Teaching calculation of inductance of power transmission lines

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Abstract The conventional derivation of the expression for inductance of power transmission lines given in books is more a recipe than a derivation starting from a law of physics. The serious shortcomings of the traditional approach are that the definition of flux linkage lacks strong connection to Faraday's law, and unnecessary assumptions are made. This paper gives a complete rigorous derivation of the expression for inductance of power transmission lines. This paper will help instructors as well as students, since it addresses the deficiencies present in the derivation given in textbooks.

Keywords flux linkage; inductance calculation; power transmission line

The concepts of flux linkage and inductance of a circuit are subtle if the circuit is not filamentary. In the author's view the derivation of the expression for inductance of power transmission lines given in textbooks is unnecessarily abstract and incomplete. The derivation can be found in Refs 1 and 2 and many other textbooks.

The derivation given in the textbooks has the following deficiencies:

1. The derivation of the flux linkage of a straight conductor of circular cross-section, due to flux within the conductor, is abstract. Flux linkage is defined for a coaxial tube of differential width, within the conductor and the flux linkage of the conductor is given as sum of the flux linkages of all the tubes in the conductor. The definition of flux linkage of a tube and the point that flux linkage of the conductor is sum of the flux linkages of all the tubes in the conductor (obtained by integration), are not all clear. This point is made even in an earlier publication.[3] In Ref. 2, the following statement is made regarding the expression for internal flux linkage: 'The result, although intuitively reasonable, is by no means obvious'.
2. Flux linkage of a straight conductor due to current in another parallel conductor is derived by making an unnecessary assumption that distance between the two conductors is very large compared to their radius. This assumption gives an impression that the derived result is approximate, though it is not so. This point is made in Ref. 1: 'In fact, it can be shown that calculations made on this assumption are correct even when D is small', and Ref. 4: 'this assumption leads to an exact result even if they are closely spaced'.
3. The derivation of the expression for inductance of power transmission line starts from the special case of single circuit line with conductors of circular cross-section. This is followed by the general case of composite conductors (for example multi-circuit line or bundled conductors) for which the expression for inductance is given directly in terms of geometric mean distances, without any derivation. In some books, for example Ref. 1, this is preceded

by the derivation of the expression for inductance of single phase line with composite conductors; in this derivation, transposition within the composite conductor is not explicitly mentioned without which the expression does not hold.

These deficiencies can be avoided if the derivation starts from Faraday's law (one of the Maxwell's equations in integral form), which is strictly applicable only to a filamentary closed path. This paper gives a complete rigorous derivation of the expression for inductance of a general three-phase overhead power transmission line, starting from Faraday's law.

The derivation assumes that the three-phase system is balanced.

Faraday's law

The e.m.f. e induced in a filamentary closed path is given by Faraday's law.

$$e = \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{1}$$

B is magnetic flux density and S is any surface such that the closed path is along its perimeter. This equation can be generalised for a closed path which is not filamentary as

$$e = \frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{2}$$

where ψ is the flux linkage of the closed path. In the special case of a filamentary closed path, the flux linkage is the flux passing through the surface whose perimeter coincides with the closed path. The expression for flux linkage in general (when the closed path is not filamentary) is derived in the next section.

Flux linkage of a straight conductor

Flux linkage of a closed path consisting of a straight filamentary conductor and a parallel straight path at infinite distance is known as flux linkage of the conductor; this is nothing but the flux enclosing the conductor. If the conductor is not filamentary, then the flux linkage of the conductor is due to both the flux within the conductor and the flux external to the conductor.

Consider parallel straight conductors 1 and 2 (shown in Fig. 1), carrying currents i_1 and i_2 respectively. The cross-sectional area of the conductors is $A = \pi r^2$. Let it be assumed that the current density in the conductors is uniform.

A conductor is made up of infinite number of filaments. Consider a coaxial tube of radius $x < r$ and thickness dx in conductor 1. Consider a filament in this tube with cross-sectional area $\mathrm{d}A = x \times \mathrm{d}x \times \mathrm{d}\theta$, carrying current $i_1\mathrm{d}A/A$. $\mathrm{d}\theta$ is the angle subtended at the axis of conductor 1 by the filament and D is the distance of this filament from the axis of conductor 2. The e.m.f. e induced in this filament is given by the following equation.

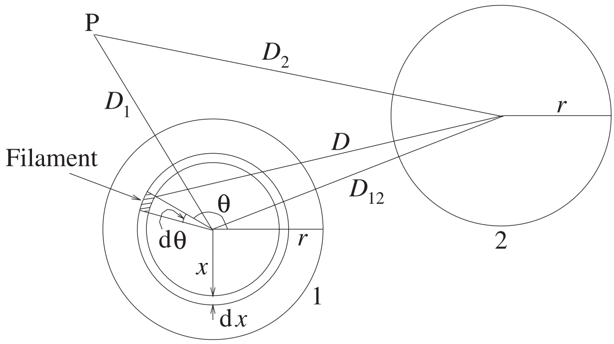

Fig. 1 *Cross-section of two parallel straight conductors.*

$$e = \frac{\mathrm{d}(\psi_{f1} + \psi_{f2})}{\mathrm{d}t} = \frac{\partial \psi_{f1}}{\partial i_1}\frac{\mathrm{d}i_1}{\mathrm{d}t} + \frac{\partial \psi_{f2}}{\partial i_2}\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{3}$$

where ψ_{f1} and ψ_{f2} are the flux linkages of the filament due to i_1 and i_2 respectively. For the conductor (which is usually aluminium) and the air in which the magnetic flux is set up, the permeability is constant and almost equal to that of free space $\mu_0 = 4\pi \times 10^{-7}$ H/m. Therefore $\partial\psi_{f1}/\partial i_1 = \psi_{f1}/i_1$ and $\partial\psi_{f2}/\partial i_2 = \psi_{f2}/i_2$. Hence

$$e = \frac{\psi_{f1}}{i_1}\frac{\mathrm{d}i_1}{\mathrm{d}t} + \frac{\psi_{f2}}{i_2}\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{4}$$

The power delivered to this filament is given by

$$\mathrm{d}p = ei_1\frac{\mathrm{d}A}{A} = \psi_{f1}\frac{\mathrm{d}i_1}{\mathrm{d}t}\frac{\mathrm{d}A}{A} + \psi_{f2}\frac{i_1}{i_2}\frac{\mathrm{d}i_2}{\mathrm{d}t}\frac{\mathrm{d}A}{A} \tag{5}$$

The power delivered to conductor 1 is

$$p = \int_A \mathrm{d}p = \frac{\mathrm{d}i_1}{\mathrm{d}t}\frac{1}{A}\int_A \psi_{f1}\mathrm{d}A + \frac{i_1}{i_2}\frac{\mathrm{d}i_2}{\mathrm{d}t}\frac{1}{A}\int_A \psi_{f2}\mathrm{d}A \tag{6}$$

It is to be noted that the e.m.f.s induced in all the filaments of the conductor are equal. If the flux linkage of conductor 1 is $\psi = \psi_{c1} + \psi_{c2}$, where ψ_{c1} and ψ_{c2} are the flux linkages of conductor 1 due to currents i_1 and i_2 respectively, then

$$p = \frac{\mathrm{d}\psi}{\mathrm{d}t}i_1 = \frac{\partial\psi_{c1}}{\partial i_1}\frac{\mathrm{d}i_1}{\mathrm{d}t}i_1 + \frac{\partial\psi_{c2}}{\partial i_2}\frac{\mathrm{d}i_2}{\mathrm{d}t}i_1 = \psi_{c1}\frac{\mathrm{d}i_1}{\mathrm{d}t} + \psi_{c2}\frac{i_1}{i_2}\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{7}$$

From eqns (6) and (7), by equating the terms on the RHS, the following equation is obtained:

$$\psi = \frac{1}{A}\int_A (\psi_{f1} + \psi_{f2})\mathrm{d}A \tag{8}$$

Hence the flux linkage of the conductor is the average of the flux linkages of all the filaments of the conductor.

The flux densities B_1 and B_2 due to i_1 and i_2 respectively can be obtained from Ampere's law.

$$B_1 = \begin{cases} \mu_0 i_1 x'/2\pi r^2, \; x' \le r \\ \mu_0 i_1/2\pi x', \; x' \ge r \end{cases} \tag{9}$$

$$B_2 = \frac{\mu_0 i_2}{2\pi D'}, \; D' \ge r \tag{10}$$

x' and D' are the distances from the axis of conductors 1 and 2 respectively. The flux linkage per unit length of the filament (shown in Fig. 1) due to flux up to point P is $\psi'_{f1} + \psi'_{f2}$.

$$\psi'_{f1} = \int_x^{D_1} B_1 dx' = \frac{\mu_0 i_1}{2\pi}\left(\frac{1}{2} - \frac{x^2}{2r^2} + \ln\frac{D_1}{r}\right) \tag{11}$$

$$\psi'_{f2} = \int_D^{D_2} B_2 dD' = \frac{\mu_0 i_2}{2\pi}\ln\frac{D_2}{D} \tag{12}$$

where $D = (D_{12}^2 + x^2 - 2D_{12}x\cos\theta)^{1/2}$ and $D_2 > D_{12} + r$. From eqn (8), the flux linkage per unit length of conductor 1 due to flux up to point P is

$$\psi' = \frac{1}{A}\int_A (\psi'_{f1} + \psi'_{f2})\,\mathrm{d}A = \frac{\mu_0 i_1}{2\pi}\ln\frac{D_1}{r'} + \frac{\mu_0 i_2}{2\pi}\ln\frac{D_2}{D_{12}} \tag{13}$$

where $r' = \mathrm{e}^{-1/4}r$. The first and second terms on the RHS of eqn (13) are the flux linkages due to i_1 and i_2 respectively. The derivation of the second term is given in the Appendix.

If there are n parallel straight conductors 1, 2, . . . n of radius r and carrying currents i_1, i_2, . . . i_n respectively, then from eqn (13), the flux linkage per unit length of conductor k due to flux up to a remote point P is

$$\psi'_k = \frac{\mu_0}{2\pi}\sum_{l=1}^{n} i_l \ln\frac{D_l}{D_{kl}} \tag{14}$$

where D_l is the distance between the axis of conductor l and point P, $D_{kk} = r'$ and $D_{kl}(k \ne l)$ is the distance between axes of conductors k and l. The total flux linkage per unit length of conductor k is obtained by letting point P move infinitely far away. As P recedes to infinity, if $\sum_{l=1}^{n} i_l = 0$, the total flux linkage per unit length of conductor k is

$$\psi_k = \frac{\mu_0}{2\pi}\sum_{l=1}^{n} i_l \ln\frac{1}{D_{kl}} \tag{15}$$

Inductance of power transmission line

Consider a three-phase transmission line consisting of a composite conductor in each phase, as shown in Fig. 2. A composite conductor consists of two or more individual conductors of same radius in parallel. Each phase consists of n individual conductors of radius r.

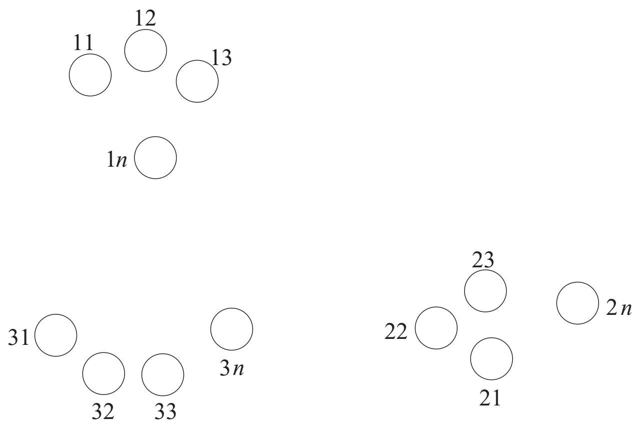

Fig. 2 *A three-phase line with composite conductors.*

It is assumed that the conductors are straight. In practice, the towers supporting the conductors may not be along a straight line, and between two consecutive towers the conductor is not straight due to the sag.

In order to have a balanced system, the three phases should be transposed (if not placed symmetrically); the line is divided into three sections of equal lengths and each phase occupies each of the three positions (1, 2, 3) for one third of the line length. Let phases a, b, c occupy positions 1, 2, 3 respectively in the first section, positions 2, 3, 1 respectively in the second section and positions 3, 1, 2 respectively in the third section. The position of each individual conductor is identified by two numbers as in Fig. 2; the first number is that of the position of the phase and the second number is that of the position of the individual conductor. The currents in the three phases are i_a, i_b, i_c. Let the current in the individual conductors of phases a, b, c be i_a/n, i_b/n, i_c/n respectively. This is true only if the individual conductors in each phase are placed symmetrically or are transposed so that each individual conductor occupies each of the n positions for equal lengths along the section. From eqn (8), the flux linkage of a phase is the average of the flux linkages of the individual conductors in that phase. Since $i_a + i_b + i_c = 0$, from eqn (15), the flux linkage per unit length of phase a in section 1 is

$$\psi_{a1} = \frac{\mu_0}{2\pi n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(i_a \ln \frac{1}{D_{1k,1l}} + i_b \ln \frac{1}{D_{1k,2l}} + i_c \ln \frac{1}{D_{1k,3l}} \right) \tag{16}$$

where $D_{pk,ql}$ (p and q are 1, 2 or 3, and $pk \neq ql$) is the distance between the axes of conductors at locations pk and ql; $D_{pk,pk} = r'$. Similarly the flux linkage per unit length of phase a in sections 2 and 3 are

$$\psi_{a2} = \frac{\mu_0}{2\pi n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(i_a \ln \frac{1}{D_{2k,2l}} + i_b \ln \frac{1}{D_{2k,3l}} + i_c \ln \frac{1}{D_{2k,1l}} \right) \tag{17}$$

$$\psi_{a3} = \frac{\mu_0}{2\pi n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(i_a \ln \frac{1}{D_{3k,3l}} + i_b \ln \frac{1}{D_{3k,1l}} + i_c \ln \frac{1}{D_{3k,2l}} \right) \tag{18}$$

The average flux linkage per unit length of phase a is

$$\psi_a = \frac{1}{3}(\psi_{a1} + \psi_{a2} + \psi_{a3}) \tag{19}$$

The self inductance per unit length of each phase is

$$L = \frac{\partial \psi_a}{\partial i_a} = \frac{\mu_0}{6\pi n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \ln \frac{1}{D_{1k,1l} D_{2k,2l} D_{3k,3l}} \tag{20}$$

The mutual inductance per unit length between any two phases is

$$M = \frac{\partial \psi_a}{\partial i_b} = \frac{\mu_0}{6\pi n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \ln \frac{1}{D_{1k,2l} D_{2k,3l} D_{3k,1l}} \tag{21}$$

If the three-phase system is balanced, then a single phase analysis is adequate; the self inductances of the phases and the mutual inductances between phases can be replaced by an equivalent self inductance in each phase. Substituting $i_b + i_c = -i_a$ in eqn (19),

$$\psi_a = \frac{\mu_0 i_a}{6\pi n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \ln \frac{D_{1k,2l} D_{2k,3l} D_{3k,1l}}{D_{1k,1l} D_{2k,2l} D_{3k,3l}} \tag{22}$$

The equivalent self inductance per unit length of each phase is

$$L_1 = \frac{\psi_a}{i_a} = \frac{\mu_0}{2\pi} \ln \frac{D_m}{D_s} \tag{23}$$

where

$$D_m = \left(\prod_{k=1}^{n} \prod_{l=1}^{n} D_{1k,2l} D_{2k,3l} D_{3k,1l} \right)^{1/3n^2} \tag{24}$$

$$D_s = \left(\prod_{k=1}^{n} \prod_{l=1}^{n} D_{1k,1l} D_{2k,2l} D_{3k,3l} \right)^{1/3n^2} \tag{25}$$

D_m is known as mutual geometric mean distance (GMD) and D_s is known as self GMD. L_1 is positive sequence inductance per unit length as well as negative sequence inductance per unit length of the transmission line.

Examples of composite conductors

Multi-circuit line

For the single circuit line shown in Fig. 3(a),

$$D_m = (D_{12} D_{23} D_{31})^{1/3} \tag{26}$$

$$D_s = r' \tag{27}$$

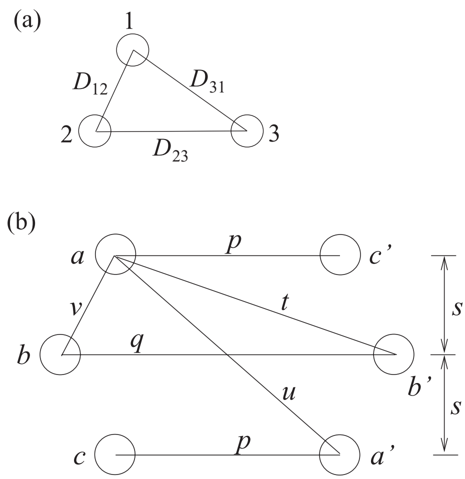

Fig. 3 *Single circuit and double circuit lines.* (a) *Single circuit line;* (b) *double circuit line.*

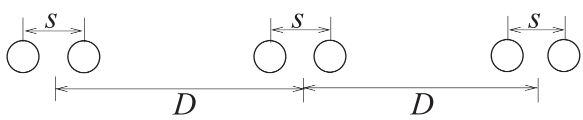

Fig. 4 *Line with bundled conductors.*

For the double circuit line shown in Fig. 3(b),

$$D_m = (2v^2t^2sp)^{1/6} \tag{28}$$

$$D_s = (\mathrm{e}^{-3/4}r^3u^2q)^{1/6} \tag{29}$$

For hexagonal spacing ($p = v$ and $q = u$), since the three phases and the two individual conductors of a phase are placed symmetrically, transposition of the three phases is not necessary to have a balanced system and the expressions for mutual and self GMD given by eqns (28) and (29) hold even if the individual conductors in a phase are not transposed.

Bundled conductors

For the line with bundled conductors shown in Fig. 4, where each bundle consists of two conductors,

$$D_m = \left[4D^6(D^2 - s^2)^2(4D^2 - s^2)\right]^{1/12} \tag{30}$$

$$D_s = (\mathrm{e}^{-1/4}rs)^{1/2} \tag{31}$$

Stranded conductors

For the line with stranded conductors shown in Fig. 5,

$$D_m = \left[D^7 (D^6 - 4096r^6)(D^6 - 64r^6)^4 (D^6 + 1728r^6)^2 \right]^{1/49} \tag{32}$$

$$D_s = 2(364.5)^{1/49} \mathrm{e}^{-1/28} r \tag{33}$$

Discussion and conclusions

The deficiencies in the derivation of inductance, mentioned in the introduction, are eliminated as follows:

1 The relation between flux linkage of a conductor and those of the filaments within the conductor, is derived. The derivation uses the fact that power delivered to the conductor is sum of the powers delivered to all filaments in the conductor; this approach is used in Ref. 3. It is shown that the flux linkage of a conductor is not the sum of the flux linkages of the filaments (or tubes) within the conductor, but their average, as given by eqn (8). Flux linkage due to flux within the conductor and that due to flux outside the conductor need not be considered separately. Further, the flux linkage of a conductor due to its own current and that due to current in another conductor need not be considered separately.

2 The expression for inductance is derived for a general three-phase line with composite conductors and then the derived expression is used for special cases.

In practice only stranded conductors are used, and the strands are spiralled which results in a slight increase in inductance.[4]

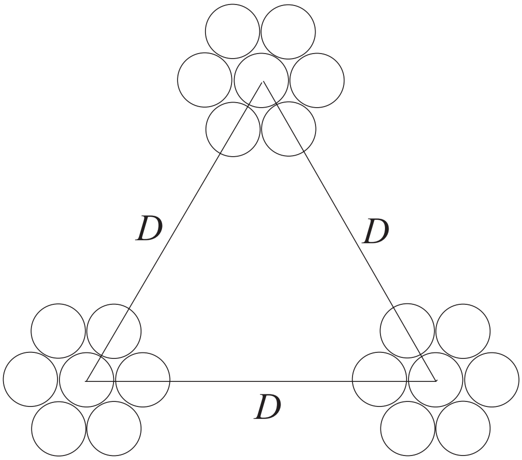

Fig. 5 *Line with stranded conductors.*

The assumption that currents in the individual conductors of a composite conductor are equal is in agreement with the assumption that current density in the conductor is uniform. Strictly current density is not uniform due to skin effect and proximity effect, which result in slight decrease in inductance. These effects are not significant for power frequency of 50 or 60 Hz and hence can be ignored. Proximity effect, which is negligibly small for the spacing in an overhead line, is decreased by spiralling of the strands.

The derivation given in the paper is not applicable to steel-reinforced aluminium conductors since the conductivity and permeability of steel are different from those of aluminium.

The transposition of the individual conductors in the composite conductor is not necessary for the three-phase system to be balanced; but it helps in easily obtaining the expression for inductance in terms of GMD. Even if the transposition within the composite conductor and the transposition of the three phases are not done, the expression derived for inductance gives a sufficiently accurate value.

The GMD is in general defined for areas in a plane.[4] The expressions for mutual and self GMD given by eqns (24) and (25) are applicable only if the conductors (or strands) have circular cross-section with same radius. The derivation of the expression for self GMD of a circular area is involved[5,6] and the proposed method avoids this derivation.

The conventional approach and the proposed derivation were used by the author to teach different sets of students and the students' responses to these two methods were different. In a class, there are at least 10% of the students who raise questions as and when there is lack of clarity in what is being taught. These questions sometimes serve as a useful immediate feedback to the instructor if there is a lacuna in the presentation. The author experienced this while teaching derivation of the expression for transmission line inductance. While teaching the conventional derivation of the expression for the transmission line inductance as a part of the course 'Field Theory' to about 65 fourth semester undergraduate students of M.S. Ramaiah Institute of Technology Bangalore, many questions were raised by the students. On the contrary, while teaching the proposed derivation to another batch of about 65 fourth semester undergraduate students of the same institute and also to about 100 fifth semester undergraduate students of Indian Institute of Technology Madras as a part of the course 'Power System Practice', there were no questions from the students. The major reason for the difference in response is that the students are familiar with the underlying Faraday's law from school days and they find it easy to relate the proposed derivation to the law. On the other hand, the conventional derivation starts by defining flux linkage of a conductor in terms of flux linkages of the components of the conductor and the students find this definition difficult to accept since it is neither explicitly stated as a law of physics nor related to Faraday's law. The other reason is that the conventional derivation obtains the exact expression for inductance in spite of making an unnecessary assumption/approximation! The students find this difficult to understand since it is obvious to presume that an assumption/approximation should lead to an approximate expression and not the exact expression. The proposed derivation does not make any assumption/approximation. Overall the proposed derivation is transparent and easy for the students to understand.

Appendix

$$\int_0^{2\pi} \ln D \mathrm{d}\theta = 2\pi \ln D_{12} + \frac{1}{2}\int_0^{2\pi} \ln(1+m^2-2m\cos\theta)\mathrm{d}\theta \tag{34}$$

where $m = x/D_{12}$. Define $z = \mathrm{e}^{j\theta} = \cos\theta + j\sin\theta$. Then $\cos\theta = (z + z^{-1})/2$, $\ln(1 + m^2 - 2m\cos\theta) = \ln(1 - mz) + \ln(1 - mz^{-1})$.[7] Since $m \leq 0.5$, by Taylor's theorem, the functions $\ln(1 - mz)$ and $\ln(1 - mz^{-1})$ can be expressed as follows:

$$\ln(1-mz) = -\sum_{k=1}^{\infty}\frac{m^k z^k}{k} \tag{35}$$

$$\ln(1-mz^{-1}) = -\sum_{k=1}^{\infty}\frac{m^k z^{-k}}{k} \tag{36}$$

Adding the two series,

$$\ln(1+m^2-2m\cos\theta) = -2\sum_{k=1}^{\infty}\frac{m^k\cos(k\theta)}{k} \tag{37}$$

Substituting in eqn (34),

$$\int_0^{2\pi} \ln D\,\mathrm{d}\theta = 2\pi\ln D_{12} \tag{38}$$

Hence

$$\frac{1}{\pi r^2}\int_{x=0}^{r}\int_{\theta=0}^{2\pi} \ln D x\,\mathrm{d}x\,\mathrm{d}\theta = \ln D_{12} \tag{39}$$

Eqn (13) follows from eqn (39).

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