AIP Journal of Applied Physics

Transformation of coordinates associated with linearized supersonic motions

T. S. Shankara and Kamal Kanti Nandi

Citation: Journal of Applied Physics **49**, 5783 (1978); doi: 10.1063/1.324582
View online: http://dx.doi.org/10.1063/1.324582
View Table of Contents: http://scitation.aip.org/content/aip/journal/jap/49/12?ver=pdfcov
Published by the AIP Publishing

Transformation of coordinates associated with linearized supersonic motions

T. S. Shankara and Kamal Kanti Nandi

Theoretical Physics Division, Department of Mathematics, Indian Institute of Technology, Madras-600036, India
(Received 11 March 1977; accepted for publication 29 March 1978)

A Lorentz-like transformation (LLT) is introduced in the supersonic regime which keeps the wave equation invariant and simultaneously fixes the coordinate system to the body, producing small disturbances. Its implications, which appear to be far reaching, are briefly discussed.

PACS numbers: 47.40.Ki, 43.20.Bi

I. INTRODUCTION

The Lorentz transformation (LT) has been used in aerodynamics since 1940 when Küssner introduced it to solve the linearized unsteady equation of subsonic motion in the coordinate system fixed to the body.[1-4] As of today, this method has not been extended to linearized spacelike (supersonic) motion for the obvious reason that, as such, the Lorentz factor $(1-v^2/a^2_0)^{1/2}$ in LT would become imaginary for velocities v greater than the sound speed a_0. This algebraic difficulty is extensively discussed in the recent developments of relativity theory; but in aerodynamics, what one does instead (following the Ackeret theory) is to replace the factor $(1-v^2/a^2_0)^{1/2}$ by $(u^2/a^2_0-1)^{1/2}$, with $u>a_0$. Though this replacement has the advantage that the new Lorentz factor is real, the corresponding new LT fails to keep the wave equation invariant. Therefore, one needs an appropriate LT which will fix the coordinate system to the supersonically moving body and simultaneously keep the wave equation invariant. In some recent discussions of relativity theory, such Lorentz-like transformations (LLT) for spacelike motions, are being envisaged. Jones[5] has constructed conformal transformations in which the light velocity appears not as an upper limit of velocities but only as a singular velocity and in which the LLT does not become imaginary. More recently, LLT in $(\vec{r},t)$ space for which the Lorentz factor is not imaginary but which still keeps the wave equation invariant has been constructed and studied extensively.[6-11] Since spacelike motions have received little notice in relativity theory, the utility of this transformation remains a speculation; but in analogy to relativity with sound speed replacing the speed of light,[2] the usual LT has been used profitably in subsonic aerodynamics and supersonic motions do exist. Furthermore, it is also found that a large field of practical application does exist within the framework of small perturbation theory involving conventional linear analysis. Therefore, it is desirable to consider how much of this new transformation would meet the needs of the linearized equations of supersonic aerodynamics. In this paper, we briefly indicate the physical implications of this new transformation for the supersonic motion of a slender body creating only small disturbances.

The above analysis implies a multiplicity of the transformations which leave the wave equation invariant, and this raises the disturbing question whether, in that case, LT is not unique. Naturally, the unique form of LT should be based on specific hypotheses other than the invariance of the wave equation since this invariance alone is not sufficient to fix the LT uniquely, and efforts are sometimes made to formulate their physical and mathematical aspects. A definitive answer to this question is due to Church,[12] who long ago formulated the postulates which uniquely determine the LT. Obviously, the LLT for spacelike motions needs a different set of postulates which we enumerate. This becomes important if ambiguities are to be avoided.

In Sec. II, the LLT for supersonic motion is derived, assuming that the disturbances are small so that the equations are linear. This contains an unexpected term for the relative velocity whose physical significance is discussed in Sec. III. The constancy of sound speed in a fixed medium with respect to relative motion between the source and the observer is demonstrated in Sec. IV. Section V contains a short discussion of the linearized equations of motion, and Sec VIII contains the kinematical view of the shock stand-off distance for a sphere. In Sec. IX, the hypotheses leading to the uniqueness of LLT are enumerated.

II. LLT FOR SPACELIKE MOTIONS

It has long been customary in linearized subsonic aerodynamics (steady flow) to employ LT. The usual LT unique under Church's postulates is

$$x'=\frac{x-vt}{[1-(v^2/a^2_0]^{1/2}},$$

$$t'=\frac{t-(vx/a^2_0)}{[1-(v^2/a^2_0]^{1/2}}, \tag{1}$$

where a_0 is the sound speed and $v<a_0$. This is applicable to timelike regions,[3] viz., $ds^2=a^2_0\,dt^2-dx^2>0$ (Fig. 1). In constructing similar transformations for spacelike motions, i.e., for an $u>a_0$ or equivalently $ds^2=a^2_0\,dt^2-dx^2<0$, we note that the line element ds^2 has a sign opposite to that of timelike motion. This may be incorporated into the transformation by taking in the form

$$x'=\frac{x-ut}{\{(u^2/a^2_0-1\}^{1/2}},$$

$$t'=\frac{t-(ux/a^2_0)}{[(u^2/a^2_0)-1]^{1/2}}. \tag{2}$$

 0021-8979/78/4912-5783$01.10

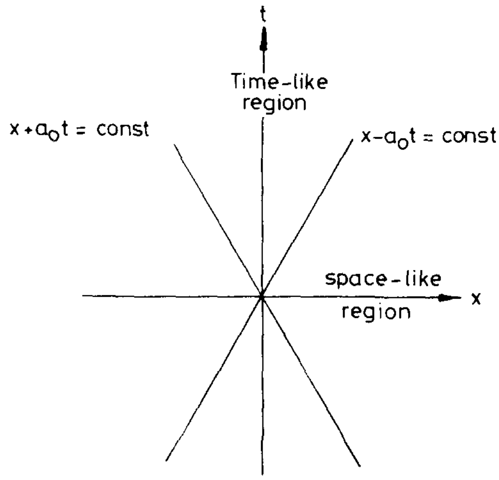

FIG. 1. The space-time diagram of subsonic motion drawn using sound signals.

This is the modified LT of Miles[4] which is obtained by the replacement of $1-v^2/a^2_0$ with $v<a_0$ in Eq. (1) by u^2/a^2_0-1 with $u>a_0$. While this prevents the transformation from becoming imaginary in the passage to the supersonic regime, it fails to keep the wave equation invariant. In order to recover this invariance, we interchange x and a_0t (Fig. 2) (the significance of which is explained in Sec. V) and write finally

$$x'=\frac{a_0t-(ux/a_0)}{[(u^2/a^2_0)-1]^{1/2}},$$

$$t'=\frac{(x/a_0)-(ut/a_0)}{[(u^2/a^2_0)-1]^{1/2}}, \qquad (3)$$

This is the LLT which will fix the coordinates on the body moving with a speed $u>a_0$.

III. PHYSICAL SIGNIFICANCE OF RELATIVE VELOCITY

Rearranging Eq. (3) in the form

$$x'=\frac{x-(a^2_0t/u)}{[1-(a^2_0/u^2)]^{1/2}},$$

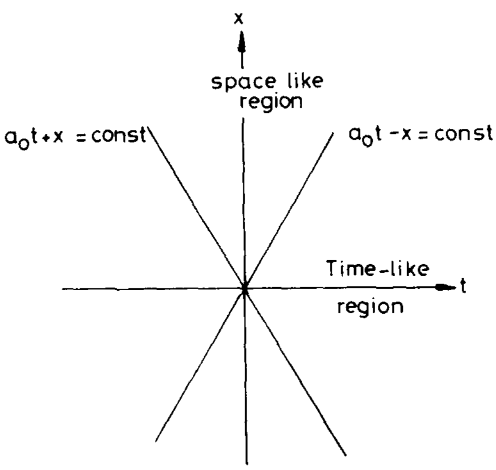

FIG. 2. The space-time diagram of supersonic motion drawn using sound signals.

$$t'=\frac{t-(x/u)}{[1-(a^2_0/u^2)]^{1/2}}, \qquad (4)$$

it may be readily noted that the supersonic and subsonic transformations (1) and (4) are connected by Prandtl's normal shock relation (with $\gamma=1$),

$$uv=a^2_o. \qquad (5)$$

This has great physical significance: The relative velocity between the frames is v in Eq. (1) and a^2_0/u in Eq. (4). One would normally expect that the relative velocity in Eq. (4) to be u and not a^2_0/u. The answer to this anomaly is to be sought by analyzing in depth the concept of relative velocity. In subsonic motion, the relative velocity between two observers as observed by sight is the same as observed by using sound signals. This means that the ratio of the distance covered *as seen* to the transit time is equal to the velocity as measured by using only acoustic signals. The situation is altogether different in supersonic motion. Supersonic speeds are measured only by sight, and they cannot be measured at all by using sound signals; observer's signals chasing a supersonic body never reach it, and those that reach it from the front never come back to the observer. This is the underlying reason for adopting Schlierren photography to locate the shock. Therefore, when one refers to a supersonic speed u, it is tacitly understood as the one measured by sight; but if an observer O_1 (Fig. 3), whom we may call "sonic observer," insisted on using only sound signals to measure the velocity, he would precisely measure it as a^2_0/u from within the shock and not at all from outside (Fig. 3). In other words, the velocity with respect to the inside fluid contiguous to the body is only $a^2_0/u=v<a_0$. It is this relative velocity which appears in Eq. (4).

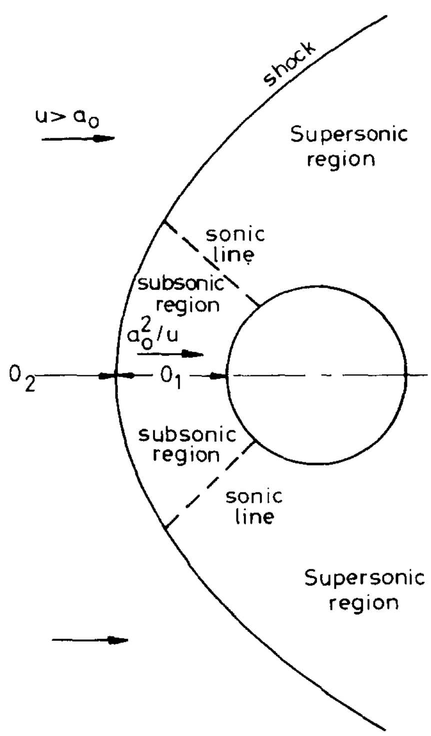

FIG. 3. The different relative velocities measured by different observers in a spacelike motion.

IV. CONSTANCY OF SOUND SPEED

It is well known that the speed of sound in a gas is not a constant but varies in a nonlinear fashion with the amplitude of the disturbance. Thus, in the case of a sphere traveling at twice the sound speed in the undisturbed medium, the bow wave ahead of the sphere is going at twice the normal sound speed, and small disturbances behind the bow wave travel much faster than the normal sound speed because of the high temperatures. Therefore, one has to restrict the application of transformations such as Eq. (4) [or Eq. (1)] to the motion of slender bodies which create only small disturbances. Specifically, it should be presupposed that the motion of the body does not alter the temperature of the gas so that the sound speed is unchanged. Under these circumstances, it is well known that the speed of sound, with respect to the medium at *rest*, is constant in the sense that it does not depend on the speed of the source or the observer. This is a statement made in any linear theory of compressible fluids but seldom pursued in the latter development of the theory. Incorporating this constancy into the theory is of paramount importance because it is the sound signal which, as the carrier of information, dictates the evolution of the gas. Of course, LT (1) vindicates this in the case of subsonic speeds since the velocity transformation following from Eq. (1) is

$$w' = \frac{w - v}{1 - (wv/a^2_0)}, \tag{6}$$

which gives $a'_o = a_0$. LLT (4), which is envisaged for observers moving relatively at supersonic speeds in a fixed medium, also demonstrates this constancy explicitly: the velocity transformation following from Eq. (4) is

$$w' = \frac{w - (a^2_0/u)}{1 - (w/u)}, \tag{7}$$

which yields $a^2_o = a_0$. Therefore, in the linear approximation, the propagation of the sound signals from a supersonic body is automatically limited to the inside of the Mach cone, so that it becomes the active zone and the rest of the space is the silent zone (see Sec. V for details).

V. COMPARISON OF THE EQUATIONS OF MOTION

There is another gain made in the foregoing analysis. In the passage from Eq. (1) to Eq. (4) two changes were made: (a) $1-(v^2/a^2_0) \to (u^2/a^2_0) - 1$ and (b) $x \to a_0 t$. The y and z variables being unaffected, change (b) alters the wave equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{1}{a^2_0}\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{8}$$

to the form

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} - \frac{1}{a^2_0}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{9}$$

which is the prescribed equation of supersonic motion. The characteristics in this case being hyperboloids of two sheets, the Mach cone is already imbedded in the theory. (a) retrieves the wave equation, which means that the timelike domain of the wave equation is the inside of the cone where it is also invariant. Thus, in the further computation of the aerodynamic variables, the range of integration of the equation of motion is automatically limited to the interior of the Mach cone. However, because of change (b), the usual boundary value problem for Eq. (9) is converted into an initial value problem for Eq. (8). It is demonstrated elsewhere that the two approaches give equivalent answers.

VI. ACOUSTICAL ABERRATION

A general impression seems to be that there is no aberration in the case of sound propagation as in the case of optics. Therefore, in this section we first obtain the aberration formula purely from well-known methods of subsonic kinematics and then point out how the same formula may be obtained more simply by a direct use of LT (1). In Sec. VII we shall obtain the Mach-cone geometry by considering the singularities of the aberration in the corresponding case of supersonic kinematics. The considerations of this section and Sec. VII are intended to spell out the nature of the role of "sonic relativity" in acoustic kinematics.

Consider[13] a sound source moving along the x axis with Mach number $m = (v/a_0) < 1$. Let O : (0,0,0,0) be the emission space-time point of a sound signal and let O' : (x,y,z,t) be the space-time point of observation. From Fig. 4 we obtain

$$R^2 = \{x - v[t - (R/a_0)]\}^2 + y^2 + z^2, \tag{10}$$

giving

$$R = \frac{(v/a_0)(x - vt) + R_1}{1 - m^2}, \tag{11}$$

where

$$R_1 = +[(x - vt)^2 + (1 - m^2)(y^2 + z^2)]^{1/2}. \tag{12}$$

From Fig. 4 we also have

$$x - vt = R(\cos\theta - m). \tag{13}$$

Substituting this expression in Eq. (11) and rearranging, we

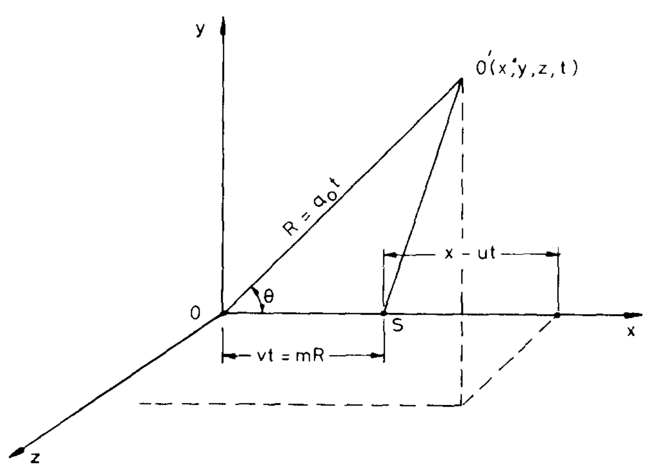

FIG. 4. Kinematics of subsonic motion of a point source.

obtain

$$R_1 = R(1 - m\cos\theta). \tag{14}$$

Dividing Eq. (13) by Eq. (14) we obtain

$$\frac{\cos\theta - m}{1 - m\cos\theta} = \frac{x - vt}{R_1}. \tag{15}$$

Further using Eq. (14), we also have similar equations along the y and z axes, viz.,

$$\frac{y(1-m^2)^{1/2}}{R(1-m\cos\theta)} = \frac{y(1-m^2)^{1/2}}{R_1}, \tag{16}$$

$$\frac{z(1-m^2)^{1/2}}{R(1-\cos\theta)} = \frac{z(1-m^2)^{1/2}}{R_1}. \tag{17}$$

That these are actually the aberration formulas for "sonic relativistic observers" may be explicitly demonstrated by using the transformations of sonic relativity. Note that

$$\frac{R_1}{(1-m^2)^{1/2}} = (x'^2 + y'^2 + z'^2)^{1/2} = R', \tag{18}$$

where

$$x' = \frac{x - vt}{(1-m^2)^{1/2}},$$

$$y' = y, \quad z' = z \tag{19}$$

is the spatial part of the Lorentz transformation. Introducing the direction cosines $(\lambda,\mu,\nu) = (x/R, y/R, z/R)$ and $(\lambda',\mu',\nu') = (x'/R', y'/R', z'/R')$, we can rewrite Eqs. (15)–(17) as

$$\lambda' = \frac{x'}{R'} = \frac{\lambda - m}{1 - m\lambda}, \tag{20}$$

$$\mu' = \frac{y'}{R'} = \frac{\mu(1-m^2)^{1/2}}{1 - m\lambda}, \tag{21}$$

$$\nu' = \frac{z'}{R'} = \frac{\nu(1-m^2)^{1/2}}{1 - m\lambda}. \tag{22}$$

Thus, (λ',μ',ν') turn out to be the direction cosines of the line OO' : (λ,μ,ν) as "heard" by a sonic observer moving with the source S. Now it is well known how these formulas can be obtained directly by transforming the plane wave $\exp(i\omega/a_0)(\lambda x + \mu y + \nu z - a_0 t)$ with the LT without the foregoing detailed calculations.

VII. MACH-CONE GEOMETRY

In this section we shall discuss what information can be extracted from the aberration formulas obtained in Sec. VI as adopted to the supersonic case.

Let $M = (u/a_0) > 1$. Then, both the roots of R_1 of Eq. (12) will be acceptable for R to be real provided $x < ut$, that is, when the point of observation is behind the source. Further, we continue to have the relation

$$\lambda' = \frac{\cos\theta - M}{1 - M\cos\theta}$$

$$= \pm\frac{x - ut}{R_1}. \tag{23}$$

In the corresponding situation of Eq. (15), $m < 1$, and therefore θ could assume all values; but in the present case $M > 1$, and at $\cos\theta_0 = 1/M$, the components of aberration are all discontinuous. Correspondingly, $R_1 = 0$ at this value of $\cos\theta$. Further, as may be seen from Eq. (11), the two roots of R, viz., $R_\pm$ will now coincide at the value

$$R_0 = \frac{M(ut - x)}{M^2 - 1}.$$

The two equivalent sources $E_\pm$ and the equivalent angles $\theta_\pm$ corresponding to $R_\pm$ will also coincide. This implies that this coincident point of observation lies on the surface of a cone of semivertex angle $\varphi = \frac{1}{2}\pi - \theta_0$, where θ_0 corresponds to the value R_0. Equivalently,

$$\sin\varphi = \cos\theta_0 = 1/M, \tag{24}$$

which is precisely the Mach-cone angle (Fig. 5).

However, there is another significant complication with regard to the aberration phenomena in the supersonic case: We have $|\lambda'| < 1$ for all values of λ in Eq. (23). Therefore,

$$(x - ut)^2 < R^2{}_1 = (x - ut)^2 - (M^2 - 1)(y^2 + z^2), \tag{25}$$

which is absurd. This is a warning that the phenomenon of aberration, in contrast to the subsonic case, is not acoustically observable in the supersonic case.

VIII. ROLE OF SONIC RELATIVITY IN SHOCK SHAPE

Throughout this paper we have been using the words "constancy of sound speed" in the sense that, relative to the air, sound speed has a fixed value which does not depend on the relative motion between the source and the observer, as long as this motion is linear and does not interfere with the fixed temperature of the air. The constancy of sound speed in such a situation is an unqualified testimony of the fact that the medium itself is equipped with rods and clocks whose spatial coordinate readings and time instants transform according to LT. In other words, the correct interpretation of the applicability of LT in linearized theory is that which

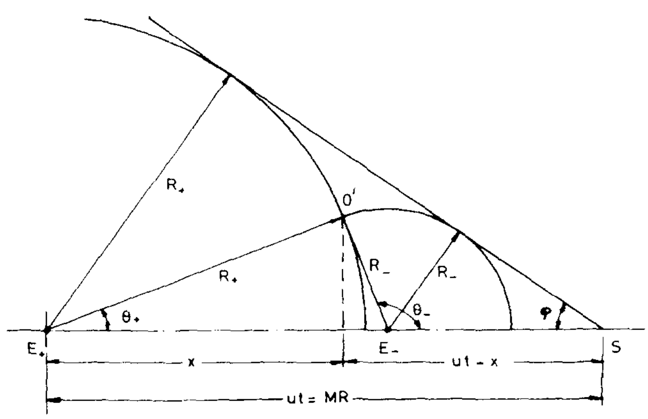

FIG. 5. Kinematics of supersonic motion of a point source.

Lorentz attributes to it; it is not the Einsteinian interpretation. This further implies that the phenomena of Lorentz contraction and time dilatation are absolute for an observer fixed to the medium (as in Lorentz's interpretations). This point of view yields an astounding feature of the theory leading to a formula for the shock stand-off distance for a sphere in supersonic flight.[14]

From relativity, it is known that LT (1) leads to a Lorentz contraction of the length of an object along the direction of relative motion. This means that a moving sphere contracts to an oblate spheroid with respect to the other with its diameter reduced by the Lorentz factor $[1-(v^2/c^2)]^{1/2}$, where c is the speed of light. However, by analogy, there does not exist any contraction of a rigid sphere in subsonic aerodynamics. This appears to be explainable on the basis that the increase in the aerodynamic pressures due to the motion of the body as observed by a "sonic observer" fixed to the medium is negligible compared to the elastic forces of the rigid body.[15] If one had a spherical balloon in place of a rigid sphere, then the increase in pressure would change it to an oblate spheroid. The exact change in the diameter would obviously depend on the elastic properties of the balloon, and an experimental study of this may be profitable in the understanding of "sonic relativity."

Another consequence of Eq. (1) is a time dilatation by the same factor, meaning that time intervals in a moving frame become longer by the factor $[1-(v^2/c^2)]^{1/2}$. While we defer a more complete discussion of time dilatation in sonic relativity, we will now note that with the interchange $x \leftrightarrow a_0 t$, LLT (4) records a space dilatation. This dilatation has a curious connection with the shock geometry of a blunt body in supersonic motion. As already noted in Sec. IV, this motion is highly nonlinear. Therefore, the validity of the foregoing linear theory to this problem becomes very doubtful. However, the formula we have derived in Sec. IX for the shock goemetry of a sphere in supersonic motion coincides remarkably accurately with the one obtained numerically by Belotserkovskiy[16] using the complete system of equations of gasdynamics and relaxation equations. Therefore, it is tempting to speculate that nonlinear aspects of the problems may refer only to its dynamical part, though its kinematic part is amenable to the linear theory. In other words, the complete dynamical equations are *sine qua non* to obtain the thermodynamic variables downstream of the shock and the foregoing theory cannot handle the problem; but just to obtain the shock geometry, the linear theory of "sonic relativity" appears to be sufficient, and a solution of the complete system of the equations of gasdynamics may be too much for the purpose.

Consider a sphere of diameter D and free-stream Mach number $M > 1$. Let the curvature of the shock at the tip be $2/D'$. Let O_2 be an observer anywhere ahead of the shock along the line of symmetry and equipped with only sound signals (Fig. 3). The observer's signals along this line fail to reach the sphere, but travel only up to the tip of the shock. Thus, the observer's signals locate the sphere at this point as a surface of curvature $2/D'$. On the other hand, consider an observer O_1 equipped with only sound signals and on the line of symmetry inside the shock. This observer's signals reach the nose of the sphere without hindrance and hence locate the sphere as seen, viz., as a surface of curvature $2/D$ at the stagnation point. In order to find the relation between D and D', we recall that O_1 and O_2 are related by the interchange $x \leftrightarrow a_0 t$ of Sec. V. Therefore, these lengths are to be discussed in terms of equivalent time intervals. A convenient presentation of these ideas can be made using Lorentz diagrams.[17] In Fig. 6 $x, a_0 t$ and $x', a_0 t'$ represent oblique axes corresponding to the two frames in relative motion. $x, a_0 t'$ and $x', a_0 t$ are chosen to be mutually perpendicular such that $\sin\alpha = m < 1$. Suppose a length L is in the unprimed frame which is moving. Then, the length L' in the other frame is measured by reading the two ends simultaneously in the primed frame, i.e., at the same value of t'. The projection on the x' axis corresponding to these simultaneous events is $L' = L(1-m^2)^{1/2}$, giving the Lorentz contraction. Similarly a clock in the $x, a_0 t$ frame ticks off the events E_1 and E_2 of time interval τ. The same events have a projection $\tau' = \tau/(1-m^2)^{1/2}$ in the $x', a_0 t'$ frame (Fig. 7).

The supersonic case is obtained, according to our construction, by interchanging x' and $a_0 t'$ of the subsonic case. Length transformation is now to be calculated in terms of equivalent time intervals. Thus, if O_1 were at rest with respect to the sphere, the observer's sound signals would measure the diameter D in a time interval $\tau = D/a_0$. On the other hand, if the observer were moving uniformly with the fluid this time interval would appear dilated to the observer as $\tau = \tau'/(1-m^2)^{1/2}$. Here, $m = v/a$ is the subsonic Mach number along the stagnation stream line inside the shock. When this m is expressed in terms of the free-stream Mach number $M = u/a$ using the Prandtl relation $Mm = a_*^2/a^2$ as in Sec. III the expression for τ' assumes the following significance: It is the time for the sound signal to cover a distance D' as measured by O_2 (we also assume the shock thickness to be zero). The difference in the two times $\Delta\tau = \tau' - \tau$ is clearly the time required by the signal to traverse the stand-off distance

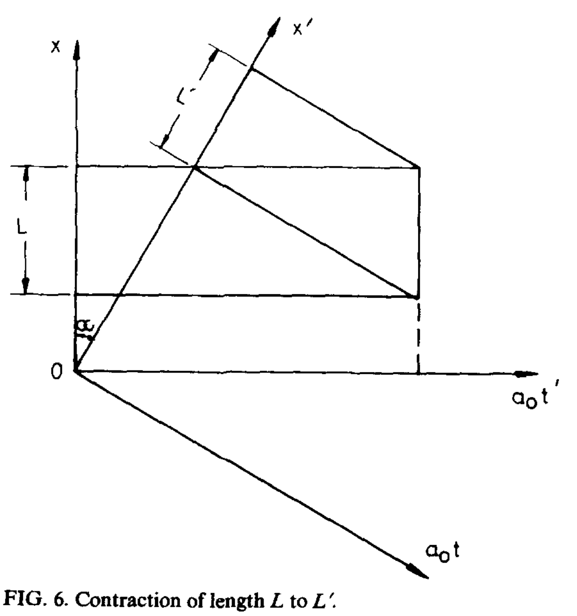

FIG. 6. Contraction of length L to L'.

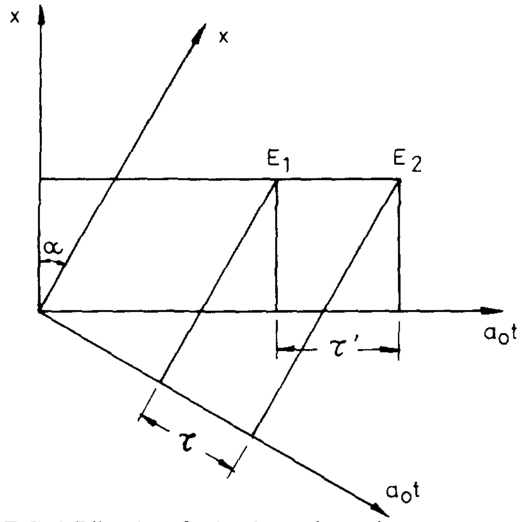

FIG. 7. Dilatation of a time interval τ to τ'.

$\Delta = D' - D$ as measured by O_1. Thus

$$-\Delta/D = \Delta\tau/\tau$$
$$= [1-(a_*^4/a^4M^2)]^{1/2} - 1$$

where Δ carries the negative sign since it is measured opposite to the direction of motion. However, in Ref. 14 when Δ/D was compared with the numerical calculations of Van Dyke[18] and Lomax,[19] it was found necessary to replace a by a_0 and write the final expression as

$$\frac{\Delta}{D} = 1 - \left[1 - \frac{4}{(\gamma+1)^2}\left(\frac{1}{m^2} + \frac{\gamma-1}{2}\right)\right]^{1/2}. \tag{26}$$

This amounts to saying that the moving observer's readings in sonic relativity are related to those of the stationary observer with reference to the stagnation conditions.

The foregoing considerations may be extended heuristically as follows: What a stationary sonic observer recognizes as a point on the sphere is the same as what a sonic observer moving with the fluid at the radially opposite point $P(\vec{r})$ upstream of the shock records as a point on the shock. Therefore, the distance of the point $P(\vec{r})$ from the sphere is obtained by replacing in Eq. (26) the Mach number M by the component $M\cos\theta$ along the line joining the center of the sphere to the point $P(\vec{r})$, i.e.,

$$\frac{\Delta(\theta)}{D} = 1 - \left[1 - \frac{4}{(\gamma+1)^2}\left(\frac{1}{m^2\cos^2\theta} + \frac{\gamma-1}{2}\right)\right]^{1/2}. \tag{27}$$

The agreement of this expression with the numerical calculations of Belotserkovskiy[16] is good enough to encourage further investigation of "sonic relativity."

IX. THE POSTULATES OF LLT

In order to go to the system attached to the body, LLT and LT have been employed in the supersonic and subsonic regimes, respectively, since they keep the wave equation invariant. In fact, there exists a large class of transformations which keep the wave equation invariant, and therefore we will have to fix the transformation uniquely only on the basis of experimental requirement. Indeed, LT is to be chosen in the subsonic case and LLT in the supersonic case. Therefore, it becomes necessary to examine the postulates which determine a transformation uniquely. This analysis has been done by Church[12] in the case of LT, and in this section we do it for LLT. The postulates are

(a) x', t' are functions of x, t and $u > a_0$; all these variables are real.

(b) For no value of u is $(\partial t'/\partial t) < 0$ for every value of x and t.

(c) $dx/dt = a_0$ implies that $dx'/dt' = a_0$.

(d) $dx/dt = a_0^2/u$ implies that $dx'/dt' = 0$.

(e) The inverse transformation is obtainable from the direct transformation by the changes $x \leftrightarrow x'$, $t \leftrightarrow t'$, and $u \rightarrow -u$. Here, postulate (a) is an algebraic requirement, and the physical meanings of the last four postulates are as follows:

(b) t' never flows backwards with respect to t;

(c) sound speed is independent of the motion of the source or the observer (when the medium is fixed);

(d) the origin in the original coordinate system attached to the body has a velocity a_0^2/u along the x axis. Its implications are discussed in Sec. III;

(e) the axis of the Mach cone is reversed when the direction of the supersonic motion of the body is reversed.

We will now derive LLT (4) using these postulates. Let the transformation be (a),

$$x' = \varphi(x,t,u), \qquad t' = \psi(x,t,u).$$

Then

$$\frac{dx'}{dt'} = \left(\varphi_x \frac{dx}{dt} + \varphi_t\right)\left(\psi_x \frac{dx}{dt} + \psi_t\right)^{-1}. \tag{28}$$

Setting $dx/dt = a_0^2/u$, we obtain (d),

$$(a_0^2/u)\varphi_x + \varphi_t = 0. \tag{29}$$

Using (d) and (e), $dx/dt = 0$ implies that $dx'/dt' = -a_0^2/u$, which when substituted in Eq. (28) gives

$$(a_0^2/u)\psi_t + \varphi_t = 0. \tag{30}$$

From Eq. (29) and (30) we obtain

$$\varphi_x = \psi_t. \tag{31}$$

Using (c) in Eq. (28),

$$a_0(a_0\psi_x + \psi_t) = a_0\varphi_x + \varphi_t,$$

and because of Eq. (31),

$$a_0^2\psi_x = \varphi_t. \tag{32}$$

Substituting in Eq. (29) and integrating,

$$(1/u)\varphi + \psi = T. \tag{33}$$

Integrating Eq. (30),

$$\varphi + (a_0^2/u)\psi = X, \tag{34}$$

where T is indepenent of x and X is independent of t.

Solving Eqs. (33) and (34),

$$\varphi=\frac{x-(a_0^2/u)T}{1-(a_0^2/u^2)},$$

$$=X_1+T_1$$

$$\psi=\frac{T-(1/u)X}{1-(a_0^2/u^2)}=X_2+T_2 .$$

Now, using Eqs. (31) and (32), we obtain

$$\frac{dX_1}{dx}=\frac{dT_2}{dt},\quad \frac{dX_2}{dx}=\frac{dT_1}{dt},$$

which is true only if all of them are independent of x and t, i.e., X_1, X_2, and T_1, T_2 are of first degree in x and t, respectively.

Thus,

$$\varphi=p_1x+q_1t+r_1,\quad \psi=p_2x+q_2t+r_2.$$

Using Eqs. (30)—(32), respectively,

$$(a^2{}_0/u)q_2+q_1=0,\quad p_1=q_2,\quad a^2{}_0p_2=q_1.$$

Therefore,

$$\varphi=q_2[x-(a^2{}_0t/u)]+r_1\quad \psi=q_2[t-(x/u)]+r_2. \quad (35)$$

Solving for x and t,

$$x=\frac{\varphi+(a_0^2/u)\psi}{[1-(a_0^2/u^2)]q_2}-\frac{r_1-(a_0^2/u)r_2}{[1-(a_0^2/u^2)]q_2}, \quad (36)$$

$$t=\frac{\psi+(1/u)\varphi}{[1-(a_0^2/u^2)]q_2}-\frac{r_2+(1/u)r_1}{[1-(a_0^2/u^2)]q_2};$$

but using (e) on Eq. (35) to obtain the inverse

$$x=q_2[\varphi+(a^2{}_0/u)\psi]+r_1,\quad t=q_2[\psi+(1/u)\varphi]+r_2. \quad (37)$$

Comparing Eqs. (36) and (37),

$$r_1=\frac{r_1+(a_0^2/u)r_2}{[1-(a_0^2/u^2)]q_2},$$

$$r_2=-\frac{r_2+(r_1/u)}{[1-(a_0^2/u^2)]q_2},$$

$$q_2=\frac{1}{[1-(a_0^2/u^2]q_2},$$

which give

$$r_1=r_2=0,$$

$$q_2=\pm\frac{1}{[1-(a_0^2/u^2)]^{1/2}},$$

and (b) fixes the positive sign for q_2. On substituting these values in Eq. (35), we obtain LLT (4).

X. CONCLUSION

The LT already in use in subsonic aerodynamics is inadequate to cope with the supersonic case because of the form of the Lorentz factor involved in the LT; but we take the viewpoint that the form of the transformation is to be decided by the experimental needs, and the needs of subsonic aerodynamics demand the unique form of LT (1). The physics of the situation in the supersonic case determines the form of the LT as given by Eq. (4). We have shown that the crucial point for this fixation comes from the distinction between the two values of the relative velocity, one obtained optically and the other acoustically. We shall show elsewhere how the mixing up of these two values has given rise to various rules for obtaining correct answers in the theory of compressible fluids. The constancy of sound velocity with respect to the relative motion between the source and the observer in a fixed medium is a demonstration of the fact that the medium itself is equipped with clocks whose relative times transform according to the Lorentz transformations. Therefore, all transactions which are mediated by the sound signal have to be formulated in terms of "sonic relativity," and the Galilean transformation $\vec{r}'=\vec{r}-\vec{u}t$, $t'=t$, is permissible only when fast signals, such as light, are used.

[1]H.G. Küssner, NACA Tech. Memo. 979 (1940).
[2]R.W. Truitt, North Carolina State Dept. of Engineering Research Bulletin No. 44, 1949 (unpublished); No. 47, 1950 (unpublished).
[3]W.R. Sears, *General Theory of High Speed Aerodynamics*, edited by W.R. Sears (Princeton U.P., Princeton, N.J., 1954), Vol. VI C.
[4]J.W. Miles, *The Potential Theory of Unsteady Supersonic Flow* (Cambridge U.P., London, 1959).
[5]R.T. Jones, J. Franklin Inst. **275**, 1 (1963).
[6]K.H. Mariwalla, MATSCIENCE preprint No. 9, 1968 (unpublished).
[7]T.S. Shankara, Found. Phys. **4**, 97 (1974).
[8]E.A. Lord and T.S. Shankara, Found. Phys. **7**, 459 (1977).
[9]T.S. Shankara, J. Math. Anal. Apl. **51**, 577 (1975).
[10]Kamal Kanti Nandi and T.S. Shankara, J. Math. Anal. Appl. (to be published).
[11]T.S. Shankara, Lett. Nuovo Cimento (to be published).
[12]A. Church, Am. Math. Mon. **31**, 376 (1924).
[13]P.M. Morse and K.U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968).
[14]T.S. Shankara and A.K. Sreekanth, J. Appl. Phys. **48**, 1765 (1977).
[15]E.L. Feinberg, Usp. Fiz. Nauk **18**, 624 (1975).
[16]O.M. Belotserkovskiy, NASA Tech. Transl. F-453 (1967).
[17]F.W. Sears and R.W. Brehme, *Introduction to the Theory of Relativity* (Addison-Wesley, Reading, Mass., 1968).
[18]M.D. Van Dyke and H.D. Gordon, NASA Tech. Rep. R−1 (1959).
[19]H. Lomax and M. Inoye, NASA Tech. Rep. R-204 (1964).