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Syzygies, Betti Numbers, and Regularity of Cover Ideals of Certain Multipartite Graphs

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Abstract: Let G be a finite simple graph on n vertices. Let $J_G \subset K[x_1, \ldots, x_n]$ be the cover ideal of G. In this article, we obtain syzygies, Betti numbers, and Castelnuovo–Mumford regularity of J_G^s for all $s \geq 1$ for certain classes of graphs G.

Keywords: syzygy; Betti number; Castelnuovo-Mumford regularity; bipartite graph; multipartite graph

1. Introduction

In this article, we compute the resolution, syzygies, and Betti numbers of powers of certain classes of squarefree monomial ideals. As a by-product, we obtain the expression for the regularity of powers of these classes of ideals. Computation of minimal free resolution and syzygies of ideals and modules over polynomial rings have always attracted researchers in commutative algebra and algebraic geometry. Recently, there has been much interest in studying the homological aspects of squarefree monomial ideals in polynomial rings. As these ideals have strong combinatorial connections, problems in this area have attracted both commutative algebrists and combinatorists. Even in this case, there are only very few cases of ideals for which explicit computation of the resolution, including the complete description of syzygies, is done. The main open problems in this area are to find/construct minimal free resolutions in more cases, and to introduce new ideas and structures ([1] Remark 2.6). There are even less results on the resolution of powers of ideals. If I is generated by a regular sequence, then I^s is a determinantal ideal and hence the minimal free resolution of I^s can be obtained by using the Eagon–Northcott complex [2]. As the maps in this resolution may not be degree preserving, the computation of graded Betti numbers and more generally the computation of the syzygies may not be possible. In the work by the authors of [3], they explicitly compute the graded Betti numbers of I^s if I is a homogeneous complete intersection in a polynomial ring.

Among the resolutions, linear resolutions are possibly the simplest to describe. Let $R = K[x_1, \ldots, x_n]$, where K is a field. If I is the defining ideal of the rational normal curve in $\mathbb{P}^{n-1}$, then Conca proved that I^s has a linear resolution for all s [4]. Herzog, Hibi, and Zheng proved that if I is a monomial ideal generated in degree 2, then I has linear resolution if and only if I has linear quotients if and only if I^s has a linear resolution for all $s \geq 1$ [5]. It was proved by Conca and Herzog [6], that if I is a polymatroidal ideal in R, then I has linear quotients. Moreover, they proved that the product of polymatroidal ideals are polymatroidal. Therefore, if I is polymatroidal, then I^s has linear resolution for all $s \geq 1$. If I is a lexsegment ideal, then Ene and Olteanu proved that I has linear resolution if and only if I has linear quotients if and only if I^s has linear resolution for all $s \geq 1$ if and only if I^s has linear quotients for all $s \geq 1$ [7].

In all the above-mentioned results, the authors do not compute the syzygy modules. In general, it is a nontrivial task to compute the syzygy modules, even when the resolution is linear. It was shown by Kodiyalam [8], and independently by Cutkosky, Herzog, and Trung [9], that if I is a homogeneous ideal in a polynomial ring, then there exist non-negative integers d, e and s_0 such that $\operatorname{reg}(I^s) = ds + e$ for all $s \geq s_0$, where $\operatorname{reg}(-)$ denote the Castelnuovo–Mumford regularity. Kodiyalam proved that $d \leq \deg(I)$, where $\deg(I)$ denotes the largest degree of a homogeneous minimal generator of I. In general, the stability index s_0 and the constant term e are hard to compute. There have been discrete attempts in identifying s_0 and e for certain classes of ideals.

Recently, there have been a lot of activity in studying the interplay between the combinatorial properties of graphs and the algebraic properties of ideals associated to graphs. For a finite simple graph G on the vertex set $\{x_1, \ldots, x_n\}$, let $J_G \subset R = K[x_1, \ldots, x_n]$, where K is a field, denote the cover ideal of G (see Section 2 for the definition). It may be noted that $J_G = \cap_{\{x_i, x_j\} \in E(G)} (x_i, x_j)$ is the Alexander dual of the edge ideal of I (see Section 2 for the definition). Although the connection between algebraic properties of the edge ideal and combinatorial properties of the graph has been studied extensively, not much is known about the connection between the properties of the cover ideal and the graph. In [10], Seyed Fakhari studied certain homological properties of symbolic powers of cover ideals of very well covered and bipartite graphs. It was shown that if G is a very well covered graph and J_G has a linear resolution, then $J_G^{(s)}$ has a linear resolution for all $s \geq 1$. Furthermore, it was proved that if G is a bipartite graph with n vertices, then for $s \geq 1$,

$$\operatorname{reg}(J_G^s) \leq s \deg(J_G) + \operatorname{reg}(J_G) + 1.$$

Hang and Trung [11] studied unimodular hypergraphs and proved that if $\mathcal{H}$ is a unimodular hypergraph on n vertices and rank r and $J_{\mathcal{H}}$ is the cover ideal of $\mathcal{H}$, then there exists a non-negative integer $e \leq \dim(R/J_{\mathcal{H}}) - \deg(J_{\mathcal{H}}) + 1$, such that

$$\operatorname{reg} J_{\mathcal{H}}^s = \deg(J_{\mathcal{H}})s + e$$

for all $s \geq \frac{rn}{2} + 1$. Since bipartite graphs are unimodular, their results hold true in the case of bipartite graphs as well. While the first result gives an upper bound for the constant term, the later result gives the upper bound for both the stability index and the constant term.

In this article, we obtain the complete description of the minimal free resolution, including the syzygies, of J_G^s for some classes of multipartite graphs. The paper is organized as follows. In Section 2, we collect the preliminaries required for the rest of the paper. We study the resolution of powers of cover ideals of certain bipartite graphs in Section 3. If G is a complete bipartite graph, then J_G is a regular sequence and hence the minimal graded free resolution of J_G^s can be obtained from (Theorem 2.1 in the work by the authors of [3]). We then move on to study some classes of bipartite graphs which are not complete. We obtain the resolution, syzygies and Betti numbers of powers of cover ideals of certain bipartite graphs, and as a by-product, we obtain expression for the regularity of powers of these ideals. Section 4 is devoted to the study of resolution and regularity of powers of cover ideals of certain complete multipartite graphs.

The main results of this article are the following: When G is the cycle of length three or the complete graph on 4 vertices, we describe the graded minimal free resolution of J_G^s for all $s \geq 1$. This allow us to compute the Betti numbers, Hilbert series and the regularity of J_G^s for all $s \geq 1$. As a consequence, for cover ideals of complete tripartite and 4-partite graphs, we obtain precise expressions for the Betti numbers and the regularity of J_G^s. We conclude our article with a conjecture on the resolution of J_G^s for all $s \geq 1$, where G is a complete multipartite graph.

2. Preliminaries

In this section, we set the notation for the rest of the paper. All the graphs that we consider in this article are finite, simple, and without isolated vertices. For a graph G, $V(G)$ denotes the set of all

vertices of G and $E(G)$ denotes the set of all edges of G. A graph G is said to be a "complete multipartite" graph if $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ for some $k \geq 2$, such that $\{x, y\} \in E(G)$ if and only if $x \in V_i$ and $y \in V_j$ for $i \neq j$. When $k = 2$, the graph is called a "complete bipartite" graph. If $k = 2$ with $|V_1| = m$ and $|V_2| = n$, we denote the corresponding complete bipartite graph by $K_{m,n}$ or by K_{V_1,V_2}. If G and H are graphs, then $G \cup H$ denote the graph on the vertex set $V(G) \cup V(H)$ with $E(G \cup H) = E(G) \cup E(H)$. A graph G is called a "bipartite" graph if $V(G) = V_1 \sqcup V_2$ such that $\{x, y\} \in E(G)$ only if $x \in V_1$ and $y \in V_2$. A subset $w = \{x_{i_1}, \ldots, x_{i_r}\}$ of $V(G)$ is said to be a *vertex cover* of G if $w \cap e \neq \emptyset$ for every $e \in E(G)$. A vertex cover is said to be minimal if it is minimal with respect to inclusion.

Let G be a graph with $V(G) = \{x_1, \ldots, x_n\}$. Let K be a field and $R = K[x_1, \ldots, x_n]$. The edge ideal of G is defined to be $I(G) = \langle \{x_i x_j : \{x_i, x_j\} \in E(G)\} \rangle \subset R$ and the cover ideal of G is defined to be $J_G = \langle x_{i_1}, \ldots, x_{i_r} : \{x_{i_1}, \ldots, x_{i_r}\}$ is a (minimal) vertex cover of $G \rangle$ It can also be seen that J_G is the Alexander dual of $I(G)$.

Let $S = R/I$, where R is a polynomial ring over K and I a homogeneous ideal of R. For a finitely generated graded S-module $M = \oplus M_i$, set

$$t_i^S(M) = \max\{j : \operatorname{Tor}_i^S(M, K)_j \neq 0\},$$

with $t_i^S(M) = -\infty$ if $\operatorname{Tor}_i^S(M, K) = 0$. The Castelnuovo–Mumford regularity, denoted by $\operatorname{reg}_S(M)$, of an S-module M is defined to be

$$\operatorname{reg}_S M = \max\{t_i^S(M) - i : i \geq 0\}.$$

3. Bipartite Graphs

In this section, we study the regularity of powers of cover ideals of certain bipartite graphs. We begin with a simple observation concerning the vertex covers of a bipartite graph.

Proposition 1. *Let G be a bipartite graph on $n + m$ vertices. Then G is a complete bipartite graph if and only if J_G is generated by a regular sequence that has disjoint supports.*

Proof. Let $V(G) = X \sqcup Y$ be the partition of the vertex set of G with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_{n+1}, \ldots, y_{n+m}\}$. First, note that J_G is generated by a regular sequence if and only if for any two minimal vertex covers w, w', $w \cap w' = \emptyset$. If $G = K_{n,m}$, then $J_G = (x_1 \cdots x_n, y_{n+1} \cdots y_{n+m})$ which is a regular sequence. Conversely, suppose G is not a complete bipartite graph. As G is a bipartite graph, note that $\prod_{x_i \in X} x_i, \prod_{y_j \in Y} y_j \in J_G$ are minimal generators of J_G. Therefore, there exist $x_{i_0} \in X$ and $y_{i_0} \in Y$ such that $\{x_{i_0}, y_{i_0}\} \notin E(G)$. Then $w = \{x_i, y_j : i \neq i_0$ and $y_j \in N_G(x_{i_0})\}$ is a minimal vertex cover of G that intersects X as well as Y nontrivially. Therefore J_G is not a complete intersection. □

First we discuss the regularity of powers of cover ideals of complete bipartite graphs. As the cover ideal of a complete bipartite graph is a complete intersection, the result is a consequence of (Theorem 2.1 in the work by the authors of [3]).

Remark 1. *Let $J = J_{K_{m,n}}$ be the cover ideal of the complete bipartite graph $K_{m,n}$, $m \leq n$. Then* $\operatorname{reg}(J^s) = sn + m - 1$ *for all $s \geq 1$.*

Proof. Consider the ideal $I = (T_1, T_2) \subset R = K[T_1, T_2]$ with $\deg T_1 = m$ and $\deg T_2 = n$. It follows from (Theorem 2.1 in the work by the authors of [3]) that the resolution of I^s is

$$0 \to \bigoplus_{\substack{a_1+a_2=s+1 \\ a_i \geq 1}} R(-a_1 m - a_2 n) \to \bigoplus_{a_1+a_2=s} R(-a_1 m - a_2 n) \to I^s \to 0. \tag{1}$$

Note that $J = (x_1 \cdots x_m, y_{m+1} \cdots y_{m+n})$. Set $x_1 \cdots x_m = T_1$ and $y_{m+1} \cdots y_{m+n} = T_2$. Then, J^s has the minimal free resolution as in (1). If $m \leq n$, then $\operatorname{reg}(J^s) = sn + m - 1$. □

It follows from Remark 1 that in the case of the complete bipartite graph $K_{m,n}$, the stability index is 1 and the constant term is $\tau - 1$, where τ is the size of a minimum vertex cover.

If the graph is not a complete bipartite graph, then the cover ideal is no longer a complete intersection. If G is a Cohen–Macaulay bipartite graph, then it was shown by F. Mohammadi and S. Moradi that the vertex cover ideals are weakly polymatroidal. Therefore, they have linear quotients and hence all the powers have linear resolution (Theorem 2.2 in the work by the authors of [12]). It would be quite a challenging task to obtain the syzygies and Betti numbers of powers of cover ideals of all bipartite graphs. Therefore, we restrict our attention to some structured subclasses of bipartite graphs.

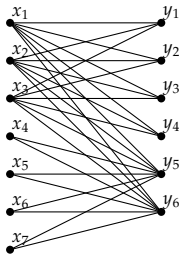


Figure 1. $K_{U_1,V} \cup K_{U_2,V_2}$.

Notation 1. *For disjoint vertex sets U and V, let $K_{U,V}$ denote the complete bipartite graph on $U \sqcup V$. Let $U_1 = \{x_1, \ldots, x_k\}$, $U_2 = \{x_{k+1}, \ldots, x_n\}$, $V_1 = \{y_1, \ldots, y_r\}$, and $V_2 = \{y_{r+1}, \ldots, y_m\}$. Let $U = U_1 \cup U_2$ and $V = V_1 \cup V_2$. In the following, we consider the bipartite graph G on the vertex set $U \sqcup V$ with edges $E(G) = E(K_{U_1,V}) \cup E(K_{U_2,V_2})$. An illustrative figure can be seen in Figure 1 above. Note that although the vertex sets of $K_{U_1,V}$ and K_{U_2,V_2} are not disjoint, the edge sets are. The figure on the left is an example of such a graph with $U_1 = \{x_1, x_2, x_3\}, U_2 = \{x_4, x_5, x_6, x_7\}$, $V_1 = \{y_1, y_2, y_3, y_4\}$ and $V_2 = \{y_5, y_6\}$.*

Theorem 1. *Let $U = U_1 \sqcup U_2$ and $V = V_1 \sqcup V_2$ be a collection of vertices with $|U| = n$, $|U_i| = n_i$, $|V| = m$, $|V_i| = m_i$ and $1 \leq n_i, m_i$ for $i = 1, 2$. Let G be the bipartite graph $K_{U_1,V} \cup K_{U_2,V_2}$. Let $R = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Let $J_G \subset R$ denote the cover ideal of G. Then the graded minimal free resolution of R/J_G is of the form*

$$0 \longrightarrow R(-n - m_2) \oplus R(-m - n_1) \longrightarrow R(-(n_1 + m_2)) \oplus R(-m) \oplus R(-n) \longrightarrow R.$$

In particular,

$$\operatorname{reg}(J_G) = \max\{n + m_2 - 1, m + n_1 - 1\}$$

Proof. It can easily be seen that the cover ideal J_G is generated by $g_1 = x_1 \cdots x_n$, $g_2 = y_1 \cdots y_m$, and $g_3 = x_1 \cdots x_{n_1} y_{m_1+1} \cdots y_m$. Set $X_1 = x_1 \cdots x_{n_1}$; $X_2 = x_{n_1+1} \cdots x_n$; $Y_1 = y_1 \cdots y_{m_1}$ and $Y_2 = y_{m_1+1} \cdots y_m$. Then we can write $g_1 = X_1X_2$, $g_2 = Y_1Y_2$ and $g_3 = X_1Y_2$.

Consider the minimal graded free resolution of R/J_G over R:

$$\cdots \longrightarrow F \xrightarrow{\partial_2} R(-n) \oplus R(-m) \oplus R(-(n_1 + m_2)) \xrightarrow{\partial_1} R,$$

where $\partial_1(e_1) = g_1$, $\partial_1(e_2) = g_2$, and $\partial_1(e_3) = g_3$.

Let $ae_1 + be_2 + ce_3 \in \ker \partial_1$. Then $aX_1X_2 + bY_1Y_2 + cX_1Y_2 = 0$. Solving the above equation, it can be seen that,

$$\ker \partial_1 = \operatorname{Span}_R\{Y_2e_1 - X_2e_3, X_1e_2 - Y_1e_3\}.$$

Also, it is easily verified that these two generators are R-linearly independent. Therefore, $\ker \partial_1 \cong R^2$. Note that $\deg(Y_2e_1 - X_2e_3) = n + m_2$ and $\deg(X_1e_2 - Y_1e_3) = n_1 + m$. Therefore, we get the minimal free resolution of R/J_G as

$$0 \longrightarrow R(-n-m_2) \oplus R(-m-n_1) \xrightarrow{\partial_2} R(-(n_1+m_2)) \oplus R(-m) \oplus R(-n) \xrightarrow{\partial_1} R,$$

where $\partial_2(a,b) = a(Y_2e_1 - X_2e_3) + b(X_1e_2 - Y_1e_3)$. The regularity assertion follows immediately from the resolution. □

Our aim is to compute the syzygies and the Betti numbers of J_G^s, where J_G is the cover ideal discussed in Theorem 1. For this purpose, we first study the resolution of powers of the ideal (X_1X_2, X_1Y_2, Y_1Y_2) and obtain the resolution and regularity of the cover ideal as a consequence. It is known from the work by the authors of [5] that all the powers of this ideal has a linear resolution. We explicitly compute the syzygies and Betti numbers for the powers of this ideal.

Theorem 2. *Let $R = K[X_1, X_2, Y_1, Y_2]$ and $J = (X_1X_2, X_1Y_2, Y_1Y_2)$ be an ideal of R. Then, for $s \geq 2$, the minimal free resolution of R/J^s is of the form*

$$0 \longrightarrow R^{\binom{s}{2}} \longrightarrow R^{2\binom{s+1}{2}} \longrightarrow R^{\binom{s+2}{2}} \longrightarrow R,$$

and $\operatorname{reg}(J^s) = 2s$.

Proof. Denote the generators of J by $g_1 = X_1X_2$, $g_2 = X_1Y_2$, and $g_3 = Y_1Y_2$. Note that J is the edge ideal of P_4, the path graph on the vertices $\{X_1, X_2, Y_1, Y_2\}$. Since P_4 is chordal, it follows that R/J^s has a linear resolution for all $s \geq 1$ [5]. Now we compute the Betti number of the R/J^s. Write

$$(g_1,g_2,g_3)^s = (g_1^s, g_1^{s-1}(g_2,g_3), g_1^{s-2}(g_2,g_3)^2, \ldots, g_1(g_2,g_3)^{s-1}, (g_2,g_3)^s)$$

where

$$(g_2,g_3)^t = (g_2^t, g_2^{t-1}g_3, g_2^{t-2}g_3^2, \ldots, g_2g_3^{t-1}, g_3^t).$$

For $i \geq j$, set $M_{i,j} = g_1^{s-i}g_2^{i-j}g_3^j = (X_1X_2)^{s-i}Y_2^iY_1^jX_1^{i-j}$. It follows that $\mu(J^s) = \frac{(s+1)(s+2)}{2}$.

Set $\beta_1 = \frac{(s+1)(s+2)}{2}$. Let $\{e_{p,q} \mid 0 \leq p \leq s; 0 \leq q \leq p\}$ denote the standard basis for R^{β_1}. Let $\partial_1 : R^{\beta_1} \longrightarrow R$ be the map $\partial_1(e_{p,q}) = M_{p,q}$. As g_i's are monomials, the kernel is generated by binomials of the form $m_{i,j}M_{i,j} - m_{k,l}M_{k,l}$, where $m_{p,q}$'s are monomials in R. Since the resolution of R/J^s is linear, it is enough to find the linear syzygy relations among the generators of $\ker(\partial_1)$. To find these linear syzygies, we need to find conditions on i,j,k,l such that $\frac{M_{i,j}}{M_{k,l}}$ is equal to $\frac{X_p}{Y_q}$ or $\frac{Y_q}{X_p}$ for some p,q. First of all, note that for such linear syzygies, $|i-k|, |j-l| \leq 1$. If $i = k$ and $j = l+1$, then $\dfrac{M_{i,j}}{M_{i,j+1}} = \dfrac{g_3}{g_2} = \dfrac{Y_1}{X_1}$. We get the same relation if $i = k$ and $j = l-1$. If $i = k+1$ and $j = l$, then $\dfrac{M_{i,j}}{M_{i-1,j}} = \dfrac{g_2}{g_1} = \dfrac{Y_2}{X_2}$. As before, $i = k-1$ yields the same relation. Therefore, the kernel is minimally generated by

$$\left\{Y_1e_{i,j} - X_1e_{i,j+1}, X_2e_{i,j} - Y_2e_{i-1,j} \mid 0 \leq j < i \leq s\right\}.$$

Hence, $\mu(\ker \partial_1) = 2\binom{s+1}{2}$. Write the basis elements of $R^{2\binom{s+1}{2}}$ as

$$\{e_{1,i,p}, e_{2,i,q} \mid 1 \leq i \leq s, 0 \leq p < i \text{ and } 0 \leq q < i\}$$

and define $\partial_2 : R^{2\binom{s+1}{2}} \longrightarrow R^{\beta_1}$ by

$$\begin{aligned} \partial_2(e_{1,i,p}) &= Y_1 e_{i,p} - X_1 e_{i,p+1} \\ \partial_2(e_{2,i,q}) &= X_2 e_{i,q} - Y_2 e_{i-1,q}. \end{aligned}$$

By (Proposition 3.2 in the work by the authors of [13]), $\operatorname{pdim}(R/J^s) = 3$ for all $s \geq 2$. Hence we conclude that the minimal graded free resolution of R/J^s is of the form

$$0 \longrightarrow R^{\beta_3} \longrightarrow R^{2\binom{s+1}{2}} \xrightarrow{\partial_2} R^{\binom{s+2}{2}} \xrightarrow{\partial_1} R.$$

Therefore,

$$\beta_3 - 2\binom{s+1}{2} + \binom{s+2}{2} - 1 = 0,$$

so that $\beta_3 = \binom{s}{2}$. Now we compute the generators of the second syzygy. Again, as the resolution is linear, it is enough to compute linear generators. First, note that

$$B = \{Y_2 e_{1,i,j} - X_2 e_{1,i+1,j} + Y_1 e_{2,i+1,j} - X_1 e_{2,i+1,j+1} \mid 0 \leq j < i < s\} \subseteq \ker \partial_2.$$

It can easily be verified that B is R-linearly independent. As $\mu(\ker \partial_2) = \binom{s}{2}$, B generates $\ker \partial_2$. Let $\{E_{i,j} \mid 0 \leq j < i < s\}$ denote the standard basis for $R^{\binom{s}{2}}$. Define $\partial_3 : R^{\binom{s}{2}} \to R^{s(s+1)}$ by

$$\partial_3(E_{i,j}) = Y_2 e_{1,i,j} - X_2 e_{1,i+1,j} + X_1 e_{2,i+1,j} - Y_1 e_{2,i+1,j+1}.$$

Therefore, we get the minimal free resolution of R/J^s as

$$0 \to R^{\binom{s}{2}} \xrightarrow{\partial_3} R^{2\binom{s+1}{2}} \xrightarrow{\partial_2} R^{\binom{s+2}{2}} \xrightarrow{\partial_1} R.$$

As the resolution is linear, $\operatorname{reg} J^s = 2s$. □

As an immediate application of the previous theorem, we obtain resolution and regularity of powers of the cover ideals of graphs discussed in Theorem 1.

Corollary 1. *Let $U = U_1 \sqcup U_2$ and $V = V_1 \sqcup V_2$ be a collection of vertices with $|U| = n$, $|U_i| = n_i$, $|V| = m$, $|V_i| = m_i$ and $1 \leq n_i, m_i$ for $i = 1, 2$. Let G be the bipartite graph $K_{U_1,V} \cup K_{U_2,V_2}$. Let $R = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Let $J_G \subset R$ denote the cover ideal of G. Then the minimal free resolution of R/J_G^s is of the form*

$$0 \longrightarrow R^{\binom{s}{2}} \longrightarrow R^{2\binom{s+1}{2}} \longrightarrow R^{\binom{s+2}{2}} \longrightarrow R.$$

Moreover,

$$\operatorname{reg} J_G^s = \max \begin{cases} (s-j)n_1 + (s-i)n_2 + jm_1 + im_2 & \text{for } 0 \leq j \leq i \leq s \\ (s-j)n_1 + (s-i)n_2 + (j+1)m_1 + im_2 - 1 & \text{for } 0 \leq j < i \leq s \\ (s-j)n_1 + (s-i+1)n_2 + jm_1 + im_2 - 1 & \text{for } 0 \leq j < i \leq s \\ (s-j)n_1 + (s-i)n_2 + (j+1)m_1 + (i+1)m_2 - 2 & \text{for } 0 \leq j < i < s \end{cases}.$$

Proof. Let $R = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Then $J_G = (x_1 \cdots x_n, y_1 \cdots y_m, x_1 \cdots x_{n_1} y_{m_1+1} \cdots y_m)$. Set $X_1 = x_1 \cdots x_{n_1}, X_2 = x_{n_1+1} \cdots x_n, Y_1 = y_1 \cdots y_{m_1}$ and $Y_2 = y_{m_1+1} \cdots y_m$. Then $J_G = (X_1X_2, Y_1Y_2, X_1Y_2)$. Therefore, it follows from Theorem 1 that J_G^s has the given minimal free resolution.

To compute the regularity, we need to obtain the degrees of the syzygies. Following the notation in the proof of Theorem 1, we can see that

$$\begin{aligned}\deg e_{i,j} &= (s-j)n_1+(s-i)n_2+jm_1+im_2,\\ \deg e_{1,i,j} &= (s-j)n_1+(s-i)n_2+(j+1)m_1+im_2,\\ \deg e_{2,i,j} &= (s-j)n_1+(s-i+1)n_2+jm_1+im_2,\\ \deg E_{i,j} &= (s-j)n_1+(s-i)n_2+(j+1)m_1+(i+1)m_2.\end{aligned}$$

Therefore, the assertion on the regularity follows. □

3.1. Discussion

It has been proved by Hang and Trung [11] that if G is a bipartite graph on n vertices and J_G is the cover ideal of G, then there exists a non-negative integer e, such that for $s \geq n+2$, $\operatorname{reg}(J_G^s) = \deg(J_G)s + e$, where $\deg(J_G)$ denote the maximal degree of minimal monomial generators of J_G. It follows from Remark 1 that if $G = K_{n,m}$ with $n \geq m$, then $e = m-1$ and the index of stability is 1. If the graph is not a complete bipartite graph, then e does not uniformly represent a combinatorial invariant associated to the graph as can be seen in the computations below. We compute the polynomial $\operatorname{reg}(J_G^s)$ for some classes of bipartite graphs that are considered in Corollary 1. We see that e depends on the relation between the integers n_1, n_2, m_1 and m_2. Just to illustrate the computation of the polynomial from Corollary 1, we compute $\operatorname{reg}(J_G^s)$ in some cases below:

1. If $m_1 = m_2 = 1$, then it can be seen that for $s \geq 2$,

$$\operatorname{reg}(J_G^s) = \max\{(s-j)n_1+(s-i+1)n_2+jm_1+im_2-1 \ : \ 0 \leq j < i \leq s\}.$$

   Since $(s-j)n_1+(s-i+1)n_2+jm_1+im_2-1 = s(n_1+n_2)+j(1-n_1)+i(1-n_2)+n_2-1$ and $n_1 > 1$, $n_2 > 1$, this expression attains maximum when i and j attain minimum, i.e., if $j = 0$ and $i = 1$. Therefore, $\operatorname{reg}(J_G^s) = ns$. Thus, in this case, $e = 0$. It can also be noted that, since $n \geq 2$, it follows from Theorem 1 that $\operatorname{reg}(J_G) = n$. Therefore, in this case, the stability index is also equal to 1.
2. If $n_1 = n_2 = m_1 = m_2 = \ell > 1$, then for $s \geq 2$,

$$\operatorname{reg}(J_G^s) = \max\{(s-j)n_1+(s-i)n_2+(j+1)m_1+(i+1)m_2-2 \ : \ 0 \leq j < i \leq s\}.$$

   Therefore, $\operatorname{reg}(J_G^s) = 2\ell s + (2\ell-2)$, and therefore $e = 2\ell-2$. Note that in this case, the stability index is 2.
3. $n_1 \geq m_2 \geq n_2 = m_1$: Note that, in this case, $\deg(J_G) = n_1+m_2$. We have

$$\operatorname{reg}(J_G^s) = \max\{(s-j)n_1+(s-i)n_2+(j+1)m_1+(i+1)m_2-2 \ : \ 0 \leq j < i < s\}.$$

   Since $(s-j)n_1+(s-i)n_2+(j+1)m_1+(i+1)m_2-2 = s(n_1+n_2)+j(m_1-n_1)+i(m_2-n_2)+m_1+m_2-2$. Since $n_1 \geq m_1$ and $m_2 \geq n_2$, the above expression attains the maximum when i attains the maximum and j attains the minimum, i.e., if $i = s-1$ and $j = 0$. Therefore, $\operatorname{reg}(J_G^s) = (n_1+m_2)s + (n_2+m_1-2)$ and hence $e = n_2+m_1-2$.
4. $n_1 \geq n_2 = m_1 \geq m_2$: In this case, $\deg(J_G) = n$ and

$$\operatorname{reg}(J_G^s) = \max\{(s-j)n_1+(s-i)n_2+(j+1)m_1+(i+1)m_2-2 \ : \ 0 \leq j < i < s\}.$$

   As in the previous case, one can conclude that the maximum is attained when $i = 1$ and $j = 0$. Therefore, $\operatorname{reg}(J_G^s) = ns + (2m_2-2)$. Thus $e = 2m_2-2$.
5. $n_2 = m_1 \geq n_1 \geq m_2$: As done earlier, one can conclude that $\operatorname{reg}(J_G)^s = ns + (2m_2-2)$, and therefore $e = 2m_2-2$.

Remark 2. *If $G = K_{U_1,V} \cup K_{U_2,V_2} \cup K_{U_3,V_3}$, for some set of vertices U_i, V_i, then one can still describe the complete resolution and the regularity of J_G^s using a similar approach. However, the resulting syzygies are not so easy to describe though the generating sets are similar. Therefore, we restrict ourselves to the above discussion.*

4. Complete Multipartite Graphs

In this section our goal is to understand the resolution and regularity of the powers of cover ideals of complete m-partite graphs. Let G be a complete m-partite graph and let J_G be the cover ideal of G. Let $R = K[x_1, \ldots, x_m]$. It is known that the cover ideal J_G of complete graph $G = K_m$ is generated by all squarefree monomials $x_1x_2 \cdots \hat{x_i} \cdots x_m$ of degree $m-1$. Moreover one can also identify this cover ideal with the squarefree Veronese ideal $I = I_{m,m-1}$, and thus it is a polymatroidal ideal [14]. Therefore, by the results of Conca-Herzog [6] and Herzog-Hibi [15], we have

Remark 3. *The cover ideal J_G of the complete graph $G = K_m$ has linear quotients and hence has linear resolution. Moreover J_G^s has linear resolution for all $s \geq 1$.*

If I is an ideal of R all of whose powers have linear resolution, then depth R/I^k is a non-increasing function of k and depth R/I^k is constant for all $k \gg 0$, ([14] Proposition 2.1). Further, we have,

Remark 4. *(Corollary 3.4 in the work of [14]) Let $R = K[x_1, \cdots, x_m]$ and J_G be the cover ideal of $G = K_m$. Then*

$$\operatorname{depth} R/J_G^s = \max\{0, m-s-1\}.$$

In particular, $\operatorname{depth} R/J_G^s = 0$ *for all $s \geq m-1$.*

4.1. Complete Tripartite:

We first describe the graded minimal free resolution of $J_{K_3}^s$ for all $s \geq 1$. We also obtain the Hilbert series of the powers.

Theorem 3. *Let $R = K[x_1, x_2, x_3]$ and $I = (x_1x_2, x_1x_3, x_2x_3)$. Then, the graded minimal free resolution of R/I is of the form*

$$0 \to R(-3)^2 \to R(-2)^3 \to R.$$

For $s \geq 2$, the graded minimal free resolution of R/I^s is of the form:

$$0 \to R(-2s-2)^{\binom{s}{2}} \to R(-2s-1)^{2\binom{s+1}{2}} \to R(-2s)^{\binom{s+2}{2}} \to R$$

so that $\operatorname{reg}_R(I^s) = 2s$. *Moreover, the Hilbert series of R/I^s is given by*

$$H(R/I^s, t) = \frac{1 + 2t + 3t^2 + \cdots + 2st^{2s-1} - \left(\binom{s+2}{2} - 2s - 1\right) t^{2s}}{(1-t)}.$$

Proof. It is clear that the resolution of I is as given in the assertion of the theorem. Since I is the cover ideal of K_3, I^s has linear minimal free resolution for all $s \geq 1$. Note also that by ([16] Lemma 3.1), depth $R/I^s = 0$ for all $s \geq 2$ so that pdim $R/I^s = 3$ for all $s \geq 2$. Therefore, the minimal free resolution of R/I^s is of the form

$$0 \to R(-2s-2)^{\beta_3} \xrightarrow{\partial_3} R(-2s-1)^{\beta_2} \xrightarrow{\partial_2} R(-2s)^{\beta_1} \xrightarrow{\partial_1} R.$$

Now we describe the syzygies and Betti numbers of R/I^s for $s \geq 2$. Let $g_1 = x_1x_2, g_2 = x_1x_3$ and $g_3 = x_2x_3$. The generators of I^s are of the form $g_1^{\ell_1}g_2^{\ell_2}g_3^{\ell_3}$, where $0 \leq \ell_1, \ell_2, \ell_3 \leq s$ and $\ell_1 + \ell_2 + \ell_3 = s$. Set $f_{\ell_1,\ell_2,\ell_3} = g_1^{\ell_1}g_2^{\ell_2}g_3^{\ell_3} = x_1^{s-\ell_3}x_2^{s-\ell_2}x_3^{s-\ell_1}$. It is easy to see that $\mu(I^s) = \binom{s+2}{2}$, as the cardinality

of a minimal generating set of I^s is same as the total number of non-negative integral solution of $\ell_1+\ell_2+\ell_3=s$, which is $\binom{s+2}{2}$.

Let $\{e_{\ell_1,\ell_2,\ell_3} : 0\le \ell_1,\ell_2,\ell_3\le s; \ell_1+\ell_2+\ell_3=s\}$ denote the standard basis for $R^{\binom{s+2}{2}}$ and consider the map $\partial_1 : R^{\binom{s+2}{2}} \to R$ defined by $\partial_1(e_{\ell_1,\ell_2,\ell_3}) = f_{\ell_1,\ell_2,\ell_3}$. As f_{ℓ_1,ℓ_2,ℓ_3}'s are monomials, the kernel is generated by binomials of the form $m_{\ell_1,\ell_2,\ell_3}f_{\ell_1,\ell_2,\ell_3} - m_{t_1,t_2,t_3}f_{t_1,t_2,t_3}$, where $m_{i,j,k}$'s are monomials in R. Also, as the minimal free resolution is linear, the kernel is generated in degree 1. Note that $\frac{f_{\ell_1,\ell_2,\ell_3}}{f_{t_1,t_2,t_3}} = x_1^{t_3-\ell_3}x_2^{t_2-\ell_2}x_3^{t_1-\ell_1}$. Therefore, for $\frac{f_{\ell_1,\ell_2,\ell_3}}{f_{t_1,t_2,t_3}}$ to be a linear fraction, $|t_i-\ell_i|\le 1$ for $i=1,2,3$.

Let $t_3=\ell_3$, $t_2=\ell_2+1$ and $t_1=\ell_1-1$. The corresponding linear syzygy relation is

$$x_3\cdot f_{\ell_1,\ell_2,\ell_3} - x_2\cdot f_{\ell_1-1,\ell_2+1,\ell_3} = 0, \tag{2}$$

where $1\le\ell_1\le s$, and $0\le\ell_2,\ell_3\le s-1$. Note that the number of such relations is equal to the number of integral solution to $(\ell_1-1)+\ell_2+\ell_3=s$, i.e., $\binom{s+1}{2}$. Similarly, if $t_3=\ell_3+1$, $t_2=\ell_2-1$ and $t_1=\ell_1$, then we get the corresponding linear syzygy relation as

$$x_2\cdot f_{\ell_1,\ell_2,\ell_3} - x_1\cdot f_{\ell_1,\ell_2-1,\ell_3+1} = 0, \tag{3}$$

where $0\le\ell_1,\ell_3\le s-1$ and $1\le\ell_2\le s$. Note that the linear syzygy relation obtained by fixing ℓ_2 and taking $|\ell_i-t_i|=1$ for $i=1,2$

$$x_3\cdot f_{\ell_1,\ell_2,\ell_3} - x_1\cdot f_{\ell_1-1,\ell_2,\ell_3+1} = 0$$

can be obtained from Equations (2) and (3) by setting the ℓ_i's appropriately. Therefore,

$$\ker\partial_1 = \langle x_3\cdot e_{\ell_1,\ell_2,\ell_3} - x_2\cdot e_{\ell_1-1,\ell_2+1,\ell_3},\ x_2\cdot e_{\ell_1-1,\ell_2+1,\ell_3} - x_1\cdot e_{\ell_1-1,\ell_2,\ell_3+1} : 1\le\ell_1\le s,\ 0\le\ell_2,\ell_3\le s-1\rangle.$$

As there are $\binom{s+1}{2}$ minimal generators of type (2) and (3), $\mu(\ker\partial_1)=\beta_2=2\binom{s+1}{2}$. Write the standard basis of $R^{2\binom{s+1}{2}}$ as

$$B_2 = \left\{ \begin{array}{l} e_{1,(\ell_2+1,\ell_3),\ell_1-1} \\ e_{2,(\ell_1,\ell_2),\ell_3} \end{array} : 1\le\ell_1\le s,\ 0\le\ell_2,\ell_3\le s-1 \right\}$$

and define $\partial_2 : R^{2\binom{s+1}{2}} \to R^{\binom{s+2}{2}}$ by

$$\begin{aligned} \partial_2(e_{1,(\ell_2+1,\ell_3),\ell_1-1}) &= x_2\cdot e_{\ell_1-1,\ell_2+1,\ell_3} - x_1\cdot e_{\ell_1-1,\ell_2,\ell_3+1}, \\ \partial_2(e_{2,(\ell_1,\ell_2),\ell_3}) &= x_3\cdot e_{\ell_1,\ell_2,\ell_3} - x_2\cdot e_{\ell_1-1,\ell_2+1,\ell_3}. \end{aligned}$$

From the equation, $\beta_3 - 2\binom{(s+1)}{2} + \binom{s+2}{2} - 1 = 0$, it follows that $\beta_3=\binom{s}{2}$. Now, we describe $\ker\partial_2$. For $2\le\ell_1\le s$ and $0\le\ell_2,\ell_3\le s-2$, set

$$E_{\ell_1,\ell_2,\ell_3} = -x_3\cdot e_{2,(\ell_1-1,\ell_2),\ell_3+1} + x_2\cdot e_{2,(\ell_1-1,\ell_2+1),\ell_3} + x_2\cdot e_{1,(\ell_2+2,\ell_3),\ell_1-2} - x_3\cdot e_{1,(\ell_2+1,\ell_3),\ell_1-1}.$$

Note that

$$B_3 = \{E_{\ell_1,\ell_2,\ell_3} : 2\le\ell_1\le s,\ 0\le\ell_2,\ell_3\le s-2\} \subset \ker\partial_2.$$

Let $N=\ker\partial_2$. We claim that $\bar{B}_3=\{\bar{E}_{\ell_1,\ell_2,\ell_3}\}$ is a basis for $N/\mathfrak{m}N$. It is enough to prove that $\bar{B}_3$ is $R/\mathfrak{m}$-linearly independent. Suppose $\sum_{\ell_1,\ell_2,\ell_3}\alpha_{\ell_1,\ell_2,\ell_3}\bar{E}_{\ell_1,\ell_2,\ell_3}=\bar{0}$. Since $B_3\subset N_1$ and $\mathfrak{m}N\subset\oplus_{r\ge2}N_r$, the above equation implies that $\sum\alpha_{\ell_1,\ell_2,\ell_3}E_{\ell_1,\ell_2,\ell_3}=0$. In the above equation, the coefficient of $e_{2,(i,j),k} = -x_3\alpha_{i+1,j,i-1} + x_2\alpha_{i+1,j-1,k}$ and coefficient of $e_{1,(i,j),k} = x_2\alpha_{k+2,j-2,i} - x_3\alpha_{k+1,j-1,i}$. As the set B_2 is linearly independent in $R^{2\binom{s+1}{2}}$, all of these coefficients have to be zero, i.e., $\alpha_{i,j,k}=0$ for all i,j,k.

This proves our claim and hence $N = \langle B_3 \rangle$. Let $\{H_{\ell_1,\ell_2,\ell_3} : 2 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3 \leq s-2\}$ denote the standard basis of $R^{\binom{s}{2}}$. Define $\partial_3 : R^{\binom{s}{2}} \to R^{2\binom{s+1}{2}}$ by $\partial_3(H_{\ell_1,\ell_2,\ell_3}) = E_{\ell_1,\ell_2,\ell_3}$. As $\operatorname{pdim}(R/I^s) = 3$, this map is injective and the resolution is complete.

It can also be seen that $\deg e_{\ell_1,\ell_2,\ell_3} = 2s$, $\deg e_{1,(\ell_2+1,\ell_3),\ell_1-1} = \deg e_{2,(\ell_1,\ell_2),\ell_3} = 2s+1$ and $\deg H_{\ell_1,\ell_2,\ell_3} = 2s+2$. Therefore, we get the minimal graded free resolution of R/I^s as

$$0 \to R(-2s-2)^{\binom{s}{2}} \xrightarrow{\partial_3} R(-2s-1)^{2\binom{s+1}{2}} \xrightarrow{\partial_2} R(-2s)^{\binom{s+2}{2}} \xrightarrow{\partial_1} R.$$

Therefore, the Hilbert series of R/I^s is given by

$$\begin{aligned} H(R/I^s,t) &= \frac{1-\binom{s+2}{2}t^{2s}+2\binom{s+1}{2}t^{2s+1}-\binom{s}{2}t^{2s+2}}{(1-t)^3} \\ &= (1-t)^{-2}\frac{(1-\binom{s+2}{2}t^{2s}+2\binom{s+1}{2}t^{2s+1}-\binom{s}{2}t^{2s+2})}{(1-t)}. \end{aligned}$$

By expanding $(1-t)^{-2}$ in the power series form and multiplying with the numerator, we get the required expression. □

We now proceed to compute the minimal graded free resolution of powers of complete tripartite graphs.

Notation 2. *Let G denote a complete tripartite graph with $V(G) = V_1 \sqcup V_2 \sqcup V_3$ and $E(G) = \{\{a,b\} : b \in V_i, b \in V_j, i \neq j\}$. Set $V_1 = \{x_1,\ldots,x_\ell\}$, $V_2 = \{y_1,\ldots,y_m\}$ and $V_3 = \{z_1,\ldots,z_n\}$. Let J_G denote the vertex cover ideal of G. Let $X = \prod_{i=1}^{\ell} x_i$, $Y = \prod_{j=1}^{m} y_i$ and $Z = \prod_{k=1}^{n} z_i$. It can be seen that $J_G = (XY, XZ, YZ)$.*

Theorem 4. *Let $R = K[x_1,\ldots x_{m_1}, y_1,\ldots,y_{m_2}, z_1,\ldots,z_{m_3}]$. Let G be a complete tripartite graph as in Notation 2. Let $J_G \subset R$ denote the cover ideal of G. Then for all $s \geq 2$, the minimal free resolution of R/J_G^s is of the form*

$$0 \to R^{\binom{s}{2}} \xrightarrow{\partial_3} R^{2\binom{s+1}{2}} \xrightarrow{\partial_2} R^{\binom{s+2}{2}} \xrightarrow{\partial_1} R.$$

Set $\alpha = (s-\ell_3)m_1 + (s-\ell_2)m_2 + (s-\ell_1)m_3$. Then for all $s \geq 2$,

$$\operatorname{reg}(J_G^s) = \max \begin{cases} \alpha, & \text{for } 0 \leq \ell_1,\ell_2,\ell_3 \leq s, \\ \alpha + m_3 - 1, & \text{for } 1 \leq \ell_1 \leq s,\ 0 \leq \ell_2,\ell_3 \leq s-1, \\ \alpha + 2m_3 - 2, & \text{for } 2 \leq \ell_1 \leq s,\ 0 \leq \ell_2,\ell_3 \leq s-2. \end{cases}$$

Proof. Taking $X_1 = x_1$, $X_2 = x_2$ and $X_3 = x_3$ in Theorem 3, it follows that the minimal free resolution of S/J_G^s is of the given form:

$$0 \to R^{\binom{s}{2}} \xrightarrow{\partial_3} R^{2\binom{s+1}{2}} \xrightarrow{\partial_2} R^{\binom{s+2}{2}} \xrightarrow{\partial_1} R.$$

We now compute the degrees of the generators and hence obtain the regularity. Let $\deg X_1 = m_1$, $\deg X_2 = m_2$, and $\deg X_3 = m_3$. Then, it follows that

$$\deg e_{\ell_1,\ell_2,\ell_3} = \deg\left(X_1^{s-\ell_3}X_2^{s-\ell_3}X_3^{s-\ell_1}\right) = (s-\ell_3)m_1 + (s-\ell_2)m_2 + (s-\ell_1)m_3.$$

We observe that

$$\begin{gathered} \deg e_{1,(\ell_2+1,\ell_3),\ell_1-1} = \deg e_{2,(\ell_1,\ell_2),\ell_3} = \deg e_{\ell_1,\ell_2,\ell_3} + \deg X_3, \\ \deg H_{\ell_1,\ell_2,\ell_3} = \deg e_{\ell_1,\ell_2,\ell_3} + 2\deg X_3. \end{gathered}$$

Therefore,

$$\operatorname{reg}(J_G^s) = \max \begin{cases} \deg e_{\ell_1,\ell_2,\ell_3}, & \text{for } 0 \le \ell_1, \ell_2, \ell_3 \le s, \\ \deg e_{\ell_1,\ell_2,\ell_3} + \deg(X_3) - 1, & \text{for } 1 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3 \le s-1, \\ \deg e_{\ell_1,\ell_2,\ell_3} + 2\deg(X_3) - 2, & \text{for } 2 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3 \le s-2, \end{cases}$$

where $\ell_1 + \ell_2 + \ell_3 = s$. Let $\alpha = \deg e_{\ell_1,\ell_2,\ell_3}$. Then the regularity assertion follows. □

Observe that the above expression for $\operatorname{reg}(J_G^s)$ is not really in the form $ds + e$. Given a graph, one can compute d and e by studying the interplay between the cardinality of the partitions. For example, suppose the graph is unmixed, i.e., all the partitions are of same cardinality. Then, $\operatorname{reg}(J_G^s) = 2\ell s + (2\ell - 2)$ for all $s \ge 2$, where $\ell = m_1 = m_2 = m_3$. Note also that the stabilization index in this case is 2. As in Section 3.1, one can derive various expressions for $\operatorname{reg}(J_G^s)$ for different cases as well. Consider the arithmetic progression $m_1 = m + 2r$, $m_2 = m$, and $m_3 = m + r$:

Corollary 2. *Let m, r be any two positive integers. Let $m_1 = m + 2r$, $m_2 = m$, and $m_3 = m + r$ in Theorem 4. Then for all $s \ge 2$, we have*

$$\operatorname{reg}(J_G^s) = s(2m + 3r) + 2m - 2.$$

Proof. By Theorem 4, $\alpha = (s - \ell_3)m_1 + (s - \ell_2)m_2 + (s - \ell_1)m_3$. Therefore, we get

$$\alpha = s(3m + 3r) - \ell_3(m + 2r) - \ell_2(m) - \ell_1(m + r).$$

Using Theorem 4, we have for all $s \ge 2$,

$$\operatorname{reg}(J_G^s) = \max \begin{cases} \alpha, & \text{for } 0 \le \ell_1, \ell_2, \ell_3 \le s, \\ \alpha + (m + r) - 1, & \text{for } 1 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3 \le s-1, \\ \alpha + 2(m + r) - 2, & \text{for } 2 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3 \le s-2, \end{cases}$$

where $\ell_1 + \ell_2 + \ell_3 = s$. As regularity $\operatorname{reg}(J_G^s)$ is maximum of all the numbers, we need to maximize the value of α. For this to happen, negative terms in α should be minimum. The coefficient of ℓ_3 is largest among the negative terms in α, so ℓ_3 should be assigned the least value. After ℓ_3, assign the minimum value to ℓ_1, and finally take $\ell_2 = s - \ell_1 - \ell_3$. For example, to get the maximum of α when $2 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3 \le s - 2$, put $\ell_3 = 0, \ell_1 = 2$, and $\ell_2 = s - 2$. We get for all $s \ge 2$

$$\operatorname{reg}(J_G^s) = \max \begin{cases} s(2m + 3r), & \text{for } 0 \le \ell_1, \ell_2, \ell_3, \ell_4 \le s, \\ s(2m + 3r) + m - 1, & \text{for } 1 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-1, \\ s(2m + 3r) + 2m - 2, & \text{for } 2 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-2. \end{cases}$$

For all $m \ge 1$, and for all $s \ge 2$, we get

$$\operatorname{reg}(J_G^s) = s(2m + 3r) + 2m - 2.$$

□

4.2. Complete 4-Partite Graphs

We now describe the resolution and regularity of powers of cover ideals of 4-partite graphs. For this purpose, we first study the resolution of powers of cover ideal of the complete graph K_4.

Theorem 5. *Let $R = K[x_1, x_2, x_3, x_4]$ and $I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4)$. Then, for $s \ge 3$, the minimal graded free resolution of R/I^s is of the form*

$$0 \to R(-3s-3)^{\beta_4} \longrightarrow R(-3s-2)^{\beta_3} \longrightarrow R(-3s-1)^{\beta_2} \longrightarrow R(-3s)^{\beta_1} \longrightarrow R,$$

where

$$\beta_1 = \binom{s+3}{3},\ \beta_2 = 3\binom{s+2}{3},\ \beta_3 = 3\binom{s+1}{3},\ \textit{and}\ \beta_4 = \binom{s}{3}.$$

In particular, $\operatorname{reg}_R(I^s) = 3s$. *Moreover the Hilbert series of* R/I^s *is given by*

$$\begin{aligned} H(R/I^s,t) &= \frac{1+2t+3t^2+4t^3+\cdots+3st^{3s-1}-\left(\binom{s+3}{3}-3s-1\right)t^{3s}+\binom{s}{3}t^{3s+1}}{(1-t)^2} \\ &= \frac{\sum_{i=0}^{3s-1}(i+1)t^i-\left(\binom{s+3}{3}-3s-1\right)t^{3s}+\binom{s}{3}t^{3s+1}}{(1-t)^2}. \end{aligned}$$

Proof. By Remark 3, I^s has linear resolution for all $s \geq 1$. Moreover, by Remark 4, $\operatorname{depth} R/I^s = 0$ for all $s \geq 3$, and therefore $\operatorname{pdim} R/I^s = 4$ for all $s \geq 3$. Therefore, the minimal graded free resolution of R/I^s, for $s \geq 3$, is of the form

$$0 \to R(-3s-3)^{\beta_4} \xrightarrow{\partial_4} R(-3s-2)^{\beta_3} \xrightarrow{\partial_3} R(-3s-1)^{\beta_2} \xrightarrow{\partial_2} R(-3s)^{\beta_1} \xrightarrow{\partial_1} R.$$

Let $g_1 = x_1x_2x_3, g_2 = x_1x_2x_4, g_3 = x_1x_3x_4$ and $g_4 = x_2x_3x_4$. The minimal generators of I^s are of the form $g_1^{\ell_1}g_2^{\ell_2}g_3^{\ell_3}g_4^{\ell_4}$ where $0 \leq \ell_i \leq s$ for every i and $\ell_1+\ell_2+\ell_3+\ell_4 = s$. The number of non-negative integral solution to the linear equation $\ell_1+\ell_2+\ell_3+\ell_4 = s$ is $\binom{s+3}{3}$. Therefore, we have $\mu(J^s) = \beta_1 = \binom{s+3}{3}$. Note that $g_1^{\ell_1}g_2^{\ell_2}g_3^{\ell_3}g_4^{\ell_4} = x_1^{s-\ell_4}x_2^{s-\ell_3}x_3^{s-\ell_2}x_4^{s-\ell_1}$ with $0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s$. Set $f_{\ell_1,\ell_2,\ell_3,\ell_4} = x_1^{s-\ell_4}x_2^{s-\ell_3}x_3^{s-\ell_2}x_4^{s-\ell_1}$. Let

$$\{e_{\ell_1,\ell_2,\ell_3,\ell_4} : 0 \leq \ell_1,\ell_2,\ell_3,\ell_4 \leq s \text{ and } \ell_1+\ell_2+\ell_3+\ell_4 = s\}$$

denote the standard basis of $R^{\binom{s+3}{3}}$ and consider the map $\partial_1 : R^{\binom{s+3}{3}} \to R$ defined by $\partial_1(e_{\ell_1,\ell_2,\ell_3,\ell_4}) = f_{\ell_1,\ell_2,\ell_3,\ell_4}$. Now we find the minimal generators for $\ker\partial_1$.

Let $f_{\ell_1,\ell_2,\ell_3,\ell_4}, f_{t_1,t_2,t_3,t_4}$ be any two minimal monomial generators of I^s. It is known that the kernel of ∂_1 is generated by binomials of the form $m_{\ell_1,\ell_2,\ell_3,\ell_4}f_{\ell_1,\ell_2,\ell_3,\ell_4} - m_{t_1,t_2,t_3,t_4}f_{t_1,t_2,t_3,t_4}$, where $m_{i,j,k,l}$'s are monomials in R. As the syzygy is generated by linear binomials, we need to find conditions on $\ell_1, \ell_2, \ell_3, \ell_4$ such that $\frac{f_{\ell_1,\ell_2,\ell_3,\ell_4}}{f_{t_1,t_2,t_3,t_4}}$ is equal to $\frac{x_i}{x_j}$ for some i, j. Observe that

$$\frac{f_{\ell_1,\ell_2,\ell_3,\ell_4}}{f_{t_1,t_2,t_3,t_4}} = x_1^{t_4-\ell_4}x_2^{t_3-\ell_3}x_3^{t_2-\ell_2}x_4^{t_1-\ell_1}.$$

Suppose $t_4 = \ell_4+1$, $t_3 = \ell_3-1$ and $t_j = \ell_j$ for $j = 1,2$. Then we get the linear syzygy relation

$$x_2f_{\ell_1,\ell_2,\ell_3,\ell_4} - x_1f_{\ell_1,\ell_2,\ell_3-1,\ell_4+1} = 0. \tag{4}$$

Fixing t_3 and t_4 with $1 \leq t_3+t_4 \leq s$, there are as many linear syzygies are there as the number of solutions of $t_1+t_2 = s-(t_3+t_4)$. Therefore, for the pair (t_3,t_4), there are $\binom{s+1}{2}+\binom{s}{2}+\cdots+\binom{2}{2} = \binom{s+2}{3}$ number of solutions. Similarly, for each pair $(t_4,t_2),(t_4,t_1),(t_3,t_2),(t_3,t_1)$ and (t_2,t_1), we get $\binom{s+2}{3}$ linear syzygies. Note that the syzygies $x_4f_{t_1+1,t_2-1,t_3,t_4} - x_3f_{t_1,t_2,t_3,t_4}$ and $x_3f_{t_1,t_2,t_3,t_4} - x_2f_{t_1,t_2-1,t_3+1,t_4}$ give rise to another linear syzygy $x_4f_{t_1+1,t_2-1,t_3,t_4} - x_2f_{t_1,t_2-1,t_3+1,t_4}$. The same linear syzygy can also be obtained from a combination of linear syzygies that arise out of the pairs (t_1,t_4) and (t_3,t_4). Therefore, to get a minimal generating set, we only need to consider the linear syzygies corresponding to the pairs (t_1,t_2), (t_2,t_3), and (t_3,t_4). For each such pair, we have $\binom{s+2}{3}$ number of linear syzygies. Therefore, $\beta_2 = 3\binom{s+2}{3}$.

Write the basis elements of $R^{3\binom{s+2}{3}}$ as

$$B_2 = \left\{ \begin{array}{l} e_{(1,\ell_1-1,\ell_2),\ell_3+1,\ell_4} \\ e_{(2,\ell_1-1,\ell_4),\ell_2+1,\ell_3} \\ e_{(3,\ell_3,\ell_4),\ell_1,\ell_2} \end{array} : 1 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-1 \right\}$$

and define $\partial_2 : R^{3\binom{s+1}{2}} \longrightarrow R^{\beta_1}$ by

$$\begin{aligned} \partial_2(e_{(1,\ell_1-1,\ell_2),\ell_3+1,\ell_4}) &= x_2 e_{\ell_1-1,\ell_2,\ell_3+1,\ell_4} - x_1 e_{\ell_1-1,\ell_2,\ell_3,\ell_4+1}; \\ \partial_2(e_{(2,\ell_1-1,\ell_4),\ell_2+1,\ell_3}) &= x_3 e_{\ell_1-1,\ell_2+1,\ell_3,\ell_4} - x_2 e_{\ell_1-1,\ell_2,\ell_3+1,\ell_4}; \\ \partial_2(e_{(3,\ell_3,\ell_4),\ell_1,\ell_2}) &= x_4 e_{\ell_1,\ell_2,\ell_3,\ell_4} - x_3 e_{\ell_1-1,\ell_2+1,\ell_3,\ell_4}. \end{aligned}$$

Now, we decipher the Betti numbers β_3 and β_4 to complete the resolution. The Hilbert series of R/I^s is

$$H(R/I^s, t) = \frac{\left(1 - \beta_1 t^{3s} + \beta_2 t^{3s+1} - \beta_3 t^{3s+2} + \beta_4 t^{3s+3}\right)}{(1-t)^4} = \frac{p(t)}{(1-t)^4}.$$

As $\dim R/I^s = 2$, the polynomial $p(t)$ has a factor $(1-t)^2$. Note that $(1-t)^2$ is a monic polynomial of degree 2, therefore we can write $p(t) = (1-t)^2 \cdot q(t)$, where

$$q(t) = \beta_4 t^{3s+1} + (2\beta_4 - \beta_3) t^{3s} + a_{3s-1} t^{3s-1} + \cdots + a_1 t + 1. \tag{5}$$

On the other hand, we also have

$$\begin{aligned} \frac{p(t)}{(1-t)^2} &= \left(\sum_{n \geq 0} (n+1) t^n \right) \left(1 - \beta_1 t^{3s} + \beta_2 t^{3s+1} - \beta_3 t^{3s+2} + \beta_4 t^{3s+3}\right) \qquad (6) \\ &= \sum_{n=0}^{3s-1} (n+1) t^n + (3s + 1 - \beta_1) t^{3s} + (3s + 2 - 2\beta_1 + \beta_2) t^{3s+1} + \sum_{j \geq 3s+2} a_j t^j \end{aligned}$$

For the expressions in Equations (5) and (6) to be equal, their respective coefficients should be equal. In particular, we should have that

$$\beta_4 = 3s + 2 - 2\beta_1 + \beta_2 \text{ and } 2\beta_4 - \beta_3 = 3s + 1 - \beta_1.$$

On substituting $\beta_1 = \binom{s+3}{3}$ and $\beta_2 = 3\binom{s+2}{3}$, we get $\beta_4 = \binom{s}{3}$ from the first equation and substituting the value in the second equation, we get $\beta_3 = 3\binom{s+1}{3}$. Now we verify that $a_j = 0$ for all $j \geq 3s+2$. Note that for $r \geq 2$, the coefficient of t^{3s+r} in Equation (6) is

$$a_{3s+r} = (3s + r + 1) + (r-2)\beta_4 - (r-1)\beta_3 + r\beta_2 - (r+1)\beta_1.$$

As $1 - \beta_1 + \beta_2 - \beta_3 + \beta_4 = 0$, this equation is reduced to $a_{3s+r} = (3s+3) - \beta_3 + 2\beta_2 - 3\beta_1$. Applying the binomial identify $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$ repeatedly, we get $a_{3s+r} = 0$ for all $r \geq 2$. Hence the Hilbert series of R/I^s is

$$H(R/I^s, t) = \frac{1 + 2t + 3t^2 + 4t^3 + \cdots + 3st^{3s-1} - \left(\binom{s+3}{3} - 3s - 1\right) t^{3s} + \binom{s}{3} t^{3s+1}}{(1-t)^2}.$$

We now complete the description of the resolution. Write the basis elements of $R^{3\binom{s+1}{3}}$ as

$$B_3 = \left\{ \begin{array}{l} E_{1,\ell_1,\ell_2,\ell_3,\ell_4}; \\ E_{2,\ell_1,\ell_2,\ell_3,\ell_4}, \\ E_{3,\ell_1,\ell_2,\ell_3,\ell_4}, \end{array} : 2 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-2 \right\}.$$

Note that $|B_3| = 3\binom{s+1}{3}$. Now define the map $\partial_3 : R^{3\binom{s+1}{3}} \longrightarrow R^{3\binom{s+2}{3}}$ by

$$\begin{aligned}
\partial_3(E_{1,\ell_1,\ell_2,\ell_3,\ell_4}) &= x_4 e_{(2,\ell_1-1,\ell_4),\ell_2+1,\ell_3} + x_4 e_{(1,\ell_1-1,\ell_2),\ell_3+1,\ell_4} - x_3 e_{(3,\ell_3,\ell_4),\ell_1-1,\ell_2+1} \\
&\quad - x_3 e_{(2,\ell_1-2,\ell_4),\ell_2+2,\ell_3} - x_3 e_{(1,\ell_1-2,\ell_2+1),\ell_3+1,\ell_4} + x_1 e_{(3,\ell_3,\ell_4+1),\ell_1-1,\ell_2}; \\
\partial_3(E_{2,\ell_1,\ell_2,\ell_3,\ell_4}) &= x_4 e_{(1,\ell_1-1,\ell_2),\ell_3+1,\ell_4} - x_3 e_{(1,\ell_1-2,\ell_2+1),\ell_3+1,\ell_4} - x_2 e_{(3,\ell_3+1,\ell_4),\ell_1-1,\ell_2} \\
&\quad + x_1 e_{(3,\ell_3,\ell_4+1),\ell_1-1,\ell_2}; \\
\partial_3(E_{3,\ell_1,\ell_2,\ell_3,\ell_4}) &= x_3 e_{(1,\ell_1-2,\ell_2+1),\ell_3+1,\ell_4} - x_2 e_{(2,\ell_1-2,\ell_4),\ell_2+1,\ell_3+1} - x_2 e_{(1,\ell_1-2,\ell_2),\ell_3+2,\ell_4} \\
&\quad + x_1 e_{(2,\ell_1-2,\ell_4+1),\ell_2+1,\ell_3}.
\end{aligned}$$

We now compute the kernel of ∂_3. Consider the set

$$A = \left\{ H_{\ell_1,\ell_2,\ell_3,\ell_4} : 3 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-3 \right\},$$

where

$$\begin{aligned}
H_{\ell_1,\ell_2,\ell_3,\ell_4} &= x_4 E_{3,\ell_1,\ell_2,\ell_3,\ell_4} - x_3 E_{2,\ell_1-1,\ell_2+1,\ell_3,\ell_4} - x_3 E_{3,\ell_1-1,\ell_2+1,\ell_3,\ell_4} + x_2 E_{1,\ell_1-1,\ell_2,\ell_3+1,\ell_4} \\
&\quad - x_1 E_{1,\ell_1-1,\ell_2,\ell_3,\ell_4+1} + x_1 E_{2,\ell_1-1,\ell_2,\ell_3,\ell_4+1}.
\end{aligned}$$

It can be verified that $\partial_3(H_{\ell_1,\ell_2,\ell_3,\ell_4}) = 0$, i.e., $A \subseteq \ker \partial_3$. Let $\ell_1' = \ell_1 - 3$, then one has $\ell_1' + \ell_2 + \ell_3 + \ell_4 = s - 3$. The cardinality of A is equal to the total number of non-negative integral solution of this linear equation, which is $\binom{s}{3}$. As in the proof of Theorem 3, it can be seen that $\ker \partial_3 = \langle A \rangle$. Write the basis elements of $R^{\binom{s}{3}}$ as $B_4 = \left\{ G_{\ell_1,\ell_2,\ell_3,\ell_4} :\ 3 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-3 \right\}$ and define the map $\partial_4 : R^{\binom{s}{3}} \longrightarrow R^{3\binom{s+1}{3}}$ by

$$\partial_4(G_{\ell_1,\ell_2,\ell_3,\ell_4}) = \quad H_{\ell_1,\ell_2,\ell_3,\ell_4}.$$

This is an injective map, and therefore we get the complete resolution:

$$0 \to R(-3s-3)^{\beta_4} \longrightarrow R(-3s-2)^{\beta_3} \longrightarrow R(-3s-1)^{\beta_2} \longrightarrow R(-3s)^{\beta_1} \longrightarrow R.$$

□

Note that in the above proof, we used $s \geq 3$ only to conclude that $\operatorname{pdim}(R/I^s) = 4$. By Remark 4, $\operatorname{depth}(R/I) = 2$, and therefore $\operatorname{pdim}(R/I) = 2$. Similarly, $\operatorname{depth}(R/I^2) = 1$ and hence $\operatorname{pdim}(R/I^2) = 3$. This forces ∂_2 to be injective when $s = 1$ and ∂_3 to be injective when $s = 2$. The computations of syzygies in the cases of resolution of R/I and R/I^2 remain the same as given in the above proof. Therefore, we get resolutions truncated at R^{β_2} in the case of R/I and truncated at R^{β_3} in the case of R/I^2, with the expressions for β_2 and β_3 coinciding with the ones given in the proof. Therefore, we can conclude that in this case, $\operatorname{reg}(I^s) = 3s$ for all $s \geq 1$.

As an immediate consequence, we obtain an expression for the asymptotic regularity of cover ideals of complete 4-partite graphs.

Theorem 6. *Let G denote a complete 4-partite graph with $V(G) = \sqcup_{i=1}^4 V_i$ and $E(G) = \{\{a,b\} : a \in V_i, b \in V_j, i \neq j\}$. Set $V_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, 4$. Let $J_G \subset R = K[x_{ij} : 1 \leq i \leq 4;\ 1 \leq j \leq m_i]$ denote the cover ideal of G. Then the minimal free resolution of R/J_G^s is of the form*

$$0 \to R^{\binom{s}{3}} \longrightarrow R^{3\binom{s+1}{3}} \longrightarrow R^{3\binom{s+2}{3}} \longrightarrow R^{\binom{s+3}{3}} \longrightarrow R.$$

Set $\alpha = (s-\ell_4)m_1 + (s-\ell_3)m_2 + (s-\ell_2)m_3 + (s-\ell_1)m_4$. *Furthermore, we have*

$$\operatorname{reg}(J_G^s) = \max \begin{cases} \alpha, & \text{for } 0 \le \ell_1, \ell_2, \ell_3, \ell_4 \le s, \\ \alpha + m_4 - 1, & \text{for } 1 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-1, \\ \alpha + 2m_4 - 2, & \text{for } 2 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-2, \\ \alpha + 3m_4 - 3, & \text{for } 3 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-3, \end{cases}$$

where $\ell_1 + \ell_2 + \ell_3 + \ell_4 = s$.

Proof. Let $X_i = \prod_{j=1}^{m_i} x_{ij}$. Then $J_G = (X_1X_2X_3, X_1X_2X_4, X_1X_3X_4, X_2X_3X_4)$. Then it follows from Theorem 5 that the minimal free resolution of R/J_G^s is of the given form

$$0 \to R^{\binom{s}{3}} \longrightarrow R^{3\binom{s+1}{3}} \longrightarrow R^{3\binom{s+2}{3}} \longrightarrow R^{\binom{s+3}{3}} \longrightarrow R.$$

To compute the regularity of R/J_G^s, we first need to find the degree's of the generators of the syzygies. Following the notation of Theorem 5, we have

$$\begin{array}{rcl} \deg e_{\ell_1,\ell_2,\ell_3,\ell_4} & = & \deg\left(X_1^{s-\ell_4}X_2^{s-\ell_3}X_3^{s-\ell_2}X_4^{s-\ell_1}\right) = (s-\ell_4)m_1 + (s-\ell_3)m_2 + (s-\ell_2)m_3 + (s-\ell_1)m_4, \\ \deg e_{(1,\ell_1-1,\ell_2),\ell_3,\ell_4} & = & \deg e_{(2,\ell_1-1,\ell_4),\ell_2+1,\ell_3} = \deg e_{(3,\ell_3,\ell_4),\ell_1,\ell_2} = \deg e_{\ell_1,\ell_2,\ell_3,\ell_4} + \deg(X_4), \\ \deg E_{1,\ell_1,\ell_2,\ell_3,\ell_4} & = & \deg E_{2,\ell_1,\ell_2,\ell_3,\ell_4} = \deg E_{3,\ell_1,\ell_2,\ell_3,\ell_4} = \deg e_{\ell_1,\ell_2,\ell_3,\ell_4} + 2\deg(X_4), \\ \deg G_{\ell_1,\ell_2,\ell_3,\ell_4} & = & \deg e_{\ell_1,\ell_2,\ell_3,\ell_4} + 3\deg(X_4). \end{array}$$

Therefore, by setting $\alpha = \deg e_{\ell_1,\ell_2,\ell_3,\ell_4}$, we get

$$\operatorname{reg}(J_G^s) = \max \begin{cases} \alpha, & \text{for } 0 \le \ell_1, \ell_2, \ell_3, \ell_4 \le s, \\ \alpha + \deg(X_4) - 1, & \text{for } 1 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-1, \\ \alpha + 2\deg(X_4) - 2, & \text{for } 2 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-2, \\ \alpha + 3\deg(X_4) - 3, & \text{for } 3 \le \ell_1 \le s,\ 0 \le \ell_2, \ell_3, \ell_4 \le s-3, \end{cases},$$

where $\ell_1 + \ell_2 + \ell_3 + \ell_4 = s$. □

Here also, we have obtained an expression for $\operatorname{reg}(J_G^s)$ not in the form of a linear polynomial. However, as we have demonstrated in the previous cases, this can always be derived for a given graph. Analyzing the interplay between the cardinalities of the partitions, one can obtain the polynomial expression. Let $m_1 = m_2 = m_3 = m_4 = m$. Then,

$$\begin{array}{rcl} \alpha & = & (s-\ell_4)m_1 + (s-\ell_3)m_2 + (s-\ell_2)m_3 + (s-\ell_1)m_4 \\ & = & (4s - (\ell_1 + \ell_2 + \ell_3 + \ell_4))m = 3ms. \end{array}$$

Therefore $\operatorname{reg}(J_G^s) = 3ms + (3m-3)$ for all $s \ge 3$.

Corollary 3. *Let* m, r *be any two positive integers. Consider the arithmetic progression* $m_1 = m$, $m_2 = m + r$, $m_3 = m + 2r$, *and* $m_4 = m + 3r$ *in Theorem 6. Then, for all* $s \ge 3$, *we have*

$$\operatorname{reg}(J_G^s) = s(3m + 6r) + 3m - 3.$$

Proof. We have from Theorem 6, $\alpha = (s-\ell_4)m_1 + (s-\ell_3)m_2 + (s-\ell_2)m_3 + (s-\ell_1)m_4$. On substituting the values of m_i's in α, we get

$$\alpha = s(4m + 6r) - m\ell_4 - (m + r)\ell_3 - (m + 2r)\ell_2 - (m + 3r)\ell_1$$

By Theorem 6, we have for all $s \geq 3$,

$$\operatorname{reg}(J_G^s) = \max \begin{cases} \alpha, & \text{for } 0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s, \\ \alpha + (m+3r) - 1, & \text{for } 1 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-1, \\ \alpha + 2(m+3r) - 2, & \text{for } 2 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-2, \\ \alpha + 3(m+3r) - 3, & \text{for } 3 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-3, \end{cases}$$

where $\ell_1 + \ell_2 + \ell_3 + \ell_4 = s$. To achieve the maximum value of α, negative terms in α should be minimum. The coefficient of ℓ_1 in negative terms in α is largest, so ℓ_1 should be assigned the minimum value. After assigning the minimum value to ℓ_1, assign the minimum value to ℓ_2, and, similarly, the minimum value to ℓ_3. Then assign $\ell_4 = s - \ell_1 - \ell_2 - \ell_3$. For instance, to get the maximum of α when $1 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-1$, put $\ell_1 = 1, \ell_2 = 0, \ell_3 = 0$, and $\ell_4 = s-1$. With appropriate substitution, we get for all $s \geq 3$

$$\operatorname{reg}(J_G^s) = \max \begin{cases} s(3m+6r), & \text{for } 0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s, \\ s(3m+6r) + m - 1, & \text{for } 1 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-1, \\ s(3m+6r) + 2m - 2, & \text{for } 2 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-2, \\ s(3m+6r) + 3m - 3, & \text{for } 3 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-3. \end{cases}$$

Clearly for all $m \geq 1$, and for all $s \geq 3$, we get

$$\operatorname{reg}(J_G^s) = s(3m+6r) + 3m - 3.$$

□

4.3. Complete m-Partite Graphs

Let G be a complete graph on m-vertices. Then the cover ideal J_G of G is generated by $\{x_1 \cdots \hat{x}_i \cdots x_m \ : \ 1 \leq i \leq m\}$. It follows from Remark 4 that $\operatorname{depth} R/J_G^s = 0$ for all $s \geq m-1$. Moreover, by Remark 3, we know that R/J_G^s has linear resolution for all $s \geq 1$. Therefore, the minimal graded free resolution of R/J_G^s for all $s \geq m-1$ is of the form

$$0 \to R(-s(m-1) - m + 1)^{\beta_m} \longrightarrow \to \cdots \to R(-s(m-1) - 1)^{\beta_2} \longrightarrow R(-s(m-1))^{\beta_1} \longrightarrow R.$$

Let $g_1, g_2, \ldots, g_m$ be the minimal generators J_G. Then, the elements in J_G^s consist of elements $T_{\ell_1,\ell_2,\ldots,\ell_m} = g_1^{\ell_1} g_2^{\ell_2} \ldots g_m^{\ell_m}$, such that $\ell_1 + \ell_2 + \cdots + \ell_m = s$ and $0 \leq \ell_i \leq s$. Therefore the total number of elements in J_G^s is same as the total number of non-negative integral solution to the linear equation $\ell_1 + \ell_2 + \cdots + \ell_m = s$, which is $\binom{s+m-1}{m-1}$. Thus, $\mu(J_G^s) = \binom{s+m-1}{m-1}$. Therefore, $\beta_1 = \binom{s+m-1}{m-1}$.

Let $\{e_{\ell_1,\ell_2,\ldots,\ell_m} \mid 0 \leq \ell_i \leq s$; and $\ell_1 + \ell_2 + \cdots + \ell_m = s\}$ denote the standard basis for R^{β_1}. Let $\partial_1 : R^{\beta_1} \longrightarrow R$ be the map $\partial_1(e_{\ell_1,\ell_2,\ldots,\ell_m}) = T_{\ell_1,\ell_2,\ldots,\ell_m}$. As done in the proofs of Theorems 3 and 5, we can see that the first syzygy is given by the relations of the form

$$x_i \cdot T_{\ell_1,\ell_2,\cdots,\ell_{i-1},\ell_i-1,\ell_{i+1}+1,\cdots,\ell_m} - x_{i+1} \cdot T_{\ell_1,\ell_2,\cdots,\ell_{i-1},\ell_i,\ell_{i+1},\cdots,\ell_m} = 0$$

for each $1 \leq i \leq m-1$. Set $\ell_i - 1 = \ell_i'$ and $\ell_{i+1} + 1 = \ell_{i+1}'$. Then it can be seen that, for each $1 \leq i \leq m-1$, there exist as many such relations as the number of non-negative integer solutions of $\ell_1 + \cdots + \ell_i' + \cdots \ell_m = s-1$. Therefore, the total number of such linear relations is $(m-1)\binom{s+m-2}{m-1}$. Therefore $\beta_2 = \binom{m-1}{1}\binom{s+m-2}{m-1}$. However it is not very difficult to realize that writing down the higher syzygy relations are quite challenging. Based on Theorems 3 and 5 and some of the experimental results using the computational commutative algebra package Macaulay 2 [17], we propose the following conjecture.

Conjecture 1. *Let $R = K[x_1, x_2, \ldots, x_m]$ and let J be the cover ideal of the complete graph K_m. The minimal graded free resolution of R/I^s for all $s \geq m-1$ is of the form*

$$0 \to R(-s(m-1)-m+1)^{\beta_m} \longrightarrow \cdots \longrightarrow R(-s(m-1)-1)^{\beta_2} \longrightarrow R(-s(m-1))^{\beta_1} \longrightarrow R,$$

where

$$\beta_i = \binom{m-1}{i-1}\binom{s+m-i}{m-1}.$$

Notice that proving the above conjecture will give the Betti numbers of powers of cover ideals of complete m-partite graphs. We conclude our article by proposing an expression for the regularity of powers of the cover ideals of complete m-partite graphs:

Conjecture 2. *Let G denote a complete m-partite graph with $V(G) = \sqcup_{i=1}^{m} V_i$ and $E(G) = \{\{a,b\} : a \in V_i, b \in V_j, i \neq j\}$. Set $V_i = \{x_{i1}, \ldots, x_{in_i}\}$ for $i = 1, \ldots, m$. Let $J_G \subset R = K[x_{ij} : 1 \leq i \leq m;\ 1 \leq j \leq n_i]$ denote the cover ideal of G. Let $0 \leq \ell_1, \ell_2, \ldots, \ell_m \leq s$ be integers such that $\ell_1 + \ell_2 + \cdots + \ell_m = s$. Set*

$$\alpha = s \cdot (\sum_{i=1}^{m} n_i) - \sum_{i=1}^{m} n_i \ell_{m+1-i}.$$

Then for all $s \geq m-1$, one has

$$\operatorname{reg}(J_G^s) = \max \begin{cases} \alpha, & \text{for } 0 \leq \ell_1, \ell_2, \cdots, \ell_m \leq s, \\ \alpha + n_m - 1, & \text{for } 1 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \cdots, \ell_m \leq s-1, \\ \alpha + 2(n_m - 1), & \text{for } 2 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \cdots, \ell_m \leq s-2, \\ \quad \vdots & \\ \alpha + (m-1)(n_m - 1), & \text{for } m-1 \leq \ell_1 \leq s,\ 0 \leq \ell_2, \cdots, \ell_m \leq s-(m-1). \end{cases}$$

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