicative lift given in Eq. (3.4). Then*

$$\left(\Delta^{\hat{\rho}}_k(\mathbf{Z})\right)^{p_{\hat{\rho}}} = \prod_{m|p_{\hat{\rho}}} \left(B_{\frac{p_{\hat{\rho}}}{m}\psi^{\hat{\rho}_m}}(m\,\mathbf{Z}) \right) , \tag{3.10}$$

where $p_{\hat{\rho}}$ is the length of the shortest cycle in the cycle shape for the conjugacy class $\hat{\rho}$ and $B_\psi(\mathbf{Z})$ is the Siegel modular form given by Theorem 3.1.

Proof. We deal with case when $p_{\hat{\rho}} = 1$ before considering $p_{\hat{\rho}} > 1$.

$\boxed{\mathbf{p}_{\hat{\rho}} = \mathbf{1}}$

This occurs for all cases when $g \in L_2(11)_B$ and the conjugacy classes of order 1, 5, and 11 in $L_2(11)_A$. In all these cases, there is precisely one term in the product appearing on the right hand side of Eq. (3.10).

Let N be the order of g. The Jacobi form ψ is a modular form of $\Gamma_0(N)$ with weight 0 and index 1. Considering the case when N is prime, there are only two cusps, one at $i\infty$ (which is $\Gamma_0(N)$ equivalent to $1/N$) and another at $0/1$. To prove the equality, we need to show that

$$\pi_{FE}(\psi) = \psi^{\hat{\rho}}_{0,1} , \tag{3.11}$$

$$\pi_{FE}(\psi|S) = \frac{1}{N}(\psi^{1^{12}}_{0,1} - \psi^{\hat{\rho}}_{0,1}) . \tag{3.12}$$

The first equation holds trivially since $\psi = \psi^{\hat{\rho}}_{0,1}$ has only integral powers of q in its Fourier–Jacobi expansion. The second part follows from a calculation.

$$\pi_{FE}(\psi|S)(\tau, z) = \frac{1}{N+1}\phi_{0,1}(\tau, z) + \pi_{FE}(\alpha^{(N)}|S)(\tau)\phi_{-2,1}(\tau, z) ,$$

where $\alpha^{(N)}(\tau) = \frac{N}{N+1}E_2^{(N)}(\tau)$ for $N = 2, 3, 5$. Computing $\pi_{FE}(\alpha^{(N)}|S)(\tau)$, we obtain

$$\pi_{FE}(\alpha^{(N)}|S)(\tau) = -\tfrac{1}{N(N+1)} \sum_{b=0}^{N-1} E_2^{(N)}(\tfrac{\tau+b}{N}) = -\tfrac{1}{(N+1)} E_2^{(N)}\big|_2 U_N = -\tfrac{1}{(N+1)} E_2^{(N)}(\tau) ,$$

where U_N is the Hecke operator for $\Gamma_0(N)$ (defined by Atkin and Lehner [25]) and $E_2^{(N)}$ is its eigenform with eigenvalue $+1$. Thus, we get

$$\begin{aligned}\pi_{FE}(\psi|S)(\tau, z) &= \frac{1}{N+1}\phi_{0,1}(\tau, z) - \frac{1}{N}\alpha^{(N)}(\tau)\phi_{-2,1}(\tau, z) \\ &= \frac{1}{N}\phi_{0,1}(\tau, z) - \frac{1}{N}\psi^{\hat{\rho}}_{0,1}(\tau, z) \\ &= \frac{1}{N}\left(\psi^{1^{12}}_{0,1} - \psi^{\hat{\rho}}_{0,1}\right)(\tau, z) ,\end{aligned}$$

which establishes Eq. (3.12) for prime $N = 2, 3, 5$. A similar computation holds for $N = 11$ for which $\alpha^{(11)}(\tau) = \frac{11}{6}E_2(11)(\tau) - \frac{22}{5}\eta_{1^2 11^2}(\tau)$. One can show that $\pi_{FE}(\alpha^{(11)}|S)(\tau) = -(1/11)\alpha^{(11)}(\tau)$. Thus Eq. (3.12) holds for $N = 11$ as well.

Next, considering the case of $N = 6$, where there are additional cusps at $1/3$ (with width 2) and $1/2$ (with width 3). One needs to verify that the last three conditions hold as the first condition holds trivially.

$$\begin{aligned}
\pi_{FE}(\psi) &= \hat{\psi}^{\hat{\rho}}_{0,1}(\tau, z)\\
\pi_{FE}(\psi|S) &= \frac{1}{6}(\hat{\psi}^{1^{12}}_{0,1}(\tau, z) + \hat{\psi}^{\hat{\rho}}_{0,1}(\tau, z) - \hat{\psi}^{\hat{\rho}_2}_{0,1}(\tau, z) - \hat{\psi}^{\hat{\rho}_3}_{0,1}(\tau, z))\\
\pi_{FE}(\psi|M_{1/3}) &= \frac{1}{2}(\hat{\psi}^{\hat{\rho}_2}_{0,1}(\tau, z) - \hat{\psi}^{\hat{\rho}}_{0,1}(\tau, z))\\
\pi_{FE}(\psi|M_{1/2}) &= \frac{1}{3}(\hat{\psi}^{\hat{\rho}_3}_{0,1}(\tau, z) - \hat{\psi}^{\hat{\rho}}_{0,1}(\tau, z))
\end{aligned} \tag{3.13}$$

In the above equations $\hat{\rho}=1^1 2^1 3^1 6^1$, $\hat{\rho}_2=1^3 3^3$ and $\hat{\rho}_3=1^4 2^4$. $\psi=\hat{\psi}^{\hat{\rho}}_{0,1}(\tau, z)=\frac{1}{2}Z^{1^2 2^2 3^2 6^2}(\tau, z)$, is a Jacobi form of $\Gamma_0(6)$. Then,

$$\begin{aligned}
\psi|_{0,1}S(\tau, z) &= \frac{1}{6}\phi_{0,1}(\tau, z) + \frac{1}{12}\left(E^{(2)}_2(\tau/2) + 2E^{(3)}_2(\tau/3) - 5E^{(6)}_2(\tau/6)\right)\phi_{-2,1}(\tau, z)\ ,\\
\psi|_{0,1}M_{1/2}(\tau, z) &= \frac{1}{6}\phi_{0,1}(\tau, z) + \frac{1}{12}\left(E^{(2)}_2(\tau) - 8E^{(3)}_2\left(\tfrac{2\tau+1}{3}\right) + 5E^{6}_2\left(\tfrac{\tau+2}{3}\right)\right)\phi_{-2,1}(\tau, z)\ ,\\
\psi|_{0,1}M_{1/3}(\tau, z) &= \frac{1}{6}\phi_{0,1}(\tau, z) + \frac{1}{12}\left(2E^{(3)}_2(\tau) - 9E^{(2)}_2\left(\tfrac{3\tau-1}{3}\right) + 5E^{6}_2\left(\tfrac{\tau-1}{2}\right)\right)\phi_{-2,1}(\tau, z)\ .
\end{aligned}$$

The projections π_{FE} of the Eisenstein series appearing in the above equations are given by (with $b \in \mathbb{Z}$)

$$\begin{aligned}
\pi_{FE}\left(E^{(2)}_2(\tfrac{\tau+b}{2})\right) &= E^{(2)}_2(\tau)\ ,\\
\pi_{FE}\left(E^{(3)}_2(\tfrac{\tau+b}{3})\right) &= E^{(3)}_2(\tau)\ ,\\
\pi_{FE}\left(5E^{(6)}_2(\tfrac{\tau+b}{6})\right) &= -5E^{(6)}_2(\tau) + 6E^{(3)}_2(\tau) + 4E^{(2)}_2(\tau)\ ,
\end{aligned}$$

which implies Eq. (3.13).

$\boxed{\mathbf{p}_{\hat{\rho}} > \mathbf{1}}$

There are three conjugacy classes of $L_2(11)_A$, 2^6, 3^4 and 6^2, with $p = 2, 3, 6$ respectively. First, consider $B_{\psi^{\hat{\rho}}}$ with $\psi^{\hat{\rho}}$ a Jacobi form of $\Gamma_0(N^2)$ where N is the order of the group element. For all three conjugacy classes, cusps whose width does not divide the order of the group element, do not contribute as the Fourier expansion has no integral powers of q and thus they vanish under the projection π_{FE}. The details of the computation are given in Appendix B.

2^6: We need to consider the contribution from the cusps at $i\infty$ and $1/2$ (with $h_e = 1$ $N_e = 1$). One has

$$\pi_{FE}(\psi|M_{1/2})(\tau, z) = -\hat{\psi}^{\hat{\rho}}_{0,1}(\tau, z)\ .$$

We need to consider the square of $\Delta^{2^6}_2(\mathbf{Z})$ so that $\frac{h_e}{N_e}c_{f/e}(n, \ell) \in \mathbb{Z}$ if $4n - \ell^2 \leq 0$.

$$B_{2\,\psi^{2^6}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} \Big(1-q^n r^\ell s^m\Big)^{2c^1(nm,\ell)} \Big(1-q^{2n} r^{2\ell} s^{2m}\Big)^{-c^1(nm,\ell)} ,$$

which is a meromorphic Siegel modular form at level 4 with weight -1. This Siegel modular form does not account for the following terms in Eq. (3.5):

$$q^{\frac{1}{2}} r^{\frac{1}{2}} s^{\frac{1}{2}} \prod_{(n,\ell,m)>0} (1-q^{2n} r^{2\ell} s^{2m})^{\frac{c^0(nm,\ell)}{2}} .$$

The square of this term is $B_{\psi^{1^{12}}}(2\mathbf{Z})$. We obtain

$$(\Delta_2^{2^6}(\mathbf{Z}))^2 = B_{2\,\psi^{2^6}}(\mathbf{Z})\ B_{\psi^{1^{12}}}(2\mathbf{Z}) .$$

Thus, $(\Delta_2^{2^6}(\mathbf{Z}))^2$ is a Siegel modular form of weight 4 at level 4. The transformation of the Siegel modular form is the one induced from $(\phi^{2^6}_{2,1/2}(\tau,z))^2$.

$\mathbf{3^4}$: We need to consider the contribution from the cusps at $i\infty$, $1/3$ (with $h_e=1$ $N_e=3$) and $2/3$ ($h_e=1$ $N_e=3$). The contributions from the cusps at $1/3$ and $2/3$ have cube roots of unity. However, the contributions of the two cusps add in the product formula to give integral coefficients. One finds

$$\pi_{FE}(\psi|M_{1/3}+\psi|M_{2/3})(\tau,z) = -\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z) ,$$

leading to

$$B_{3\,\psi^{3^4}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} \Big(1-q^n r^\ell s^m\Big)^{3c^1(nm,\ell)} \Big(1-q^{3n} r^{3\ell} s^{3m}\Big)^{-c^1(nm,\ell)} ,$$

which is a Siegel modular form at level 9 with weight $-2/3$. This Siegel modular form does not account for the following terms in Eq. (3.5):

$$q^{\frac{1}{2}} r^{\frac{1}{2}} s^{\frac{1}{2}}\ (1-q^{3n} r^{3\ell} s^{3m})^{\frac{c^0(nm,\ell)}{3}} .$$

The cube of this term is $B_{\psi^{1^{12}}}(3\mathbf{Z})$. Thus we obtain

$$(\Delta_1^{3^4}(\mathbf{Z}))^3 = B_{3\,\psi^{3^4}}(\mathbf{Z})\ B_{\psi^{1^{12}}}(3\mathbf{Z}) .$$

Thus, $(\Delta_1^{3^4}(\mathbf{Z}))^3$ is Siegel modular form of weight 3 at level 9.

$\mathbf{6^2}$: We need to consider the contribution from the cusps at $i\infty$, $(1/6, 5/6)$, $(1/12, 5/12)$ and $1/18$. Like in the case of 3^4, we need to pair up cusps with identical values of h_e and N_e to get integral coefficients. One obtains

$$\begin{aligned}
\pi_{FE}(\psi|M_{1/6}+\psi|M_{5/6}) &= \hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z) \\
\pi_{FE}(\psi|M_{1/12}+\psi|M_{5/12}) &= -\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z) \\
\pi_{FE}(\psi|M_{1/18}) &= -\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)
\end{aligned}$$

$$\begin{aligned}
B_{6\,\psi^{6^2}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} & \Big(1-q^n r^\ell s^m\Big)^{c^1(nm,\ell)} \Big(1-(q^n r^\ell s^m)^6\Big)^{c^1(nm,\ell)} \\
& \Big(1-(q^n r^\ell s^m)^2\Big)^{-3\,c^1(nm,\ell)} \Big(1-(q^n r^\ell s^m)^3\Big)^{-2\,c^1(nm,\ell)} , \quad (3.14)
\end{aligned}$$

which is a Siegel modular form of weight −2 at level 36. This does not give all the terms that appear in the product form given in Eq. (3.6). The missing terms can be accounted for by additional terms leading to

$$(\Delta_0^{6^2}(\mathbf{Z}))^6 = B_{6\,\psi^{6^2}}(\mathbf{Z})\ B_{2\,\psi^{2^6}}(3\mathbf{Z}) B_{3\,\psi^{3^4}}(2\mathbf{Z})\ B_{\psi^{1^{12}}}(6\mathbf{Z})\ .$$

$(\Delta_0^{6^2}(\mathbf{Z}))^6$ is a meromorphic Siegel modular form of weight zero at level 36. □

Proposition 3.3. *The modular properties of the Borcherds product formula for* $(\Delta_k^{\hat\rho}(\mathbf{Z}))^{2p_{\hat\rho}}$ *is determined by* $(\hat\phi^{\hat\rho}_{k,1/2}(\tau,z))^{2p_{\hat\rho}}$. *In particular, they are meromorphic Siegel modular forms at level* $N_{\hat\rho}$, *where* $N_{\hat\rho}$ *the order of the group element.*

Proof. It is easy to check that $(\hat\phi^{\hat\rho}_{k,1/2}(\tau,z))^{2p_{\hat\rho}}$ is the zeroth Fourier–Jacobi coefficient of $(\Delta_k^{\hat\rho}(\mathbf{Z}))^{2p_{\hat\rho}}$. From the results of Cheng and Duncan [26], we know that under $\Gamma_0(N_{\hat\rho})$ transformations, $(\hat\phi^{\hat\rho}_{k}(\tau,z))^{2}$, transforms with the following character

$$\chi(\gamma) = \exp\left(\frac{2\pi i c d}{p_{\hat\rho}\, N_{\hat\rho}}\right) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_{\hat\rho})\ .$$

The character of the $p_{\hat\rho}$-th power (of the Siegel modular form) is clearly trivial for all $\gamma \in \Gamma_0(N_{\hat\rho})$. □

4. BKM Lie algebras for $L_2(11)_A$ and $L_2(11)_B$

We have constructed Siegel modular forms, $\Delta_k^{\hat\rho}(\mathbf{Z})$ for all conjugacy classes of $L_2(11)_{A/B}$. In this section, we will establish that these Siegel modular forms appear as the Weyl–Kac–Borcherds denominator formula for BKM Lie superalgebras whose identical real simple roots, $(\delta_1, \delta_2, \delta_3)$ have the following rank three hyperbolic Cartan matrix

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}\ .$$

Each distinct conjugacy class leads to an inequivalent automorphic correction (in the sense of Gritsenko–Nikulin) to the Lorentzian Kac–Moody Lie algebra associated to the above Cartan matrix. The imaginary simple roots that appear depend on the conjugacy class.

Let w_i $(i = 1,2,3)$ denote reflections by the three real simple roots and $W = \langle w_1, w_2, w_3\rangle$ be the Weyl group generated by the real simple roots. The Weyl group acts on the future lightcone $V^+ \subset \mathbb{R}^{2,1} = \oplus \mathbb{R}\,\delta_i$. A fundamental domain (under the action of W) is a polyhedron $\mathcal{M}$, bounded by three walls. Let $\mathcal{H}_{\delta_i} := (x \in V^+ \mid (x,\delta_i) = 0)$ be the wall associated with δ_i and define the space $\mathcal{H}^+_{\delta_i} := (x \in V^+ \mid (x,\delta_i) \leq 0)$. These walls bound the polyhedron $\mathcal{M} = \cap_{i=1}^3 \mathcal{H}^+_{\delta_i}$. The dihedral group D_6 is generated by the involution $\delta_1 \leftrightarrow \delta_3$ and the cyclic permutation of the three real simple roots. Under the action of the dihedral group, $\mathcal{M}$ gets mapped to itself.

Let $Q = \oplus_i \mathbb{Z}\,\delta_i$, denote the root lattice and $Q_+ = \oplus_i \mathbb{Z}_+\,\delta_i$. The Weyl vector $\varrho = \frac{1}{2}(\delta_1 + \delta_2 + \delta_3)$ satisfies $(\varrho, \delta_i) = -1$ for $i = 1,2,3$ and is invariant under the dihedral group D_6. We make the formal identifications

$$e^{-\varrho} \sim q^{1/2} r^{1/2} s^{1/2} \quad , \quad e^{-\delta_1} \sim q\, r \quad , \quad e^{-\delta_2} \sim r^{-1} \quad , \quad e^{-\delta_3} \sim s\, r \quad .$$

Let $\alpha[n,\ell,m] = n\delta_1 + (n+m-\ell)\delta_2 + m\delta_3$. Then, one has $e^{-\alpha[n,\ell,m]} \sim q^n r^\ell s^m$ and the norm $(\alpha[n,\ell,m], \alpha[n,\ell,m]) = 2\ell^2 - 8nm$.

4.1. Properties of $\Delta_k^{\hat{\rho}}(\mathbf{Z})$

We have seen that $\Delta_k^{\hat{\rho}}(\mathbf{Z})$ is a Siegel modular form of weight $k = (k_{\hat{\rho}} - 1)$ at level $N_{\hat{\rho}}$ with character. Let $v(M)$ denote this character for M in the level $N_{\hat{\rho}}$ subgroup of $Sp(2,\mathbb{Z})$. Here $N_{\hat{\rho}}$ is the order of an element of $L_2(11)$ in the conjugacy class $\hat{\rho}$.

1. It is symmetric under the exchange $\tau \leftrightarrow \tau'$. The corresponding $Sp(2,\mathbb{Z})$ element is called V in appendix A. Thus, $v(V) = +1$.
2. It is anti-symmetric under $z \to -z$. The corresponding $Sp(2,\mathbb{Z})$ element is given by
$$\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$
Thus, $v(\delta) = -1$.
3. Under the Heisenberg group, the character is
$$v_H([\lambda,\mu;\kappa]_H) = (-1)^{\lambda+\mu+\lambda\mu+\kappa} .$$
4. Under transformations, $\widetilde{\gamma} \in Sp(2,\mathbb{Z})$ induced by $\gamma \in PSL(2,\mathbb{Z})$, one has
$$v_{\hat{\rho}}(\widetilde{\gamma}) := \chi_{\hat{\rho}}(\gamma) ,$$
where $\chi_{\hat{\rho}}(\gamma)$ is the character of the eta product $\eta_{\hat{\rho}}(\tau)$. In particular, there are two ways of understanding this character. The first one, is that the eta product is a modular form of $\Gamma_0(2N_{\hat{\rho}}\, p_{\hat{\rho}}, 2)$ possibly with Dirichlet character as given in Table 1. The second one is that the $2p_{\hat{\rho}}$-th power of the eta product is a modular form of $\Gamma_0(N_{\hat{\rho}})$. We don't give a precise formula for the character as we don't need it for our considerations. Also, Proposition 3.3 gives the character for the square of $\Delta_k^{\hat{\rho}}(\mathbf{Z})$.
5. The Siegel modular form admits the following Fourier expansion:
$$\Delta_k^{\hat{\rho}}(\mathbf{Z}) = \sum_{\substack{n,\ell,m\equiv 1 \bmod 2 \\ 4nm-\ell^2>0 \\ n,m>0}} f(nm,\ell)\, q^{n/2} r^{\ell/2} s^{m/2} . \tag{4.1}$$
The other conditions $n,\ell,m \equiv 1 \bmod 2$ and $n,m>0$ easily follow from the multiplicative lift. For $\hat{\rho} \neq 1^1 11^1, 3^4, 6^2$, the condition $(4nm - \ell^2) > 0$ follows directly from the additive lift given in Eq. (3.7). This is a condition, $\mathcal{D} > 0$, on the discriminant for terms that appear in the $\frac{m}{2}$-th Fourier–Jacobi coefficient of $\Delta_k^{\hat{\rho}}(\mathbf{Z})$.
For $\hat{\rho} = 1^1 11^1, 3^4, 6^2$, where the additive lift is not available, we conjecture that this condition is true as well. One can explicitly verify that it holds for $m = 1, 3$ as we do below.
$m=1$ The coefficient of $s^{1/2}$ is the Jacobi form $\hat{\phi}_{k,1/2}^{\hat{\rho}}(\tau,z)$. This Jacobi form has non-vanishing Fourier coefficients about the cusp at $i\infty$ when the discriminant $\mathcal{D} > 0$ which is equivalent to $4n - \ell^2 > 0$. In particular, the family of terms (that arise from $\theta_1(\tau,z)$ up to an overall pre-factor) with $q^{1/2} r^{1/2} s^{1/2} q^{y(y+1)/2} r^y$ ($y \in \mathbb{Z}$) has discriminant $\mathcal{D} = 3/4$. More generally, one can show that $\mathcal{D} \geq 3/4$.

$m = 3$ From the multiplicative lift, we see that the coefficient of $s^{3/2}$ is

$$\hat{\phi}^{\hat{\rho}}_{k,1/2}(\tau,z)\,\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)\ .$$

The Fourier expansion of the multiplicative seed $\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$ has terms with negative discriminant, $\mathcal{D} = -1$. These are of the form $q^n r^\ell s^m$ with $n = x(x+1),\ m = 1,\ \ell = 2x+1$ for $x \in \mathbb{Z}$ with $C_{i\infty}(\mathcal{D} = -1, \ell) = +1$ for all conjugacy classes $\hat{\rho}$. Combining this family of terms with $q^{1/2}r^{1/2}s^{1/2}q^{y(y+1)/2}r^y$ $(y \in \mathbb{Z})$ coming from $\hat{\phi}^{\hat{\rho}}_{k,1/2}(\tau,z)$, we obtain the term

$$[q^{1/2}r^{1/2}s^{1/2}q^{y(y+1)/2}r^y] \times [q^{x(x+1)}r^{2x+1}s]\ ,$$

which has discriminant $\mathcal{D} = (3/4) + 2(x-y)^2 > 0$. These are, potentially, the only terms that might have had a negative discriminant. Other terms from $\hat{\phi}^{\hat{\rho}}_{k,1/2}(\tau,z)$ come with higher powers of q that only increase the value of the discriminant. Thus, one has $\mathcal{D} \geq 3/4$.

For $m > 3$, we now provide a heuristic argument as to why we expect $\mathcal{D} > 0$. All terms with negative discriminant come from the expansion of exponential in Eq. (3.1). Further, they arise from the action of the Hecke operator on $\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$. Since terms with discriminant $\mathcal{D} = -1$ that appear in $\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$ are the same for all conjugacy classes, we can split, $\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$ as

$$\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z) = \psi_{\mathcal{D}=-1}(\tau,z) + \psi^{\hat{\rho}}_{\mathcal{D}\geq 0}(\tau,z)\ ,$$

where the first term which contains *all* terms with negative discriminant. Consider the product (for $y \in \mathbb{Z}$)

$$[q^{1/2}r^{1/2}s^{1/2}q^{y(y+1)/2}r^y] \times \exp\Big(\sum_{m=1}^{\infty} s^m\,\hat{\psi}_{\mathcal{D}=-1}\big|_0 T(m)(\tau,z)\Big)\ .$$

Expanding the above, we claim that all terms have discriminant $\mathcal{D} > 0$. The dependence on the conjugacy class arises solely from the terms that are not accounted above. Such terms all have $\mathcal{D} \geq 0$ and contain only non-negative powers of q and s. Thus, this ensures that all terms that appear in Eq. (4.1) must have $\mathcal{D} \geq 0$.[5] Thus, we expect the conclusion to hold for the three conjugacy classes for which we do not have an additive lift.

4.2. Establishing the Weyl–Kac–Borcherds denominator formula

The Weyl–Kac–Borcherds denominator formula takes the form for all the conjugacy classes

$$\Delta^{\hat{\rho}}_k(\mathbf{Z}) = q^{1/2}r^{1/2}s^{1/2} \prod_{(n,\ell,m)>0} \Big(1 - q^n r^\ell s^m\Big)^{c(nm,\ell)} \tag{4.2a}$$

$$= \sum_{w\in W} \det(w) \left(e^{-w(\varrho)} - \sum_{a\in Q\cap\mathcal{M}} m(\alpha)\, e^{-w(\varrho+a)} \right) \tag{4.2b}$$

[5] Let $T_1 = q^{n_1}r^{\ell_1}s^{m_1}$ and $T_2 = q^{n_2}r^{\ell_2}s^{m_2}$ be such that $n_1, n_2, m_1, m_2 \geq 0$ and $\mathcal{D}_1 > 0$, $\mathcal{D}_2 \geq 0$. Then, the discriminant, $\mathcal{D}$, of the product T_1T_2 is also positive definite, i.e., $\mathcal{D} > 0$.

1. The first line, Eq. (4.2a), follows from the multiplicative lift. In particular, Eqns. (3.5) and (3.6) are precisely of this form. Proposition 3.2 implies that $c(nm,\ell) \in \mathbb{Z}$. $c(nm,\ell)$ is the multiplicity of the positive root $\alpha[n.\ell,m]$ and negative values of $c(nm,\ell)$ corresponds to fermionic roots. The condition $(n,\ell,m) > 0$ determines all the positive roots.
2. Eq. (4.2b) is the sum side of the denominator formula.
   (a) The expansion of all Siegel modular forms has the following terms

$$\Delta_k^{\hat{\rho}}(\mathbf{Z}) = e^{-\varrho}\Big(1 - e^{-w_1(\varrho)+\varrho} - e^{-w_2(\varrho)+\varrho} - e^{-w_3(\varrho)+\varrho} + \cdots\Big)$$
$$= q^{1/2}r^{1/2}s^{1/2}\Big(1 - qr - r^{-1} - sr + \cdots\Big)\ ,$$

   where e^{ϱ} (ϱ is the Weyl vector) is identified with $q^{1/2}r^{1/2}s^{1/2}$ and the three terms shown are the terms corresponding to the real simple roots.
   (b) The Siegel modular form is invariant the cyclic $\mathbb{Z}_3$ symmetry that permutes the three real simple roots. It is generated by the $Sp(2,\mathbb{Z})$ transformation $\tilde{\gamma} = \delta \cdot V \cdot [1,0;0]_H$. The character $v(\tilde{\gamma})$ is given by

$$v(\tilde{\gamma}) = v(\delta) \times v(V) \times v([1,0;0]_H) = -1 \times 1 \times -1 = +1\ .$$

   The symmetry of the Siegel modular form under the involution $\tau \leftrightarrow \tau'$ makes it invariant under the dihedral symmetry. The antisymmetry under $\delta : z \to -z$ is equivalent to the Weyl reflection w_2. When combined with the dihedral symmetry, we see that

$$\Delta_k^{\hat{\rho}}(w_i \cdot \mathbf{Z}) = -\Delta_k^{\hat{\rho}}(\mathbf{Z})\ , \tag{4.3}$$

   for $i = 1,2,3$. This implies that for any $w \in W$, one has

$$\Delta_k^{\hat{\rho}}(w \cdot \mathbf{Z}) = \det(w)\ \Delta_k^{\hat{\rho}}(\mathbf{Z})\ , \tag{4.4}$$

   where $\det(w) = +1$ (resp. -1) if w is generated by a combination of even (resp. odd) elementary reflections.
   (c) In the following we repeat arguments due to Gritsenko and Nikulin [27, Theorem 2.3] as they are applicable here as well. We can rewrite Eq. (4.1) as

$$\Delta_k^{\hat{\rho}}(\mathbf{Z}) = \sum_{\substack{n,\ell,m\equiv 1 \bmod 2\\ ||\alpha[\frac{n}{2},\frac{\ell}{2},\frac{m}{2}]||^2<0\\ n,m>0}} f(nm,\ell)\ e^{-\alpha[\frac{n}{2},\frac{\ell}{2},\frac{m}{2}]}\ . \tag{4.5}$$

   Let $\alpha[\frac{n}{2},\frac{\ell}{2},\frac{m}{2}] = \varrho + a[n',\ell',m']$. Then $(n',m') \in \mathbb{Z}_+^2$ and $\ell' \in \mathbb{Z}$. Due to property (4.4), Eq. (4.5) can be rewritten as follows:

$$\Delta_k^{\hat{\rho}}(\mathbf{Z}) = \sum_{w\in W} \det(w)\ w\left(\sum_{\varrho+a\in\mathcal{M}\cap\frac{1}{2}Q} -m(a)\ e^{-\varrho-a}\right) \tag{4.6}$$

   where $e^{-a} \sim q^{n'}r^{\ell'}s^{m'}$ and $m(a) = -f(n'+\frac{1}{2}, m'+\frac{1}{2}, \ell'+\frac{1}{2})$ and $(\varrho+a)$ lies in the Weyl chamber $\mathcal{M}$ i.e., $(\varrho+a,\delta_i) \le 0$ for $i = 1,2,3$. However, $(\varrho+a,\delta_i) = 0$ does not happen as it needs $(n,m,\ell) = (0,0,0)$ for which $f(0,0,0) = 0$. Thus, the stronger condition $(\varrho+a,\delta_i) < 0$ holds. This implies that $(a,\delta_i) < 1$ for $i = 1,2,3$. Integrality of (n',ℓ',m') implies that (a,δ_i) must be integral and thus the stronger condition given below holds.

$$(a, \delta_i) \leq 0 \text{ for } i = 1, 2, 3\ . \tag{4.7}$$

Thus $a \in Q \cap \mathcal{M}$. The equality for all three values of i occurs only when $a = 0$ for which one has $m(0) = -1$. When $a \neq 0$, the above condition implies $(a, a) \leq 0$ – in other words these correspond to imaginary simple roots. Eq. (4.2b) follows from Eq. (4.6) after separating the $a = 0$ term from the terns with $a \neq 0$.

(d) The multiplicity of imaginary simple roots with $(a, a) = 0$ can be determined by observing that the primitive roots of this type are: $a_0 := a[1, 0, 0] = (\delta_1 + \delta_2)$, $a[0, 0, 1] = (\delta_2 + \delta_3)$ and $a[1, 2, 1] = (\delta_1 + \delta_3)$. These are permuted by the dihedral group and so it suffices to consider a_0 and integral multiples of it. The multiplicity of these imaginary simple roots are determined by the zeroth Fourier–Jacobi coefficient, $\phi_k^{\hat{\rho}}(\tau, z)$. Identifying the $\theta_1(\tau, z)$ with the denominator formula for the $\widehat{sl(2)}$ generated by δ_1 and δ_2, we see that remaining eta products determine the multiplicities by the formula:

$$q^{-3/8}\ \frac{\eta_{\hat{\rho}}(\tau)}{\eta(\tau)^3} = 1 - \sum_m m(ta_0)\ q^t\ . \tag{4.8}$$

Thus the Siegel modular forms $\Delta_k^{\hat{\rho}}(\mathbf{Z})$ indeed provide the denominator formula for a family of Borcherds–Kac–Moody super Lie algebras.

5. Concluding remarks

The main positive result of our paper is to show the existence of BKM Lie superalgebras as the end-point of a sequence of moonshines for $L_2(11)_{A/B}$ that sees a beautiful interplay involving multiplicative eta products, EOT Jacobi forms, and Siegel modular forms. It is known that on extending considerations to include CHL orbifolds preserving $\mathcal{N} = 4$ supersymmetry leads to other classes of BKM Lie superalgebras whose real simple roots differ from the examples considered here [6]. The orbifolding group for the CHL orbifolds in the type IIA picture arise from symplectic automorphisms of $K3$. These are known to be sub-groups of M_{23} from the work of Mukai [28]. The conjugacy classes of elements of this group are determined by its order with the order ≤ 8. Looking at elements of $M_{12}{:}2$ that are also elements of M_{23} picks out conjugacy classes of $M_{12}{:}2$ with at least 1 one-cycle and more than four cycles in its cycle shape. This rules out conjugacy classes of $L_2(11)_A$ with orders 2, 3 and 6 which have no one-cycles and hence are related to $M_{12}{:}2$ conjugacy classes with no one-cycles. The order 11 elements of $M_{12}{:}2$ with cycle shape $1^2 11^2$ have only 4 cycles. Thus, the simple CHL $\mathbb{Z}_N$ orbifolds occur for $N \leq 8$. In our forthcoming paper [29], we revisit these considerations and provide evidence for a new type of BKM Lie algebras that arise for the CHL $\mathbb{Z}_5$ and $\mathbb{Z}_6$ orbifolds. The cycle shapes associated with the conjugacy classes of $L_2(11)_A$ with orders 2, 3 and 6 appear in considering cases involving generalised moonshine associated with commuting pairs of elements. The Jacobi forms that appear in the multiplicative lift here are those that arise in the context of umbral moonshine [30].

The squares of the Siegel modular forms that we construct, i.e., $\Delta_k^{\hat{\rho}}(\mathbf{Z})$, are Siegel modular forms that appear in the context of M_{24}-moonshine. The construction using products of rescaled Borcherds products for the cases when the order of the group element is prime or powers of a prime number connects up to the work of Raum [24]. Other than the conjugacy classes 2^{12} and 3^8 of M_{24}, our results agree when a comparison is possible. For 3^8, this is due to an incorrect assignment of level during programming and our results correct his. For the order 6 conjugacy classes (i.e., $1^2 2^2 3^2 6^2$ and 6^4) of M_{24}, our results extend Raum's computations. The M_{12} conjugacy class, $2^1 10^1$, is one where the Jacobi form is one half of an EOT Jacobi form. In this instance, we obtain the following product formula:

$$(\Delta_0^{2^1 10^1}(\mathbf{Z}))^{10} = B_{10\psi^{2^1 10^1}}(\mathbf{Z}) B_{5\psi^{1^2 5^2}}(2\mathbf{Z}) B_{\psi^{1^{12}}}(10\mathbf{Z}) \; . \tag{5.1}$$

This is different from the formula given in Proposition 3.10 where the Siegel modular form, $\Delta_k^{\hat{\rho}}(\mathbf{Z})$, was raised to the power of the smallest cycle shape which is two in the current example. This also provides another example where the naïve additive lift fails to match the product formula. One has

$$\frac{T_3\phi^{2^1 10^1}}{\phi^{2^1 10^1}} - \hat{\psi}^{2^1 10^1} = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\Big[\tfrac{20}{3}\;\eta_{2^2 10^2}(\tau)\Big] \, .$$

We anticipate that there is a BKM Lie superalgebras associated with this Siegel modular form as well. Clearly, it appears that we should be able construct Siegel modular forms for all conjugacy classes of M_{24} that don't appear in Raum's list or the ones that we considered here. We do not pursue this here and hope to report it elsewhere.

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Appendix A. Modular forms

A modular form, of weight k and character χ, is a function $f : \mathbb{H} \to \mathbb{C}$ such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$, one has

$$f|_k\gamma(\tau) = \chi(\gamma)\; f(\tau)\; , \tag{A.1}$$

where

$$f|_k\gamma(\tau) := (c\tau+d)^{-k}\; f(\gamma\cdot\tau)\; ,$$

and $\gamma\cdot\tau = \frac{a\tau+b}{c\tau+d}$. The level N sub-group $\Gamma_0(N) \subseteq PSL(2,\mathbb{Z})$ is given by restricting to γ with $c = 0$ mod$N$.

A.1. Weight two modular forms

The Eisenstein series at level $N > 1$ and weight 2 is defined as follows:

$$E_2^{(N)}(\tau) := \tfrac{12i}{\pi(N-1)}\partial_\tau\big[\ln\eta(\tau) - \ln\eta(N\tau)\big] = 1 + \tfrac{24}{N-1}q + \cdots \; .$$

Note that $\frac{N-1}{24}E_2^{(N)}(\tau)$ has integral coefficients except for the constant term. Let $f(\tau)$ denote the following weight two modular form of $\Gamma_0(16)$:

$$f(\tau) := \frac{1}{4}\left(\eta_{4^8 8^{-4}}(\tau) - \eta_{1^4 2^2 4^{-2}}(\tau)\right) = q - 4q^3 + 6q^5 - 8q^7 + \cdots \tag{A.2}$$

An alternate formula for $f(\tau)$ is as a generalised Eisenstein series [31]

$$f(\tau) = E_{2,\chi,\chi}(\tau) := \sum_{m=0}^{\infty}\Big[\sum_{n|m} \chi(n)\,\chi(\tfrac{m}{n})\,m\Big]\,q^m\ ,$$

where $\chi(m) = \left(\frac{-4}{m}\right)$ is a real Dirichlet character modulo 4. A basis for five-dimensional space of weight two modular forms of $\Gamma_0(16)$ is given by[6]

$$E_2^{(2)}(\tau),\ E_2^{(4)}(\tau),\ E_2^{(8)}(\tau),\ E_2^{(16)}(\tau) \text{ and } f(\tau)\ . \tag{A.3}$$

The first five Fourier coefficients of any weight two modular form of $\Gamma_0(16)$ uniquely determine the modular form. A basis for weight two modular forms of $\Gamma_0(32)$ is obtained by adding three more weight two modular forms:

$$E_2^{(32)}(\tau),\ f(2\tau)\ \text{ and the cusp form } \eta_{4^2 8^2}(\tau)\ ,$$

to the $\Gamma_0(16)$ basis. Eight of the first nine Fourier coefficients of any weight two modular form of $\Gamma_0(32)$ uniquely determines the modular form.

A.2. Siegel and Jacobi forms

The group $Sp(2,\mathbb{Z})$ is the set of 4×4 matrices written in terms of four 2×2 matrices A, B, C, D (with integral entries) as $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ satisfying $AB^T = BA^T$, $CD^T = DC^T$ and $AD^T - BC^T = I$. This group acts naturally on the Siegel upper half space, $\mathbb{H}_2$, as

$$\mathbf{Z} = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \longmapsto M\cdot\mathbf{Z} \equiv (A\mathbf{Z}+B)(C\mathbf{Z}+D)^{-1}\ .$$

The level $N \in \mathbb{Z}_{>0}$ subgroup, $\Gamma_0^{(2)}(N)$, of $Sp(2,\mathbb{Z})$ is given by restricting to M such that $C = 0 \bmod N$.

The Jacobi sub-group, $\Gamma^J(N) \subset \Gamma_0^{(2)}(N)$, is the semi-direct product of the Heisenberg group and $\Gamma_0(N)$ defined as follows:

$$\Gamma^J(N) = \Gamma_0(N) \ltimes H(\mathbb{Z}), \tag{A.4}$$

where $H(\mathbb{Z})$ is given by

$$H(\mathbb{Z}) = \left\{ [\lambda,\mu;\kappa]_H := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix},\qquad \lambda,\mu,\kappa\in\mathbb{Z} \right\}\ . \tag{A.5}$$

The embedding of $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N)$ in $\Gamma_t(N)$ is given by

[6] We have used SAGE to obtain the dimension of the spaces of modular forms [32]. SAGE also provides a basis for the modular forms and we have verified that our choices are consistent with the choices given there.

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \equiv \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \; c = 0 \mod N \; . \tag{A.6}$$

Then, $\Gamma_0^{(2)}(N)$ is generated by adding the following transformation to $\Gamma^J(N)$:

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{A.7}$$

with $\det(C\mathbf{Z} + D) = -1$. This acts on $\mathbb{H}_2$ as the involution

$$(\tau, z, \tau') \longrightarrow (\tau', z, \tau) \; . \tag{A.8}$$

A Siegel modular form, of weight k with character υ with respect to $\Gamma_0^{(2)}(N)$, is a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ satisfying

$$F|_k M(\mathbf{Z}) = \upsilon(M) \; F(\mathbf{Z}) \; , \tag{A.9}$$

for all $M \in \Gamma_0^{(2)}(N)$ and the slash operation is defined as

$$F|_k M(\mathbf{Z}) := \det(C\mathbf{Z} + D)^{-k} \; F(M \cdot \mathbf{Z}) \; . \tag{A.10}$$

Jacobi Form: A holomorphic function $\phi_{k,m}(\tau, z) : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$ is called a Jacobi form of weight k and index m if the function

$$\tilde{\phi}_k(\mathbf{Z}) = \exp(2\pi i m \tau') \; \phi_{k,m}(\tau, z),$$

on $\mathbb{H}_2$ such that $\tilde{\phi}_k(\mathbf{Z})$ is a modular form of weight k with respect to the Jacobi group $\Gamma^J(N) \subseteq Sp(2, \mathbb{Z})$ with character υ i.e., it satisfies

$$\tilde{\phi}|_k \; M(\mathbf{Z}) = \upsilon(M) \; \tilde{\phi}_k(\mathbf{Z}) \qquad \forall M \in \Gamma^J(N) \; , \tag{A.11}$$

and it is holomorphic at all cusps, i.e., let $\gamma \in SL(2, \mathbb{Z})$ and $\tilde{\gamma}$, its embedding in $Sp(2, \mathbb{Z})$. Then it has a Fourier expansion

$$\tilde{\phi}|_k \tilde{\gamma}(\mathbf{Z}) = s^m \sum_{\substack{n,\ell \\ 4nm - l^2 \geq 0}} c_\gamma(n, \ell) \; q^n r^\ell =: s^m \sum_{\substack{n,\ell \\ \mathcal{D} = 4nm - l^2 \geq 0}} C_\gamma(\mathcal{D}, \ell) \; q^n r^\ell \; , \tag{A.12}$$

where $q = e^{2\pi i \tau}$, $r = e^{2\pi i z}$ and $s = e^{2\pi i \tau'}$ and $n, \ell \in \mathbb{Q}$. The combination $\mathcal{D} := (4nm - \ell^2)$ is called the *discriminant*. The Fourier coefficients of a Jacobi form depend only on the discriminant and the value of $\ell \mod 2m$.

(a) $\phi_{k,m}(\tau, z)$ is called a *cusp form* if $C_\gamma(\mathcal{D}, \ell) = 0$ unless $\mathcal{D} > 0$ at all cusps.
(b) $\phi_{k,m}(\tau, z)$ is called a *weak Jacobi form* if $C_\gamma(\mathcal{D}, \ell) = 0$ unless $n \geq 0$ at all cusps.
(c) $\phi_{k,m}(\tau, z)$ is called a *nearly holomorphic* if there exists an $n \in \mathbb{N}$ such that $\Delta^n \phi_{k,m}(\tau, z)$ is a weak Jacobi form where $\Delta = \eta(\tau)^{24}$.

The space of all Jacobi forms with character for $\Gamma^J(N)$ is denoted by $J_{k,m}(\Gamma_0(N), \upsilon)$. Similarly the space for all weak and nearly holomorphic Jacobi forms are denoted by $J^w_{k,m}(\Gamma_0(N))$ and $J^{nh}_{k,m}(\Gamma_0(N))$ respectively.

A.3. Examples

An example of Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ is the Jacobi theta function of level 8:

$$\begin{aligned}\theta_1(\tau,z) &= \sum_{m\in\mathbb{Z}}\left(-\frac{4}{m}\right)q^{m^2/8}r^{m/2} \\ &= -q^{1/8}r^{-1/2}\prod_{n\geq 1}(1-q^{n-1}r)(1-q^nr^{-1})(1-q^n)\,, \\ &= q^{1/8}\left(-\frac{1}{\sqrt{r}}+\sqrt{r}\right)+q^{9/8}\left(\frac{1}{r^{3/2}}-r^{3/2}\right)+\cdots\end{aligned} \tag{A.13}$$

This is an element of $J_{\frac{1}{2},\frac{1}{2}}(SL(2,\mathbb{Z}), v_\eta^3\times v_H)$ where v_η is the character of the Dedekind η-function and

$$v_H([\lambda,\mu:\kappa]_H) = (-1)^{\lambda+\mu+\lambda\mu+\kappa}\,. \tag{A.14}$$

Then the weight -1 index $\frac{1}{2}$ Jacobi form $\frac{\theta_1(\tau,z)}{\eta(\tau)^3}$ has character v_H.

More generally, the genus-one theta functions are defined by

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(\tau,z) = \sum_{l\in\mathbb{Z}} q^{\frac{1}{2}(l+\frac{a}{2})^2}\, r^{(l+\frac{a}{2})}\, e^{i\pi lb}\,, \tag{A.15}$$

where $a,\ \ b\in(0,1) \mod 2$. We define $\theta_1(\tau,z)\equiv\theta\begin{bmatrix}1\\1\end{bmatrix}(\tau,z)$, $\theta_2(\tau,z)\equiv\theta\begin{bmatrix}1\\0\end{bmatrix}(z_1,z)$, $\theta_3(\tau,z)\equiv\theta\begin{bmatrix}0\\0\end{bmatrix}(\tau,z)$ and $\theta_4(\tau,z)\equiv\theta\begin{bmatrix}0\\1\end{bmatrix}(\tau,z)$.

The characters of the level 1 $\mathcal{N}=4$ superconformal algebra that appear in our decomposition of the Jacobi forms of weight zero index 1 are $\mathcal{C}(\tau,z)$ and $q^{h-\frac{1}{8}}\mathcal{B}(\tau,z)$ $(h>0)$, where

$$\mathcal{C}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\frac{i}{\theta_1(\tau,2z)}\sum_{n\in\mathbb{Z}} q^{2n^2}r^{4n}\frac{1+q^nr}{1-q^nr}\,. \tag{A.16}$$

$$\mathcal{B}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\,. \tag{A.17}$$

A.4. The EOT Jacobi forms for M_{12}:2

Let $\phi_{0,1}(\tau,z)$ and $\phi_{-2,1}(\tau,z)$ denote the following Jacobi forms:

$$\phi_{0,1}(\tau,z) = 4\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}+\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}+\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right] = (r^{-1}+10+r)+O(q)\,,$$

$$\phi_{-2,1}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} = (r^{-1}-2+r)+O(q)\,.$$

These are the unique weak Jacobi forms of index 1 and weight ≤ 0. They generate the ring of weak Jacobi forms freely over the space of modular forms [33]. Thus, all weak Jacobi forms of index 1 and weight zero can be given in terms of a constant and a modular form of weight 2 at suitable level. Table 2 lists all EOT Jacobi forms that appear for all conjugacy classes of M_{12}:2. All EOT Jacobi forms have Fourier expansions about the cusp at $i\infty$ that are non-vanishing when the discriminant $\mathcal{D}\geq -1$. Further $C(\mathcal{D}-1,1)=2$ for all EOT Jacobi forms. This implies that terms with negative discriminant are identical for all of EOT Jacobi forms.

Table 2
The EOT Jacobi forms as given in [7–9] for all conjugacy classes of M_{24} that reduce to conjugacy classes of $M_{12}{:}2$.

Conj. class	$\widetilde{\rho}$	$Z^{\widetilde{\rho}}$
1a	1^{24}	$2\phi_{0,1}(\tau,z)$
2a/c	2^{12}	$(-2E_2^{(2)}(\tau)+4E_2^{(4)}(\tau))\,\phi_{-2,1}(\tau,z)$
2a/c	2^{12}	$2\eta_{1^8 2^{-4}}(\tau)\,\phi_{-2,1}(\tau,z)$
2b	$1^8 2^8$	$\frac{2}{3}\phi_{0,1}(\tau,z)+\frac{4}{3}E_2^{(2)}(\tau)\,\phi_{-2,1}(\tau,z)$
3a	$1^6 3^6$	$\frac{1}{2}\phi_{0,1}(\tau,z)+\frac{3}{2}E_2^{(3)}(\tau)\,\phi_{-2,1}(\tau,z)$
3b	3^8	$2\eta_{1^6 3^{-2}}(\tau)\,\phi_{-2,1}(\tau,z)$
4a	$1^4 2^2 4^6$	$\frac{1}{3}\phi_{0,1}(\tau,z)+(-\frac{1}{3}E_2^{(2)}(\tau)+2E_2^{(4)}(\tau))\,\phi_{-2,1}(\tau,z)$
5a	$1^4 5^4$	$\frac{1}{3}\phi_{0,1}(\tau,z)+\frac{5}{3}E_2^{(5)}(\tau)\,\phi_{-2,1}(\tau,z)$
6a/c	6^4	$2\eta_{1^2 2^2 3^2 6^{-2}}(\tau)\,\phi_{-2,1}(\tau,z)$
6b	$1^2 2^2 3^2 6^2$	$\frac{1}{6}\phi_{0,1}(\tau,z)+(-\frac{1}{6}E_2^{(2)}(\tau)-\frac{1}{2}E_2^{(3)}(\tau)+\frac{5}{2}E_2^{(6)}(\tau))\,\phi_{-2,1}(\tau,z)$
8a	$1^2 2^1 4^1 8^2$	$\frac{1}{6}\phi_{0,1}(\tau,z)+(-\frac{1}{2}E_2^{(4)}(\tau)+\frac{7}{3}E_2^{(8)}(\tau))\,\phi_{-2,1}(\tau,z)$
10a/b/c	$2^2 10^2$	$2\eta_{1^3 2^1 5^1 10^{-1}}(\tau)\,\phi_{-2,1}(\tau,z)$
11a	$1^2 11^2$	$\frac{1}{6}\phi_{0,1}(\tau,z)+(\frac{11}{6}E_2^{(11)}(\tau)-\frac{22}{5}\eta_{1^2 11^2}(\tau))\,\phi_{-2,1}(\tau,z)$
4b	$2^4 4^4$	$2\eta_{2^8 4^{-4}}(\tau)\,\phi_{-2,1}(\tau,z)$
4c	4^6	$2\eta_{1^4 2^2 4^{-2}}(\tau)\,\phi_{-2,1}(\tau,z)$
12a	12^2	$2\eta_{1^4 2^{-1} 4^1 6^1 12^{-1}}(\tau)\,\phi_{-2,1}(\tau,z)$
12b/c	$2^1 4^1 6^1 12^1$	$2\eta_{1^3 2^{-1} 3^{-1} 4^2 6^3 12^{-2}}(\tau)\,\phi_{-2,1}(\tau,z)$

Appendix B. Computations for the Borchherds product formula

B.1. *Proving Equation* (3.13)

Cycle Shape $\hat{\rho}=2^6$

The multiplicative seed, $\psi=\hat{\psi}^{2^6}_{0,1}(\tau,z)=\frac{1}{2}Z^{2^{12}}(\tau,z)$, is a Jacobi form of $\Gamma_0(4)$. The cusps and other data for $\Gamma_0(4)$ are as follows:

f/e	$i\infty$	$1/2$	$0/1$
h_e	1	1	4
N_e	1	2	4

We need the Fourier expansion of ψ about these cusps. Using $M_0=1$ and $M_{1/p}=-ST^{-p}S$, we can compute the expansion using methods described in chapter 2 of [34].

$$\hat{\psi}^{\hat{\rho}}|_{0,1}M_0(\tau,z)=-4\frac{\eta(\tau)^8}{\eta(\tau/2)^4}\phi_{-2,1}(\tau,z) \tag{B.1}$$

$$\hat{\psi}^{\hat{\rho}}|_{0,1}M_{\frac{1}{2}}(\tau,z)=-\frac{\eta(\tau)^8}{\eta(2\tau)^4}\phi_{-2,1}(\tau,z)=-\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z) \tag{B.2}$$

The Fourier expansion of the cusp about zero does not have any terms with integral powers of q and hence this cusp does not contribute to the product formula.

Cycle Shape $\hat{\rho}=3^4$

The multiplicative seed, $\psi=\hat{\psi}^{3^4}_{0,1}(\tau,z)=\frac{1}{2}Z^{3^8}(\tau,z)$, is a Jacobi form of $\Gamma_0(9)$. The cusps and other data for $\Gamma_0(9)$ are as follows:

Table 3
Character Decomposition of the M_{12} Jacobi forms.

	$\widehat{\chi}_1$	$\widehat{\chi}_2$	$\widehat{\chi}_3$	$\widehat{\chi}_4$	$\widehat{\chi}_5$	$\widehat{\chi}_6$	$\widehat{\chi}_7$	$\widehat{\chi}_8$	$\widehat{\chi}_9$	$\widehat{\chi}_{10}$	$\widehat{\chi}_{11}$	$\widehat{\chi}_{12}$	$\widehat{\chi}_{13}$	$\widehat{\chi}_{14}$	$\widehat{\chi}_{15}$
1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
q^2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
q^3	0	0	0	0	0	0	0	0	0	0	1	0	2	2	1
q^4	0	0	0	0	0	1	2	3	0	4	1	3	2	3	4
q^5	0	0	0	2	2	4	4	1	3	3	4	5	8	9	11
q^6	0	2	4	1	1	4	8	10	8	10	10	15	16	21	26
q^7	0	0	4	7	7	18	16	15	24	10	23	32	42	46	56
q^8	1	6	12	10	10	28	38	43	46	32	43	70	78	98	124
q^9	1	13	15	23	23	66	76	70	113	37	94	134	174	206	242
q^{10}	3	31	35	42	42	119	148	162	219	89	179	276	322	390	485
q^{11}	4	62	40	88	88	242	278	272	442	122	346	511	632	753	914
q^{12}	10	146	84	147	147	420	522	546	809	259	633	956	1144	1384	1699
q^{13}	19	264	102	286	286	801	938	933	1506	396	1152	1716	2102	2506	3051
q^{14}	30	500	192	484	484	1364	1664	1721	2628	768	2018	3056	3666	4420	5423
q^{15}	52	889	263	861	861	2420	2874	2896	4603	1241	3535	5263	6434	7697	9375
q^{16}	94	1579	455	1444	1444	4069	4922	5058	7754	2290	5994	9033	10886	13087	16032
q^{17}	151	2664	652	2468	2468	6920	8248	8340	13003	3773	10099	15107	18382	22027	26887
q^{18}	252	4501	1133	4020	4020	11330	13674	14000	21243	6639	16689	25077	30316	36427	44563
q^{19}	412	7330	1686	6647	6647	18681	22316	22644	34484	10950	27318	40913	49696	59567	72744
q^{20}	669	11917	2853	10649	10649	29960	36064	36844	54889	18611	44021	66134	80010	96094	117541
q^{21}	1064	18925	4427	17087	17087	48040	57526	58442	86807	30313	70371	105420	127988	153496	187481
q^{22}	1692	29831	7327	26877	26877	75625	90908	92775	135259	50001	111037	166710	201830	242298	296284
q^{23}	2622	46244	11482	42197	42197	118616	142120	144536	209293	80135	173798	260529	316064	379145	463254
q^{24}	4082	71296	18694	65174	65174	183384	220348	224690	320080	128864	269200	403992	489368	587424	718126
q^{25}	6270	108377	29259	100406	100406	282327	338446	344382	486339	202971	413792	620437	752450	902705	1103084
q^{26}	9555	163767	46683	152718	152718	429576	515886	525845	731812	319208	630341	945863	1145966	1375439	1681406
q^{27}	14433	244901	72561	231277	231277	650388	780008	793968	1094465	494269	953589	1429925	1733926	2080389	2542299
q^{28}	21711	364030	113550	346819	346819	975551	1171218	1193511	1623580	762466	1431222	2147351	2602046	3122821	3817239
q^{29}	32314	536411	174379	517616	517616	1455614	1746034	1777621	2394700	1161740	2134316	3200923	3880816	4656537	5690817
q^{30}	47909	786171	268275	766024	766024	2154660	2586488	2635260	3507492	1761600	3160915	4742013	5746832	6896777	8429971
q^{31}	70489	1143629	406567	1128391	1128391	3173388	3806978	3876453	5110133	2644483	4653450	6979350	8461124	10152616	12408027
q^{32}	103184	1655300	615246	1650225	1650225	4641456	5570988	5675403	7399554	3949188	6808400	10213676	12378560	14855132	18157233

f/e	$i\infty$	1/3	2/3	0/1
h_e	1	1	1	9
N_e	1	3	3	9

Using $M_{2/3} = -ST^{-1}ST^2S$, we obtain

$$\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)|M_{\frac{1}{3}} = e^{\frac{2\pi i}{3}}\,\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z) \tag{B.3}$$

$$\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)|M_{\frac{2}{3}} = e^{\frac{4\pi i}{3}}\,\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z) \tag{B.4}$$

Again the cusp about 0 does not have any terms with integral powers of q and hence does not contribute to the Borcherds product.

Cycle Shape $\hat{\rho} = 6^2$

The multiplicative seed, $\psi = \hat{\psi}^{6^2}_{0,1}(\tau,z) = \frac{1}{2}Z^{6^4}(\tau,z)$, is a Jacobi form of $\Gamma_0(36)$. The cusps and other data for $\Gamma_0(36)$ are as follows:

f/e	$i\infty$	0	1/2	1/3	2/3	1/4	1/6	5/6	1/9	1/12	5/12	1/18
h_e	1	36	9	4	4	9	1	1	4	1	1	1
N_e	1	36	18	12	12	9	6	6	4	3	3	2

Using $M_{\frac{5}{6}} = -ST^{-1}ST^5S$ and $M_{\frac{5}{12}} = ST^{-2}ST^2ST^{-2}S$, we obtain

$$\hat{\psi}^{\hat{\rho}}|_{0,1}M_{\frac{1}{6}}(\tau,z) = -e^{\frac{-2\pi i}{3}}\,\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$$

$$\hat{\psi}^{\hat{\rho}}|_{0,1}M_{\frac{5}{6}}(\tau,z) = -e^{\frac{-4\pi i}{3}}\,\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$$

$$\hat{\psi}^{\hat{\rho}}|_{0,1}M_{\frac{1}{12}}(\tau,z) = e^{\frac{2\pi i}{3}}\,\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$$

$$\hat{\psi}^{\hat{\rho}}|_{0,1}M_{\frac{5}{12}}(\tau,z) = e^{\frac{4\pi i}{3}}\,\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$$

$$\hat{\psi}^{\hat{\rho}}|_{0,1}M_{\frac{1}{18}}(\tau,z) = -\hat{\psi}^{\hat{\rho}}_{0,1}(\tau,z)$$

The Fourier expansions about the cusps at 0, 1/2, 1/3, 2/3, 1/4, 1/9 do not have any terms with integral powers of q and hence do not contribute to the Borcherds product.

Appendix C. Basic group theory

C.1. M_{12} and M_{12}:2

In the 12-dimensional permutation representation, M_{12} is generated as $\langle\alpha,\beta,\gamma,\delta\rangle$. $L_2(11)_A$ is a maximal subgroup of M_{12} while one has $L_2(11)_B \subset M_{11} \subset M_{12}$. The four conjugacy classes associated with elements of order 4, 8 and 10 do *not* reduce to conjugacy classes of either $L_2(11)_A$ or $L_2(11)_B$.

Let g denote an element of M_{12} in the 12-dimensional permutation representation. Let φ denote the outer automorphism of M_{12}. It acts on the generators of M_{12} as

$$\alpha^\varphi = \varphi\alpha\varphi^{-1} = \alpha^{-1} \quad , \quad \beta^\varphi = \beta \quad , \quad \gamma^\varphi = \gamma^{-1} \quad , \quad \delta^\varphi = \delta\ .$$

The 24-dimensional permutation representation of $M_{12}{:}2$ consists of two classes of elements given in block-diagonal form below:

$$(g,e) := \begin{pmatrix} g & 0 \\ 0 & \varphi(g) \end{pmatrix} \quad \text{and} \quad (g,\varphi) := \begin{pmatrix} 0 & g \\ \varphi(g) & 0 \end{pmatrix} .$$

Conjugacy classes, $\widetilde{\rho}$ of type (g,e) of $M_{12}{:}2$ descend to pairs of conjugacy classes of M_{12}. Explicitly, one has $\widetilde{\rho} = (\widehat{\rho}, \varphi(\widehat{\rho}))$, where $\widehat{\rho}$ is the conjugacy class of g in M_{12}. One has the sequence of groups

$$L_2(11)_{A/B} \subset M_{12} \xrightarrow{\varphi} M_{12}{:}2 \subset M_{24} .$$

C.1.1. $M_{12}{:}2$ characters from M_{12} characters

Given a group G and a $\mathbb{Z}_2$ automorphism φ, the characters of the group G are related to those of the group $G.2$ in two possible ways [35]:

1. *The splitting case:* A character $\hat{\chi}_m$ of G may give rise to two characters of $G.2$ – call them $\tilde{\chi}_a$ and $\tilde{\chi}_{a'}$. For elements of type (g,e), they are given by $\tilde{\chi}_a = \tilde{\chi}_{a'} = \hat{\chi}_m$. For elements of type (g,φ), one has $\tilde{\chi}_a + \tilde{\chi}_{a'} = 0$. Thus we have a natural pairing (a,a') of representations of $G.2$ and they are mapped to the m-th representation of G. For $M_{12}{:}2$, the splitting representations are (in the notation $m \leftrightarrow (a,a')$)

$$\begin{aligned} &1 \leftrightarrow (1,2)\ ,\ 6 \leftrightarrow (5,6)\ ,\ 7 \leftrightarrow (7,8)\ ,\ 8 \leftrightarrow (9,10)\ , \\ &11 + a \leftrightarrow (12+2a, 13+2a) \text{ for } a = 0,1,\ldots,4\ . \end{aligned} \tag{C.1}$$

2. *The fusion case:* Two characters $\hat{\chi}_m$ and $\hat{\chi}_n$ fuse to give a single character, call it $\tilde{\chi}_{m,n}$. For elements of $G.2$ of type (g,e), one has $\tilde{\chi}_a = \hat{\chi}_m + \hat{\chi}_n$ and for elements of type (g,φ), one has $\tilde{\chi}_a = 0$.
   The fusion characters of $M_{12}{:}2$ are (in the notation $(m,n) \leftrightarrow a$)

$$(2,3) \leftrightarrow 3\ ,\ (4,5) \leftrightarrow 4\ ,\ (9,10) \leftrightarrow 11\ . \tag{C.2}$$

C.2. Character tables

Character table for $L_2(11)$ obtained from the GAP database [36]

	$1a$	$2a$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
χ_1	1	1	1	1	1	1	1	1
χ_2	5	1	-1	0	0	1	$-\frac{1}{2}+\frac{i\sqrt{11}}{2}$	$-\frac{1}{2}-\frac{i\sqrt{11}}{2}$
χ_3	5	1	-1	0	0	1	$-\frac{1}{2}-\frac{i\sqrt{11}}{2}$	$-\frac{1}{2}+\frac{i\sqrt{11}}{2}$
χ_4	10	-2	1	0	0	1	-1	-1
χ_5	10	2	1	0	0	-1	-1	-1
χ_6	11	-1	-1	1	1	-1	0	0
χ_7	12	0	0	$-\frac{1}{2}+\frac{\sqrt{5}}{2}$	$-\frac{1}{2}-\frac{\sqrt{5}}{2}$	0	1	1
χ_8	12	0	0	$-\frac{1}{2}-\frac{\sqrt{5}}{2}$	$-\frac{1}{2}+\frac{\sqrt{5}}{2}$	0	1	1

The character table for M_{12} (obtained from the GAP character table database)

Label	1a	2a	2b	3a	3b	4a	4b	5a	6a	6b	8a	8b	10a	11a	11b
$\widehat{\chi}_1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\widehat{\chi}_2$	11	−1	3	2	−1	−1	3	1	−1	0	−1	1	−1	0	0
$\widehat{\chi}_3$	11′	−1	3	2	−1	3	−1	1	−1	0	1	−1	−1	0	0
$\widehat{\chi}_4$	16	4	0	−2	1	0	0	1	1	0	0	0	−1	α	α^*
$\widehat{\chi}_5$	16′	4	0	−2	1	0	0	1	1	0	0	0	−1	α^*	α
$\widehat{\chi}_6$	45	5	−3	0	3	1	1	0	−1	0	−1	−1	0	1	1
$\widehat{\chi}_7$	54	6	6	0	0	2	2	−1	0	0	0	0	1	−1	−1
$\widehat{\chi}_8$	55_R	−5	7	1	1	−1	−1	0	1	1	−1	−1	0	0	0
$\widehat{\chi}_9$	55	−5	−1	1	1	3	−1	0	1	−1	−1	1	0	0	0
$\widehat{\chi}_{10}$	55′	−5	−1	1	1	−1	3	0	1	−1	1	−1	0	0	0
$\widehat{\chi}_{11}$	66	6	2	3	0	−2	−2	1	0	−1	0	0	1	0	0
$\widehat{\chi}_{12}$	99	−1	3	0	3	−1	−1	−1	−1	0	1	1	−1	0	0
$\widehat{\chi}_{13}$	120	0	−8	3	0	0	0	0	0	1	0	0	0	−1	−1
$\widehat{\chi}_{14}$	144	4	0	0	−3	0	0	−1	1	0	0	0	−1	1	1
$\widehat{\chi}_{15}$	176	−4	0	−4	−1	0	0	1	−1	0	0	0	1	0	0

(C.3)

where $\alpha = -\frac{1}{2} + i\frac{\sqrt{11}}{2}$. Under the outer automorphism, φ, of M_{12} one has

$$\varphi : \quad \widehat{\chi}_2 \leftrightarrow \widehat{\chi}_3 \quad , \quad \widehat{\chi}_4 \leftrightarrow \widehat{\chi}_5 \quad , \quad \widehat{\chi}_9 \leftrightarrow \widehat{\chi}_{10} \, . \tag{C.4}$$

The character table for M_{12}:2 (obtained from the GAP database)

Label	1a	2a	2b	3a	3b	4a	5a	6a	6b	8a	10a	11a	2c	4b	4c	6c	10b	10c	12a	12b	12c
$\widetilde{\chi}_1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\widetilde{\chi}_2$	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1
$\widetilde{\chi}_3$	22	−2	6	4	−2	2	2	−2	0	0	−2	0	0	0	0	0	0	0	0	0	0
$\widetilde{\chi}_4$	32	8	0	−4	2	0	2	2	0	0	−2	−1	0	0	0	0	0	0	0	0	0
$\widetilde{\chi}_5$	45	5	−3	0	3	1	0	−1	0	−1	0	1	5	−3	1	−1	0	0	1	0	0
$\widetilde{\chi}_6$	45	5	−3	0	3	1	0	−1	0	−1	0	1	−5	3	−1	1	0	0	−1	0	0
$\widetilde{\chi}_7$	54	6	6	0	0	2	−1	0	0	0	1	−1	0	0	0	0	$\sqrt{5}$	$-\sqrt{5}$	0	0	0
$\widetilde{\chi}_8$	54	6	6	0	0	2	−1	0	0	0	1	−1	0	0	0	0	$-\sqrt{5}$	$\sqrt{5}$	0	0	0
$\widetilde{\chi}_9$	55	−5	7	1	1	−1	0	1	1	−1	0	0	5	1	−1	−1	0	0	−1	1	1
$\widetilde{\chi}_{10}$	55	−5	7	1	1	−1	0	1	1	−1	0	0	−5	−1	1	1	0	0	1	−1	−1
$\widetilde{\chi}_{11}$	110	−10	−2	2	2	2	0	2	−2	0	0	0	0	0	0	0	0	0	0	0	0
$\widetilde{\chi}_{12}$	66	6	2	3	0	−2	1	0	−1	0	1	0	6	2	0	0	1	1	0	−1	−1
$\widetilde{\chi}_{13}$	66	6	2	3	0	−2	1	0	−1	0	1	0	−6	−2	0	0	−1	−1	0	1	1
$\widetilde{\chi}_{14}$	99	−1	3	0	3	−1	−1	−1	0	1	−1	0	1	−3	−1	1	1	1	−1	0	0
$\widetilde{\chi}_{15}$	99	−1	3	0	3	−1	−1	−1	0	1	−1	0	−1	3	1	−1	−1	−1	1	0	0
$\widetilde{\chi}_{16}$	120	0	−8	3	0	0	0	0	1	0	0	−1	0	0	0	0	0	0	0	$\sqrt{3}$	$-\sqrt{3}$
$\widetilde{\chi}_{17}$	120	0	−8	3	0	0	0	0	1	0	0	−1	0	0	0	0	0	0	0	$-\sqrt{3}$	$\sqrt{3}$
$\widetilde{\chi}_{18}$	144	4	0	0	−3	0	−1	1	0	0	−1	1	4	0	2	1	−1	−1	−1	0	0
$\widetilde{\chi}_{19}$	144	4	0	0	−3	0	−1	1	0	0	−1	1	−4	0	−2	−1	1	1	1	0	0
$\widetilde{\chi}_{20}$	176	−4	0	−4	−1	0	1	−1	0	0	1	0	4	0	−2	1	−1	−1	1	0	0
$\widetilde{\chi}_{21}$	176	−4	0	−4	−1	0	1	−1	0	0	1	0	−4	0	2	−1	1	1	−1	0	0

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