Journal of Mathematical Physics

The brachistochrone in almost flat space

S. G. Kamath

Citation: J. Math. Phys. 29, 2268 (1988); doi: 10.1063/1.528158
View online: http://dx.doi.org/10.1063/1.528158
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v29/i10
Published by the American Institute of Physics.

The brachistochrone in almost flat space

S. G. Kamath
Department of Mathematics, Indian Institute of Technology, Madras 600 036, India

(Received 29 January 1988; accepted for publication 25 May 1988)

This paper is an extension, within the framework of general relativity, of the relativistic brachistochrone discussed recently by Goldstein and Bender [J. Math. Phys. **27**, 507 (1986)]. Assuming that the gravitational field due to a spherically symmetric source with mass M at equilibrium is weak, it is found that the brachistochrone, for a falling particle of mass m, described by $\Theta(r)$, with Θ an angle and r a distance measured from the center of symmetry, is in general a hyperelliptic integral. The latter integral can in one case be calculated exactly in terms of the normal elliptic integrals of the first and third kinds and the elementary transcendental functions. It is shown via a numerical computation using the sun's gravitational field as a reference that one can recast this exact version into a simple form, viz., $\sqrt{r}\Theta = a$, where a is a constant.

I. INTRODUCTION

In a recent paper Goldstein and Bender[1] have presented a relativistic generalization of the classic brachistochrone problem for a particle falling from rest in a uniform gravitational field. As is well known, the brachistochrone is the trajectory joining an initial position A to a final position B along which the time of transit of the falling particle is a minimum. While the classical nonrelativistic trajectory is known (with A as the origin of the coordinate system) to be a cycloid of the form $x = b(1 - \cos t)$, $y = b(t - \sin t)$, with the parameter b determined by the end point B, Goldstein and Bender (GB) show that in the relativistic case, this is just one of three possible curves. In particular, the two new solutions are very different from the nonrelativistic case in that, for both of them, $y(x)$ increases without bound as x increases. It should be noted here that the motion of the particle of rest mass m in Ref. 1 is still assumed to take place in a uniform gravitational field with the force law $\mathbf{F} = \widetilde{m}\mathbf{g}$, $\mathbf{g}$ being a constant, and $\widetilde{m}(1 - v^2/c^2)^{1/2} = m$.

In this paper we have relaxed the above-mentioned restriction to a uniform gravitational field. Specifically, we consider the motion of the particle in a weak gravitational field in the sense of general relativity. As is well known in the literature[2,3] such a weak field will be represented by a metric $g_{\alpha\beta}$ that differs very little from the Minkowski metric $\eta_{\alpha\beta}$. Thus with $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(1, -1, -1, -1)$, the gravitational field is said to be weak when $|g_{\alpha\beta} - \eta_{\alpha\beta}| \ll 1$. More precisely, with the assumption that $g_{\alpha\beta}$ can be expanded as an infinite series

$$g_{\alpha\beta} = \eta_{\alpha\beta} + g^{(1)}_{\alpha\beta} + g^{(2)}_{\alpha\beta} + \cdots, \qquad (1)$$

we limit ourselves in the first (linear) approximation, writing $g_{\alpha\beta} = \eta_{\alpha\beta} + g^{(1)}_{\alpha\beta}$ instead of Eq. (1). As described in detail in Refs. 2 and 3, the linearized Einstein field equations can then be solved for $g_{\alpha\beta}$ once the source of the gravitational field, viz., the energy momentum tensor $T_{\alpha\beta}$, is given.

In this paper, we consider a distribution of matter at equilibrium described by $T_{00} = \rho c^2$, $T_{\alpha\beta} = 0$ for $\alpha\beta \neq 00$, with the density ρ being time independent and spherically symmetric, so that $\rho = \rho(r)$, r being the distance from the center of symmetry. The resulting $g_{\alpha\beta}$ is then represented[2,3] in the interval

$$ds^2 = (1 - 2\Phi/c^2)c^2\,dt^2 - (1 + 2\Phi/c^2)(dx^2 + dy^2 + dz^2)\,. \qquad (2)$$

Here $\Phi = GM/r$, outside the material distribution, with $M = \int\rho(r)d^3x$ and G the familiar gravitational constant.

Since the problem is now posed within the framework of general relativity, there will be two important qualitative differences from the work of Ref. 1 which we shall merely mention below, relegating the details to subsequent sections in the paper.

First, the motion of a material particle of rest mass m in the gravitational field whose metric is represented in Eq. (2) will now be governed via Hamilton's principle by the Lagrangian $L = -mc\,ds/dt$. This will replace the flat-space Lagrangian discussed in GB and given there by $L = -mc^2\gamma^{-1} + \mathscr{E}$, with $\gamma^{-2} = 1 - v^2/c^2$, $\mathscr{E} = mc^2 \times(\exp(gx/c^2) - 1)$, and $\mathbf{v}^2 = (d\mathbf{r}/dt)^2$. Second, since we are interested in the path of minimum time rather than the geodesic (which is the path of extremal action), there will be one more point of departure from Ref. 1; namely, the element of spatial distance will now be given, following Eq. (2), by

$$dl = ((1 + 2\Phi/c^2)(dx^2 + dy^2 + dz^2))^{1/2},$$

instead of the familiar $dl = (dx^2 + dy^2 + dz^2)^{1/2}$ as in Ref. 1. Thus the line integral representation for the time of fall will be given by

$$T = \int_A^B \frac{dl}{v}, \qquad (3)$$

with dl as given above, v being the particle velocity. In Eq. (3), we shall regard A as lying outside the range of the weak gravitational field, namely, at infinity, so that Eq. (3) can be rewritten

$$T = \int_\infty^B \frac{dl}{v}, \qquad (3')$$

with the form of v being obtained from the principle of conservation of energy. Because of the assumption of spherical

 0022-2488/88/102268-05$02.50

symmetry, so that $\Phi = \Phi(r)$ only in (2), the Lagrangian L when written out in spherical polar coordinates should be cyclic in one of the angles, so that the motion of the particle can be taken, as in case of central force motion, to lie in a plane with its location specified by the coordinates (r,Θ), r being measured as mentioned earlier, from the center of the material distribution whose gravitational field is described by the metric $g_{\alpha\beta}$ represented in Eq. (2). The brachistochrone will then be obtained by solving the Euler equation associated with Eq. (3′) with the initial conditions $\Theta = 0$, $u(=1/r) = 0$. This program is carried through in the sections detailed below.

The plan of this paper is as follows. In Sec. II we obtain the expression for the velocity v appearing in (3′). Then by using this expression, we recover via the Euler equation in Sec. III the equation of the brachistochrone in the form usual to central force motion, viz., $\Theta = \Theta(u)$. This relation cannot in general be written out in a closed form. More precisely, it turns out, as we shall see in Sec. III, that one has to evaluate a hyperelliptic integral of the form

$$\Theta(u) = \int_0^u \frac{R(t)dt}{(P(t))^{1/2}},$$

with $P(t)$ being a fifth-degree polynomial in t. As is well known, for evaluating such hyperelliptic integrals[4] one usually has to resort to direct numerical integration, or to the use of complicated series expansions.

However, as we shall see in Sec. IV, it is possible to fudge a certain constant k (analogous to the constant k in Ref. 1) so as to obtain an exact dependence of $\Theta(u)$ on u; this is, unfortunately, possible only for one value of k. Still, this exact result given in Sec. IV is far too complicated, as it contains a linear combination of the normal elliptic integral of the first and third kinds, besides elementary functions like the natural logarithm whose arguments involve the Jacobian elliptic functions sn u, dn u. For our purposes it is useful to resort to an approximation whereby the complicated terms are rendered harmless by being extremely small, in fact, almost zero. The gravitational field outside the sun, which is regarded in the literature[3] as weak, turns out to be handy in this connection. As discussed in Sec. V it affords a more accessible version of the brachistochrone given in Sec. IV.

Finally, in Sec. VI we conclude with some comments on our results. Herein we offer, besides an exact analytic expression for the brachistochrone obtained by GB, a discussion of the more difficult problem, technically speaking, of the brachistochrone associated with the Schwarzschild metric. We hope to return to the latter in a subsequent publication.

II. ENERGY CONSIDERATIONS

Following the discussion in Sec. I, the Lagrangian for the material particle of rest mass m in the gravitational field is given by

$$L = -mc\frac{ds}{dt} = -mc^2\gamma^{-1}\left(1 - \frac{2\Phi}{c^2}\psi\right)^{1/2}, \qquad (4)$$

with $\psi = \gamma^2(1 + v^2/c^2)$. It is easy to obtain the Hamiltonian from (4). It is given by

$$H = mc^2\gamma(1 - (2\Phi/c^2)\psi)^{-1/2}g_-, \qquad (5)$$

with g_- defined below. As H in (5) is cyclic in time t by the principle of conservation of energy, we must also have

$$H = mc^2, \qquad (6)$$

since the particle is assumed to fall from rest. It is easy to obtain an expression for v from Eqs. (5) and (6) through simple manipulations. We obtain

$$g_+v^2 = 2g_-\Phi, \qquad (7)$$

where $g_\pm = (1 \pm (2/c^2)\Phi)$. With $\Phi = GM/r$, we see that g_-, for example, can be written as $g_- = (1 - uR)$ with $ru = 1$ and $c^2R = 2GM$, defining the Schwarzschild radius R for the material distribution of mass M. Since for macroscopic bodies for which the gravitational field can be regarded as weak (e.g., the sun) the Schwarzschild radius is very small compared to their actual radius, and since we are studying the motion of the material particle of rest mass m outside the material distribution causing the gravitational field, the case when $r = R$ can be safely ignored in (7).

III. THE EULER EQUATION

In terms of the polar coordinates (r,Θ) the element of spatial distance, following (2), is given by

$$dl = g_+^{1/2}(dr^2 + r^2\,d\Theta^2)^{1/2},$$

so that, using (7), Eq. (3′) can be rewritten

$$T = \int_\infty^B g_+(dr^2 + r^2\,d\Theta^2)^{1/2}D^{-1/2}$$
$$= \int_\infty^B g_+dr\left(1 + r^2\left(\frac{d\Theta}{dr}\right)^2\right)^{1/2}D^{-1/2}, \qquad (8)$$

with $D = 2g_-\Phi$. The time taken T is a functional of the path $\Theta(r)$ so that T will be minimum when the Euler equation

$$\frac{d}{dr}\left(\frac{\partial I}{\partial\Theta'}\right) = 0, \quad \Theta' = \frac{d\Theta}{dr}, \qquad (9)$$

is satisfied. Here I denotes the integrand in Eq. (8); thus one infers from (9) that

$$\frac{\partial I}{\partial \Theta'} = k, \qquad (9')$$

where k is a constant. Equation (9) is the analog of the Euler equation in GB. Substituting for I, it is easy to obtain

$$r^2g_+\Theta'(D(1 + r^2\Theta'^2))^{-1/2} = k. \qquad (9'')$$

In terms of u, (9″) can be rewritten

$$g_+\Theta' = -k(D(1 + u^2\Theta'^2))^{1/2}, \qquad (9''')$$

with Θ' being the first derivative of Θ with respect to u. Note the negative sign in (9‴). Since the particle of mass m is assumed to fall from rest at infinity (where $u = 0$) to the point B where $u > 0$, it follows that, for $k > 0$ (this choice can be made without loss of generality), the solution of (9‴), namely, $\Theta(u)$, will be a decreasing function of u.

Equation (9‴) immediately leads to the solution [with $\Theta(0) = 0$]

$$\Theta(u) = \int_0^u dt(g_+^2 - Du^2k^2)^{-1/2}(Dk^2)^{1/2}, \qquad (10)$$

which is the equation of the brachistochrone. Substituting for $g_\pm$ and Φ one can rework (10) as

$$\Theta(u) = \int_0^u dt\, t(1-tR)(P(t))^{-1/2}, \qquad (10')$$

with $P(t) = t(1-tR)((1+tR)^2 - \alpha^2 t^3(1-tR))$ and $\alpha^2 = k^2c^2R$. Note that for the integral to be real we require that (a) $uR \leqslant 1$ and (b) $(1+uR)^2/(1-uR)u^3 \geqslant \alpha^2$. The former requirement is easily met since the particle of mass m as mentioned in Sec. II is moving in the gravitational field outside a macroscopic object whose Schwarzschild radius R is very small compared to its actual radius. The latter bound, however, is an upper limit on α^2 and hence on k^2 for fixed R.

The integral in the above equation is a hyperelliptic integral, which is further manipulated as follows. Using a change of variable $s = tu$ we obtain, with $\sigma = uR$,

$$\Theta(u) = \alpha u^{3/2} \int_0^1 ds\, s(1-s\sigma)(P(s))^{-1/2}.$$

Although $P(s)$ is a sixth-order polynomial in s, it can be rewritten as a fifth-order polynomial through the use of $z = 1/s$, so that

$$\Theta(u) = \beta^{1/2} \int_1^\infty dz \times \frac{z-\sigma}{z[(z-\sigma)(z^2(z+\sigma)^2 - \beta(z-\sigma))]^{1/2}}, \qquad (11)$$

where $\beta = \alpha^2 u^3$, and we have now displayed the polynomial $P(z)$ in full in (11). In the following two sections we shall study the dependence of Θ on u.

IV. AN EXACT RESULT

We shall show herein that the polynomial $P(z) = z^2(z+\sigma)^2 - \beta(z-\sigma)$ has a double root at various values of β. The utility of this double root lies in that the hyperelliptic integral in (11) can be evaluated exactly without recourse to approximation. Naturally, for general β, Eq. (11) will represent the brachistochrone associated with a weak gravitational field. From $P(z)$ we obtain

$$\frac{dP}{dz} = 4z^3 + 6\sigma z^2 + 2z\sigma^2 - \beta.$$

Let $z = z_0$ denote the double root of $P(z)$. Then $P(z_0)$ and

$$\left.\frac{dP}{dz}\right|_{z=z_0}$$

should be zero. This yields

$$\beta = 4z_0^3 + 6\sigma z_0^2 + 2z_0\sigma^2. \qquad (12)$$

Clearly β is zero when $z_0 = 0$ and $z_0 = -\sigma$. Ignoring these cases, since $\beta = 0$ implies $k = 0$, we find the remaining solutions for z_0 using (12) and $P(z_0) = 0$. Thus

$$P(z_0) = -3z_0^4 + 5\sigma^2 z_0^2 + 2z_0\sigma^3 \equiv z_0 \tilde{P}(z_0). \qquad (13)$$

To obtain values of z_0 other than 0 and $-\sigma$, we should solve $\tilde{P}(z_0) = 0$. This can be done by standard methods.[5] We obtain with the notation $c = \cos\mu/3$, $s = \sin\mu/3$ the following three real roots:

$$3z_0^{(1)} = 2\sigma c\sqrt{5}, \qquad (14a)$$

$$3z_0^{(2)} = -\sigma(c + s\sqrt{3})\sqrt{5}, \qquad (14b)$$

$$3z_0^{(3)} = -\sigma(c - s\sqrt{3})\sqrt{5}, \qquad (14c)$$

with $9\tan\mu = 2\sqrt{11}$.

For each of these z_0 there is a corresponding β. However, as can be easily checked, only $z_0^{(1)}$ leads to a positive value of β, viz.,

$$3\beta^{(1)} = 8\sigma^3(1 + 5c^2 + 13c\sqrt{5}/6); \qquad (15)$$

$z_0^{(2)}$ and $z_0^{(3)}$ lead to $\beta^{(2)} = O(10^{-6}\sigma^3)$ and $\beta^{(3)} < 0$, respectively. We are not reproducing these tedious though straightforward calculations here. These latter values of z_0 are therefore ignored, leading to a unique value for β. Since the double root for the polynomial $P(z)$ is now given by (14a), the remaining roots are easily found by standard methods.[5] They are complex conjugates of each other and denoted by

$$z_\pm = -p \pm i(2z_0^{(1)}p)^{1/2}, \qquad (16a)$$

with $p = z_0^{(1)} + \sigma$. Thus

$$\Theta(u) = \beta^{1/2} \int_1^\infty \frac{dz(z-\sigma)}{z(z-z_0)\{(z-\sigma)(z-z_+)(z-z_-)\}^{1/2}}, \qquad (16b)$$

where we have now dropped the superscript on $z_0^{(1)}$ and $\beta^{(1)}$. The integral in (16b) can be exactly evaluated in two steps with the help of Ref. 6. Thus with $Q(z)$ denoting the polynomial under the square root in (16b) we find for the integral

$$I_1 = \int_1^\infty dz(z-z_0)^{-1}(Q(z))^{-1/2}$$

the result

$$I_1 = g\eta^{-1} \int_0^{u_1} \frac{1 - \operatorname{cn} u}{1 + q \operatorname{cn} u} du \qquad (17a)$$

$$= \frac{g}{\eta q}[-u_1 + (1-q)^{-1} \times (\pi(\phi, q^2/(q^2-1), j) - q f_1(u_1))], \qquad (17b)$$

where

$$\begin{aligned} &g = A^{-1/2}, \quad A^2 = (b_1 - \sigma)^2 + a_1^2, \\ &2j^2A = A + b_1 - \sigma, \quad 2b_1 = z_+ + z_-, \\ &4a_1^2 = -(z_+ - z_-)^2, \quad \eta q = (A - \sigma + z_0), \\ &\eta = A + \sigma - z_0, \quad u_1 = \operatorname{cn}^{-1}(\cos\phi, j) = F(\phi, j), \\ &\phi = \operatorname{am} u_1 = \cos^{-1}\omega, \ \omega(1 - \sigma + A) = 1 - \sigma - A. \end{aligned} \qquad (18)$$

In (17) it is understood that $q^2 \neq 1$, and $F(\phi, j)$ and $\pi(\phi, q^2/(q^2-1), j)$ are the normal elliptic integrals[4] of the first and third kinds, respectively. Both are zero when $\phi = 0$. The last term in (17b) is obtained from[6]

$$f_1(u_1) = \begin{cases} h\arctan(\operatorname{sd} u_1/h), & \text{if } q^2/(q^2-1) < j^2, \quad (19a) \\ \tilde{h}\ln\left(\dfrac{\operatorname{dn} u_1 + \tilde{h}\operatorname{sn} u_1}{\operatorname{dn} u_1 - \tilde{h}\operatorname{sn} u_1}\right), & \text{if } q^2/(q^2-1) > j^2, \quad (19b) \end{cases}$$

with $h^2(j^2 + (1-j^2)q^2) = 1 - q^2$, $\tilde{h}^2 = -h^2$.

In (17a) and (19), cn u, sn u, and dn u are the Jacobian elliptic functions, with $\operatorname{sd} u = \operatorname{sn} u/\operatorname{dn} u$. It is of course known that $\operatorname{cn} 0 = 1 = \operatorname{dn} 0$ and $\operatorname{sn} 0 = 0$; thus the lower limit in (17a) is not displayed in the various terms in (17b). There is one more integral denoted by I_2 below that we need in order to evaluate (16b), and it is given by

$$I_2 = \int_1^\infty dz\, z^{-1}(Q(z))^{-1/2}. \quad (20)$$

Clearly I_2 is just I_1 evaluated at $z_0 = 0$; it can be reached via (17b) *et seq*. through appropriate replacements. Thus for the value of β given by (15) we have the exact dependence of $\Theta(u)$ on u. It works out to

$$\Theta(u) = \beta^{1/2}(1 - \sigma/z_0)I_1 + \beta^{1/2}(\sigma/z_0)I_2. \quad (21)$$

V. NUMERICAL COMPUTATION

Equation (21) is clearly far too complicated to be of immediate physical interest. It would be ideal if it could be simplified considerably by application to a physical situation so that the equation for the brachistochrone would be more accessible. Happily, the gravitational field outside the sun, which is generally considered[3] as weak ($R/r = 10^{-5}$ at the surface of the sun), comes in handy in this connection. It needs to be pointed out here that it is our unfamiliarity with astrophysics that prevents us from extending the considerations of this section to other stellar objects whose gravitational fields are also deemed to be weak. Still, we believe that for such stellar objects the brachistochrone should not be vastly different from the expression derived here.

For the sun, the Schwarzschild radius R and the actual radius are about 3 km and 10^6 km, respectively (it is not necessary to use precise numbers here). Thus, as the particle of mass m falls from infinity in the sun's gravitational field, u increases from zero to about 10^{-6}, so that $uR \ll 1$ throughout the motion of the particle.

We now turn to a numerical estimation of the various entities in Eq. (18). With $9 \tan\mu = 2\sqrt{11}$ we obtain $\mu = 36.39$ deg. Thus (15) leads to $\beta = 28.042\,264\sigma^3$; clearly, as $\sigma \ll 1, \beta < 1$. We shall choose a positive square root of β in our calculations below; clearly this corresponds to a $k > 0$. Inserting the value of μ into (14a) one can calculate z_0 in units of σ, and hence b_1, a_1, and A as defined by (18). We obtained $A^2 = 19.116\,844\sigma^2$, $\eta = 3.914\,854\sigma$, and $\eta q = 4.829\,708\sigma$. Thus q, and hence q^2, is obtained; we find that $q^2/(q^2 - 1) > j^2$ as $q^2 = 1.521\,986$ and $j^2 = 0.104\,62$. We therefore get the term in Eq. (21) denoted by Θ_1 as

$$\Theta_1 = 0.164\,576[-u_1 - 4.279\,218(\pi(\phi, 2.915\,76, 0.323\,45) - 1.233\,69 f_1(u_1))], \quad (22a)$$

where $f_1(u_1)$ will be reached via (19b) since $q^2/(q^2 - 1) > j^2$. It works out to

$$f_1(u_1) = c \ln\left(\frac{\mathrm{dn}\, u_1 + c\, \mathrm{sn}\, u_1}{\mathrm{dn}\, u_1 - c\, \mathrm{sn}\, u_1}\right), \quad (22b)$$

with $c = 0.596\,429$.

The second term in (21) denoted below by Θ_2 can also be written down in an analogous fashion. We find

$$\Theta_2 = 0.515\,278[-u_1 + 2.686\,14 \times(\pi(\phi, -0.650\,249, 0.323\,45) - 0.627\,719 f_1(u_1))], \quad (23a)$$

where $f_1(u_1)$ is now given by (19a) since $q^2/(q^2 - 1) < j^2$. To reassure the reader, we find, for I_2, $\eta = (A + 1)\sigma$, $\eta q = (A - 1)\sigma$, with A^2 quoted above and $q^2 = 0.394\,031$.

For $f_1(u_1)$ we thus obtain

$$f_1(u_1) = 1.150\,971 \arctan(0.868\,832\, \mathrm{sd}\, u_1). \quad (23b)$$

Equations (22) and (23) can now be examined in the light of our observation that $\sigma \ll 1$ throughout the motion of the particle. From (18) we note that, as $\sigma, A \ll 1$, $\omega \simeq 1$, or $\phi \simeq \cos^{-1} 1$, or $\phi \simeq 0$. Thus as u increases from zero to 10^{-6}, ϕ remains almost constant at zero.

Let us now estimate Θ_1 as given by (22). The value of u_1 is reached via $u_1 = F(\phi, j)$; since ϕ is nearly zero, and[4]

$$\lim_{\phi \to 0} [F(\phi, j)/\sin\phi] = 1,$$

we infer that $u_1 \simeq \phi$. Again, as

$$\lim_{\phi \to 0} [\pi(\phi, \alpha^2, j)/\sin\phi] = 1,$$

we infer that $\pi(\phi, q^2/(q^2 - 1), j) \simeq \phi$. The elliptic functions work out as

$$\mathrm{sn}\, u_1 = (1 - \cos^2\phi)^{1/2} = 2(A(1 - \sigma))^{1/2}(1 - \sigma + A)^{-1} \simeq 2A^{1/2}$$

and

$$\mathrm{dn}\, u_1 = (1 - j^2 \sin^2\phi)^{1/2} \simeq 1.$$

Thus the logarithm in (22b) becomes

$$c \log\left(\frac{1 + c\, \mathrm{sn}\, u_1}{1 - c\, \mathrm{sn}\, u_1}\right) \simeq 2c^2\, \mathrm{sn}\, u_1.$$

We thus obtain

$$\Theta_1 = -0.868\,832\phi + 0.618\,136\, \mathrm{sn}\, u_1. \quad (24a)$$

Similar remarks apply to Θ_2 given by (23). We get, with $b = 0.868\,831$,

$$\Theta_2 = b\phi - \arctan(b\, \mathrm{sn}\, u_1). \quad (24b)$$

Thus, adding (24a) and (24b), we get

$$\Theta = 0.618\,136\, \mathrm{sn}\, u_1 - \arctan(b\, \mathrm{sn}\, u_1). \quad (25)$$

We note that the term proportional to ϕ has now canceled almost exactly. Since the argument of the arctan function is very small, we now replace the last term by its argument and get

$$4\Theta = -\mathrm{sn}\, u_1 = -4.182\sigma^{1/2}. \quad (26)$$

Equation (26) leads (with $R = 3$ km) to

$$r^{1/2}\Theta = -1.816, \quad (27)$$

as the equation for the brachistochrone with reference to the gravitational field outside the sun. More importantly, we note that as u increases from zero, Θ decreases from zero and remains negative. This is just a reflection [as mentioned in connection with (9‴)] of the decrease of $\Theta(u)$ with u. As the reader will have noticed, the considerations of this section depend on the smallness of the variable denoted by σ. Being infinitesimally different from zero, the doubly periodic property of the Jacobian elliptic functions $\mathrm{sn}\, u_1$, for example [see Eq. (26)], has not found any place in our calculation. Indeed, with $j^2 = 0.104\,62$, the standard definitions[5] of the symbols K, K' contained in the periods $(4K, 2iK')$ of $\mathrm{sn}(u, j)$ lead to $K \sim 1.612$ and $K' \sim 2.578$. Clearly the value of K is too large to be of interest as far as u_1 is concerned.

It is appropriate before concluding this paper to recall for the reader's benefit the equation for the brachistochrone obtained by GB; this will be taken up in the following section.

VI. SOME COMMENTS

There are two parts to this section. First, we obtain an exact analytic form for the brachistochrone obtained by GB. This has been done in Ref. 1 for the case $k^2 = 1$ only; for $k^2 < 1$ and $k^2 > 1$ only a graphical plot of the brachistochrone has been given. Second, we comment (quite concisely, however) on the task of obtaining the brachistochrone for the full Schwarzschild metric.[2,3] Naturally this means that, while retaining therein the assumption of spherical symmetry, we are giving up the restriction to weak gravitational fields used in Secs. III–V.

To take up the GB calculation first, we consider the integral [Eq. (19) in Ref. 1]

$$y(x) = \int_0^x dt \left[\frac{k^2(1-\exp(-2\alpha t))}{1-k^2+k^2\exp(-2\alpha t)}\right]^{1/2}. \qquad (28)$$

For $k^2 > 1$ this can be reworked as

$$y(x) = \xi \int_0^x dt \left[\frac{\exp(2\alpha t)-1}{\xi^2-\exp(2\alpha t)}\right]^{1/2}, \qquad (28')$$

with $\xi^2(k^2-1) = k^2$ and $c^2\alpha = g$. Using the substitution $\exp \alpha t = u$ we obtain from (28'), with $\psi = \exp \alpha x$,

$$\alpha y(x) = \xi \int_1^\psi u^{-1}\, du \left[\frac{u^2-1}{\xi^2-u^2}\right]^{1/2},$$

which can now be easily evaluated in terms of elementary functions. We obtain

$$2\alpha y(x) = \xi \sin^{-1}(1-w_1^2)^{1/2} + \sin^{-1}[-(1-w_2^2)^{1/2}], \qquad (29)$$

where $(\xi^2-1)w_1 = \xi^2+1-2\psi^2$, and $(\xi^2-1)w_2 = \xi^2+1-2\xi^2\psi^{-2}$. Note that in (28') we must have $\xi^2 \geqslant \psi^2$ for the integral to be real; in terms of k^2 this is rewritten $k^2 \leqslant \psi^2(\psi^2-1)^{-1}$. We recall here that Eq. (10') requires that $k^2 \leqslant (1+Ru)^2/Rc^2u^3(1-Ru)$ for the integral to be real. The counterpart of (29) when $k^2 < 1$ can also be obtained easily. We shall merely quote the result below:

$$2\alpha y(x) = \xi \ln R(x) + \sin^{-1}[-(1-w_3^2)^{1/2}], \qquad (30)$$

with

$$(\xi^2+1)R(x) = 2\psi^2 + \xi^2 - 1 + 2(\psi^4 + (\xi^2-1)\psi^2 - \xi^2)^{1/2},$$

and

$$(\xi^2+1)w_3 = \xi^2 - 1 - 2\xi^2\psi^{-2}.$$

As mentioned earlier, a graphical plot of (29) and (30) has been given in Ref. 1, but the numerical values given therein to α, defined by $c^2\alpha = g$, are in fact too large ($\alpha \sim 1$)—perhaps so large (in fact by several orders of magnitude) as to render invalid the Newtonian approximation for the gravitational field used in Ref. 1. It seems to us that had Goldstein and Bender used values of α that are consistent with the Newtonian approximation, typically $\alpha \sim 10^{-12}$ or even less, they would then have obtained, with the range of values for x used in their calculation, a plot for $y(x)$ that was quite insensitive to changes in x. In particular, the cycloidlike (periodic) behavior of the $k^2 > 1$ brachistochrone would not have been uncovered (see Figs. 7 and 8 in GB). We remind the reader here that we were also faced with a parallel situation of not finding a suitable role for the periodic behavior of the Jacobian elliptic functions $\mathrm{sn}(u,j)$ in Sec. V.

We shall now conclude this paper with a brief comment on the Schwarzschild brachistochrone. Our objective herein is merely to highlight for the reader the utility, technically speaking, of the weak-field approximation given by $g_{\alpha\beta} \simeq \eta_{\alpha\beta} + g^{(1)}_{\alpha\beta}$ that was used in the calculations in this paper. For the Schwarzschild metric,[2,3] the analog of Eq. (2) is given by

$$ds^2 = g_{00}c^2\, dt^2 + g_{ij}\, dx^i\, dx^j, \qquad (31)$$

with $g_{00} = g_-$, $g_{ij} = -\delta_{ij} - \sigma(1-\sigma)x_i x_j/r^2$.

Correspondingly, the Hamiltonian for the material particle of mass m, following Sec. II, will work out to

$$H = mc^2 g_-(g_- + g_{ij}\dot{x}^i\dot{x}^j)^{-1/2}.$$

However, because g_{ij} now has off-diagonal components, the counterpart of Eq. (7) for the velocity will have the form

$$v^2 = 2\Phi g_-^2 \left(g_- + \sigma\left(1 + u^2\left(\frac{d\Theta}{du}\right)^2\right)^{-1}\right)^{-1}, \qquad (32)$$

thus making v^2 a function of the differential of Θ with respect to u; this feature is, however, absent in Eq. (7) and makes the problem more tractable there. But with v^2 as given by (32), one finds that the counterpart of (9″) is now a fifth-order polynomial in $(d\Theta/du)^2$, thus making for a numerical, rather than exact, solution for $(d\Theta/du)^2$. The possibility of an exact solution for $(d\Theta/du)$ using the simple quadratic in (9″) is thus an attractive feature of the weak gravitational field approximation.

ACKNOWLEDGMENTS

I thank my colleagues S. H. Kulkarni and D. S. Subramanyam for several useful discussions on the mathematical aspects of this paper.

[1] H. F. Goldstein and C. M. Bender, J. Math. Phys. **27**, 507 (1986): also referred to as GB in the text.

[2] M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (Wiley, New York, 1982), pp. 205–210.

[3] A. Papapetrou, *Lectures on General Relativity* (Reidel, Dordrecht, 1974), pp. 108–114.

[4] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer, Berlin, 1954), pp. 252ff.

[5] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1968).

[6] See Ref. 4, Eqs. 241.04 and 361.60.