Synchronous behaviour of two interacting oscillatory systems undergoing quasiperiodic route to chaos

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Synchronous behaviour of two interacting oscillatory systems undergoing quasiperiodic route to chaos

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Thermoacoustic instability, caused by a positive feedback between the unsteady heat release and the acoustic field in a combustor, is a major challenge faced in most practical combustors such as those used in rockets and gas turbines. We employ the synchronization theory for understanding the coupling between the unsteady heat release and the acoustic field of a thermoacoustic system. Interactions between coupled subsystems exhibiting different collective dynamics such as periodic, quasiperiodic, and chaotic oscillations are addressed. Even though synchronization studies have focused on different dynamical states separately, synchronous behaviour of two coupled systems exhibiting a quasiperiodic route to chaos has not been studied. In this study, we report the first experimental observation of different synchronous behaviours between two subsystems of a thermoacoustic system exhibiting such a transition as reported in Kabiraj *et al.* [Chaos **22**, 023129 (2012)]. A rich variety of synchronous behaviours such as phase locking, intermittent phase locking, and phase drifting are observed as the dynamics of such subsystem change. The observed synchronization behaviour is further characterized using phase locking value, correlation coefficient, and relative mean frequency. These measures clearly reveal the boundaries between different states of synchronization. *Published by AIP Publishing.*
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The development of most practical combustors such as land-based gas turbine engines, aero-engines, and rocket motors is hampered by thermoacoustic instability. Thermoacoustic instability, which results in large amplitude acoustic oscillations, is a consequence of the positive feedback between the unsteady heat release rate and the acoustic field in a combustor. Here, we examine the interaction between these oscillations using the framework of the synchronization theory which deals with the instantaneous interactions between coupled oscillators and their dynamics. Complex dynamical systems consist of various subsystems that exhibit different collective behaviours due to their interactions. Such interactions between the subsystems are addressed using the synchronization theory. The resultant dynamics of the coupled subsystems can be periodic, quasiperiodic or chaotic. Among these dynamics, the synchronization of periodic and chaotic oscillators has been widely studied. However, the synchronization of quasiperiodic oscillators is not yet completely understood. In this study, we show different synchronous behaviours between the acoustic pressure and the unsteady heat release rate signals of a laminar thermoacoustic system exhibiting the quasiperiodic route to chaos. To that end, we examine the dynamics of the relative phase between these oscillations and characterize their synchronous behaviour using phase locking value (*PLV*), correlation coefficient, and relative mean frequency. Using these measures, we are able to detect the boundaries of different states of synchronization. Further, the measures would give robust results even for situations where the dominant frequencies of the signals are locked, but their relative phase shows a diverse dynamics.

I. INTRODUCTION

Thermoacoustic instability is a major concern for most practical combustors such as those used in rockets and gas turbines for propulsion and power generation.[1] The land-based gas turbine combustors are, in principle, prone to thermoacoustic instability while operated with a lean premixed air-fuel mixture. The susceptibility of flame properties, such as flame speed, chemical time scale or flame temperature, to equivalence ratio fluctuations becomes high near lean flammability limit.[2] Further, the fact that the flame structure often becomes acoustically compact during fuel lean conditions favours the energy transfer from combustion to the acoustic field leading to thermoacoustic instability. However, a combustor may also become unstable while operated in a nonpremixed mode as in the case of aero-combustors. Thermoacoustic instability may also occur while operated in near stiochiometric air-fuel conditions as in the case of afterburner and even in rich fuel-oxidizer conditions as in the case of rocket motors. The occurrence of thermoacoustic instability, which is mainly characterized by large amplitude oscillations of the variables such as velocity, pressure, and temperature, is caused by a positive feedback between the unsteady heat release rate and the pressure oscillations. These undesirable large-amplitude oscillations have devastating consequences such as structural damage due to excessive heat transfer and vibrations, thrust oscillations, and even catastrophic failure of the engines.[3]

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As the onset of thermoacoustic oscillations is a nonlinear phenomenon,[4] the tools from the dynamical systems theory provide a framework to study the nonlinear behaviour of the coupled subsystems of the combustor. These tools have been successfully used by various researchers in recent times, to not only explore different dynamical states of the combustor[4–8] but also to detect a transition route to the onset of thermoacoustic instability.[9,10] The approach of nonlinear dynamics has also been used to explore the dynamical transition in pulse combustors[11–13] wherein the occurrence of thermoacoustic instability is expected and maintained to obtain better performances. All the above studies have characterized the dynamical behaviour in different kinds of combustors; however, most of them have looked at one oscillatory variable, the acoustic pressure.

Further, although the necessary criterion for the onset of thermoacoustic instabilities is known over the years,[14] the mechanism pertaining to the coupled interaction between these subsystems has not yet been understood completely. Further, the prediction and the control of thermoacoustic instabilities require an understanding of the instantaneous interaction between the acoustic field and the heat release rate fluctuations observed at different dynamical states in the system.

In the present study, we examine the synchronous behaviour between two subsystems, the acoustic field and the unsteady heat release fluctuations present in the confinement, of a simple thermoacoustic system with the laminar reacting flow. These oscillators undergo a transition from the limit cycle to chaotic oscillations through quasiperiodic dynamics with a change in the flame location.[4,7] We re-examine the data presented in a paper by Kabiraj *et al.*[7]

Under favorable situations, the acoustic field of the confinement gets positively coupled with the unsteady flame oscillations resulting in self-sustained oscillatory behaviour. The necessary condition for the onset of thermoacoustic instability is known as the Rayleigh criterion.[14] According to the criterion, when the phase between acoustic pressure (p') and heat release rate fluctuations of the flame ($\dot{q}'$) lies between $-\pi/2$ and $\pi/2$, energy is periodically added to the acoustic field causing a self-sustained oscillatory behaviour.[15] We analyze these two subsystems in their coupled oscillatory state. Such a phenomenon is possibly analogous to the interactive dynamics of heart and brain of the human body.[16,17] Similar kind of interactive dynamics between two subsystems, *e.g.*, in the cardiorespiratory system,[18] heart-brain interactions,[17] have been studied using the synchronization theory. Recently, the analysis of synchronous behaviour between p' and $\dot{q}'$ in a thermoacoustic system with turbulent flow have shed light on the temporal[19] and spatio-temporal[20] transition to self-sustained oscillations.

We use the synchronization theory to investigate the instantaneous interaction between p' and $\dot{q}'$ in a laminar combustor. The synchronization theory has been used extensively to explore the collective behaviour of complex systems which undergo bifurcations leading to a wealth of dynamical states.[21–23] The possible dynamics of such systems could be periodic, quasiperiodic or chaotic. However, most of the previous studies were focused on synchronization during periodic or chaotic dynamics.[24] Quasiperiodic oscillations, which are often exhibited in different realms of science and technology,[25–29] have not received adequate attention in the context of synchronization.

The entrainment of self-sustained quasiperiodic oscillators by external forcing or through mutual coupling has been studied in the recent past.[30–34] A quasiperiodic attractor is often characterized by its winding number which is the ratio of two incommensurate frequencies of the attractor. Such winding numbers of two different quasiperiodic oscillators have been observed to be locked during synchronization[30] (akin to frequency locking in periodic oscillators). Further, forcing a quasiperiodic motion with periodic pulses (a pulse wave with very short pulse duration) resulted in a plethora of dynamical states including synchronized period-1 oscillations depending upon the period and amplitude of the external forcing.[33] More recently, synchronization in phase dynamics has also been studied theoretically for coupled quasiperiodic oscillators.[34]

However, to the best of our knowledge, there is no experimental evidence of either forced or mutual synchronization of quasiperiodic oscillations. Further, the coupled behaviour between two oscillators has not been investigated while there is a transition in the coupled dynamics from limit cycle to chaos through a quasiperiodic route. In other words, how does the synchronous behaviour of two coupled oscillators of a real system change as they exhibit the quasiperiodic route to chaos?

We here, show that during the quasiperiodic route to chaos, both p' and $\dot{q}'$ oscillations, transition from a state of synchronized periodicity to a state of desynchronized chaos. This transition happens through an intermediate state of quasiperiodic oscillations which display rich synchronous behaviour such as phase locking, intermittent phase locking and phase drifting. For calculating the instantaneous phase of the signals, we adopt an analytic signal approach[35] based on the Hilbert transform.[36] We further calculate the relative phase between the signals and quantitatively characterize different states of synchronization using various measures such as phase locking value (*PLV*), correlation coefficient (r), and relative mean frequency ($\Delta\omega$).

II. EXPERIMENTAL SETUP

The experiments were performed by Kabiraj *et al.*[7] in a laminar thermoacoustic system, where a multiple flame burner was located inside a long borosilicate glass duct (a schematic of the experimental set-up can be found in Ref. 7). A burner tube of inner diameter 16 mm, thickness 1.5 mm, and length 800 mm is used to hold the multiple flame burner on top of it. The burner consists of a perforated copper block, 18 mm in height, with seven equispaced holes of diameter 2 mm. The top of the burner is covered with a fine wire mesh which enhances the stabilization of the flames and, in turn, prevents their blowout during the onset of instability. The fuel (LPG) and air are mixed in a premixing chamber which is packed with steel wool for better mixing of the gases. The upstream of the burner tube is connected to a large cylindrical decoupling chamber to isolate the fuel and air supply system from the downstream acoustic fluctuations. The decoupler located in between the burner tip and air/fuel supply inlets dampens the acoustic waves, and therefore, flow rates are not affected. Thus, the presence of the decoupler removes the possibility of having equivalence ratio fluctuations in our system.

The glass duct (inner diameter 56.7 mm, length 800 mm) was kept open to the atmosphere at the top end (where, the acoustic pressure, $p' = 0$ as $p = \bar{p} + p' = p_{atm}$) and closed at the bottom end (where, the acoustic velocity, $u' = 0$). This results in no acoustic power flowing in or out at boundaries as the acoustic intensity, $\langle I \rangle_{time}$ or the acoustic power $\langle P \rangle_{time} = \langle p'u' \rangle_{time} = 0$ at boundaries. Further, since the bottom end of the glass duct is closed, there is no mean flow of air around the flame which could entrain and mix with the fuel. Therefore, there is no possibility of getting the air flow perturbed by acoustic fluctuations and resulting in equivalence ratio fluctuations.

The relative location of the burner, and thereby the location of flames, was varied through a traverse mechanism which was connected with the glass duct. During experimentations, the equivalence ratio is kept constant at $\phi = 0.46$ (with the uncertainty of 2.8%), by keeping the volumetric air flow rate at 4 lpm and the volumetric fuel flow rate at 56 ccm (with uncertainty in flow rates of 2% of full-scale reading). As we are interested in lean premixed combustion where gas turbine engines are prone to thermoacoustic instability, experiments were conducted at the lean equivalence ratio. Simultaneous measurements of the acoustic pressure and the heat release rate fluctuations (measured in terms of fluctuations of CH^* chemiluminescence intensity from the flame) were performed. A piezoelectric pressure microphone (PCB103B02 of sensitivity = 223.4 mV/kPa and uncertainty = $\pm$0.14 Pa) and a photomultiplier tube (Hamamatsu H5784) with a narrowband CH^* filter (432 $\pm$ 10 nm) were used to measure p' and $\dot{q}'$, respectively. The pressure microphone was mounted near the closed end of the glass duct as the amplitudes of standing waves are always the maximum at the acoustically closed end. As CH^* chemiluminescence is considered as a signature of the heat release rate,[37,38] a global variation of CH^* intensity was captured through PMT outfitted with a CH^* filter. The data was acquired using a 16-bit analog to digital conversion card (NI-6143, resolution 0.15 mV). To ensure the repeatability, the acoustic damping was maintained within the bounds of an exponential decay rate. For the present setup, a mean exponential decay rate was found to be 16/s (evaluated in cold flow). All experiments were performed only when the decay rate of the system falls within $\pm$10% of the mean value.

III. RESULTS AND DISCUSSIONS

A. Bifurcation: Quasiperiodic route to chaos

As the flame location (x_f, measured from the top end of the duct) is varied, the onset of limit cycle oscillations happens through a Hopf bifurcation[7] (at $x_f = 13.8$ cm). With further variation in x_f, the system dynamics undergoes a transition from periodic to chaotic oscillations *via* quasiperiodic dynamics. The representative time series of both p' and $\dot{q}'$ are shown in Fig. 1 for periodic, quasiperiodic, and chaotic dynamics. From the time series, it is evident that, although the dynamical states of p' and $\dot{q}'$ are the same at a particular x_f, their oscillations need not be evolved in sync all the time. Therefore, the instantaneous interaction between p' and $\dot{q}'$ needs more rigorous analysis. Further, the amplitudes of local maxima in the pressure time series are shown as a function of x_f in a bifurcation plot as shown in Fig. 2(a). This shows the dynamical transition between various states when the flame location is varied.

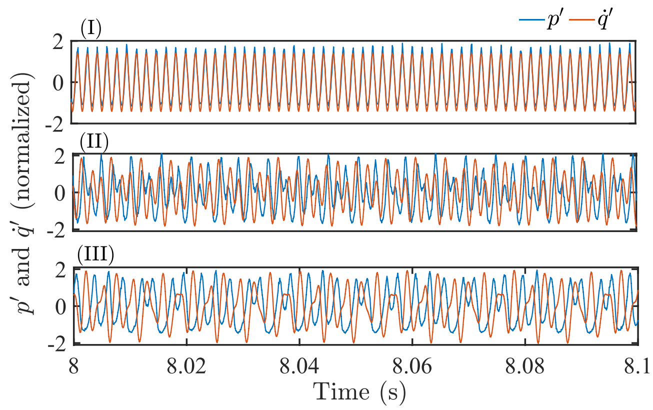

FIG. 1. The representative time series of both p' and $\dot{q}'$ are shown for (I) periodic ($x_f = 13.9$ cm), (II) quasiperiodic ($x_f = 15.6$ cm), and (III) chaotic ($x_f = 17.5$ cm) dynamics.

We further characterize the dynamical nature of these states by plotting the phase portraits and the amplitude spectra of p' as shown in Figs. 2(b) and 2(c), respectively. A closed orbit in the phase portrait [Fig. 2(b-I)] and a single dominant frequency ($f = 560$ Hz) in the frequency spectra [Fig. 2(c-I)] indicate the periodic (limit cycle) dynamics of p'. On the other hand, a toroidal structure in the phase portrait [Fig. 2(b-II)] and the presence of two incommensurate frequencies ($f_1 = 368$ Hz; $f_2 = 571$ Hz; $f_{12} = f_2 - f_1 = 203$ Hz) in the frequency spectra [Fig. 2(c-II)] confirm the quasiperiodic dynamics of p'. Such a non-harmonic state can appear due to the increased number of modes excited in the thermoacoustic system.[39,40] Further, the modal interaction can results in the onset of quasiperiodic oscillations in such systems.[41] The chaotic dynamics is affirmed from the multiple frequency peaks ($f_1 = 368$ Hz; $f_2 = 581$ Hz; $f_3 = 524$ Hz) in the frequency spectra [Fig. 2(c-III)] and the appearance of a strange attractor [Fig. 2(b-III)]. The attractor was proved to be chaotic and strange through a positive maximal Lyapunov exponent and a higher correlation dimension by Kabiraj *et al.*[7]

Depending upon the location of the flame in the duct, the acoustic admittance ($Y = \hat{u}/\hat{p}$, where $\hat{u}$ and $\hat{p}$ are the complex amplitudes of acoustic velocity and pressure, respectively), which is the assistance of a medium to a longitudinal wave motion through a duct, varies differently for different modes. In other words, for a particular mode, the acoustic admittance at the flame affects the ease with which thermal energy is transferred to the acoustic field. Therefore, for a particular location of the flame, one or more number of modes can be excited resulting in the generation of sound with a single frequency or multiple frequencies, respectively.

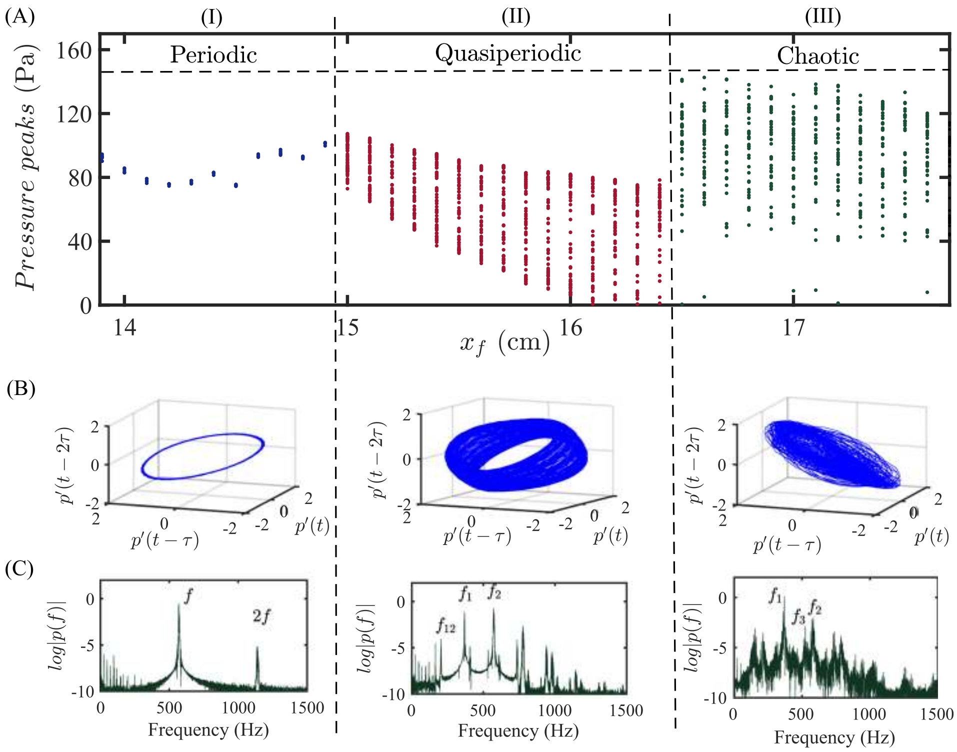

FIG. 2. (a) A bifurcation plot showing the amplitudes of local maxima in the acoustic pressure time series (p') as a function of x_f. The representative (b) phase plots and (c) amplitude spectra (the spectral bin size is 0.15 Hz) are shown for (I) $x_f = 13.9$ cm (limit cycle; $f = 560$ Hz), (II) $x_f = 15.6$ cm (quasiperiodicity; $f_1 = 368$ Hz; $f_2 = 571$ Hz; $f_{12} = f_2 - f_1 = 203$ Hz), and (III) $x_f = 17.7$ cm (chaos; $f_1 = 368$ Hz; $f_2 = 581$ Hz; $f_3 = 524$ Hz). The corresponding time series of p' and $\dot{q}'$ are shown in Fig. 1.

B. Synchronization behaviour of acoustic pressure and heat release rate oscillations

Having described the dynamical nature and the transition of the system dynamics observed in a ducted laminar flame, we turn our attention towards obtaining an understanding of coupled interactions between p' and $\dot{q}'$ signals using the synchronization theory. The temporal variation of the relative phase between the signals helps in detecting different dynamical states of synchronization. The first return maps of p' and $\dot{q}'$ [Figs. 3(II), 3(III), 4(II), and 4(III)] along with their relative phase [Figs. 3(I) and 4(I)] show that although the dynamics of the oscillations in the system are

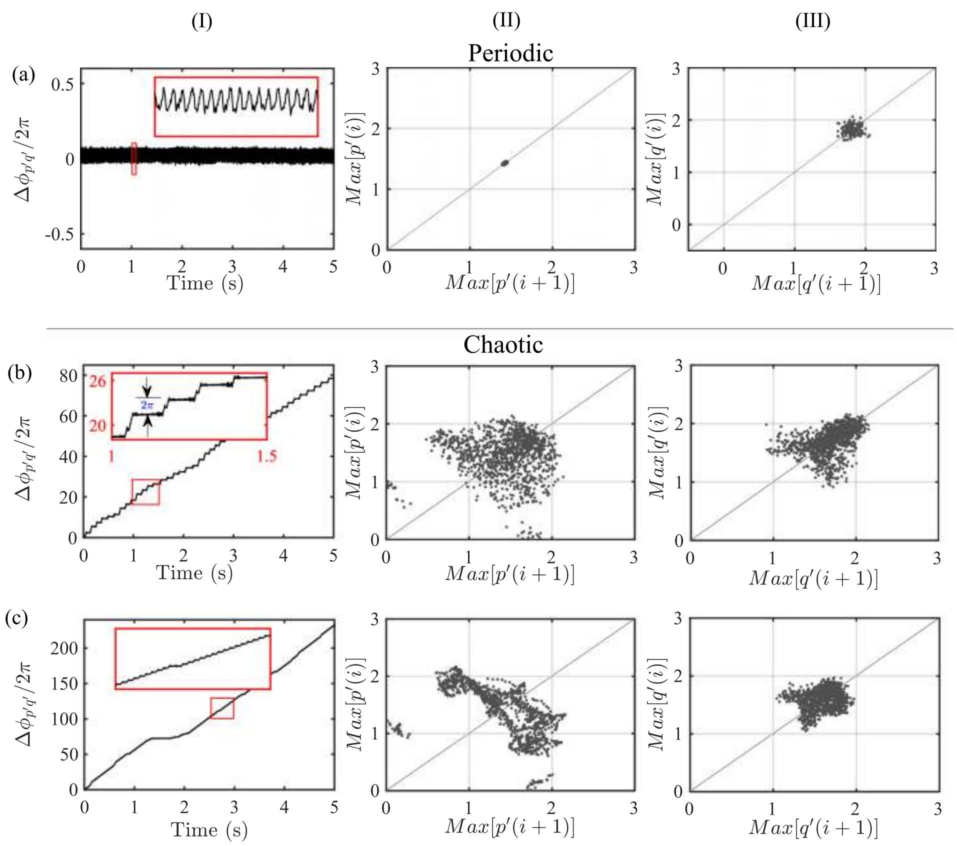

FIG. 3. (I) The temporal variation of phase difference between p' and $\dot{q}'$, (II) the first return maps of the acoustic pressure, and (III) the heat release rate signals are shown. Row "(a)" shows a representative case during limit cycle oscillations ($x_f = 13.9$ cm) whereas row "(b)" ($x_f = 16.5$ cm) and row "(c)" ($x_f = 17.7$ cm) are in the chaotic regime displaying different states of synchronous behaviour. During periodic oscillations, instantaneous phases of two oscillators are locked (a-I). Whereas, during the chaotic state, intermittent phase locking (b-I) and phase drifting (c-I) are observed depending on the values of x_f.

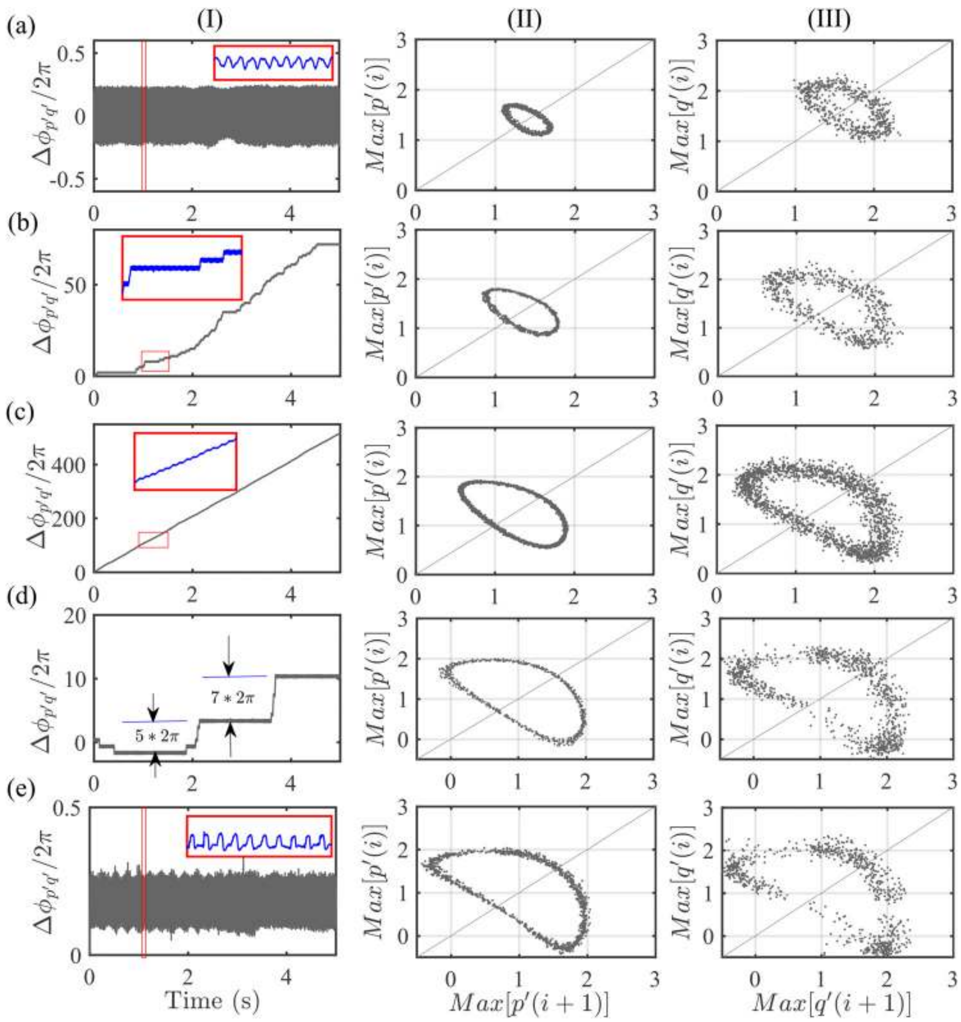

FIG. 4. (I) The temporal variation of instantaneous phase difference between p' and $\dot{q}'$ exhibiting quasiperiodic oscillations is shown for (I-a) $x_f = 15.1$ cm ($f_{p1} = f_{q1} = 366.8$ Hz; $f_{p2} = f_{q2} = 571.3$ Hz), (I-b) $x_f = 15.3$ cm ($f_{p1} = f_{q1} = 368.2$ Hz; $f_{p2} = f_{q2} = 571$ Hz), (I-c) $x_f = 15.6$cm ($f_{p1} = f_{q1} = 370$Hz; $f_{p2} = f_{q2} = 573.7$Hz), (I-d) $x_f = 16.1$cm ($f_{p1} = f_{q1} = 371.7$Hz; $f_{p2} = f_{q2} = 570.5$Hz), and (I-e) $x_f = 16.3$cm ($f_{p1} = f_{q1} = 373.1$Hz; $f_{p2} = f_{q2} = 572.2$Hz). (II) and (III) are the first return maps drawn for both p' and $\dot{q}'$ oscillations, respectively. We observe a (I-a) perfectly phase-locked, (I-b) an intermittently phase-locked, (I-c) a phase drifting, (I-d) an intermittently phase-locked, and (I-e) a perfectly phase-locked state of the quasiperiodic oscillations.

similar for both p' and $\dot{q}'$, the temporal correlation in their phases can be different for a particular dynamical state.

The phase dynamics of periodic and chaotic oscillations are examined first [Fig. 3(I)]. The occurrence of congregated cluster of points in the first return maps [Figs. 3(a-II) and 3(a-III)] reconfirms the limit cycle behaviour.[42] In an ideal situation, periodic dynamics corresponds to a single point in the return map. However, a small cluster of points in heat release rate oscillations could be due to measurement noise. We notice more noise level in $\dot{q}'$ oscillations than that in p' oscillations, and this is consistent with the observation of Kabiraj *et al.*[43] The relative phase (unwrapped instantaneous phase difference) between p' and $\dot{q}'$ is bounded and fluctuates around a constant phase value [Fig. 3(a-I)], confirming their phase locking (or synchronization) behaviour. The fluctuations of the relative phase are periodic with a dominant frequency same as the frequency of the limit cycle oscillations. Figure 3(a-I) shows the fulfillment of the Rayleigh criterion through phase locking. In other words, the acoustic driving required for a growth in the amplitude of oscillations occurs during the synchronous behaviour of p' and $\dot{q}'$.

A set of scattered points around the main diagonal in the first return maps reconfirms the presence of chaotic dynamics[42] in both p' and $\dot{q}'$ [Figs. 3(b-II) and 3(b-III) and 3(c-II) and 3(c-III)]. In the initial regime of chaotic oscillations, we witness a state of intermittent phase locking [Fig. 3(b-I), $x_f = 16.5$ cm], which transitions to a state of continuous phase drifting [Fig. 3(c-I), $x_f = 17.7$ cm] at a sufficiently higher value of x_f. During the intermittent phase locking state, the instantaneous phases of p' and $\dot{q}'$ show locking for the short epochs of the oscillations and jumps of integer multiples of 2π radians between the consecutive epochs of phase locked oscillations [see Fig. 3(b-I)]. On the other hand, during phase drifting, we notice a continuous increase in the relative phase of the signals see Fig. 3(c-I)]. Note here that the high value of relative phase shown in Figs. 3(I) and 4(I) does not provide any information about the (instantaneous) phase delay between signals, but it indicates the cumulative phase difference along the duration of the signal.

We will now shift our attention towards the quasiperiodic regime. A closed loop structure in the first return maps, shown in Figs. 4(II)–4(III), indicates the presence of quasiperiodic oscillations in both acoustic pressure and heat release rate signals.[42] However, in Figs. 4(d-III) and 4(e-III), we notice the distorted closed loop structures of $2D$ torus in the first return maps of heat release rate oscillations, which possibly reflect the transition from quasiperiodic to chaotic oscillations. In other words, the appearance of scattered points in the return maps [Fig. 4(d-III) and 4(e-III)] is indicative of breaking of the two-torus structure observed during a quasiperiodic state. Although the frequencies of both oscillators vary slightly due to change in the flame location (x_f), we find that the frequencies are always perfectly locked for both p' and $\dot{q}'$. In such a situation, where the dominant frequencies of a system of coupled oscillators are perfectly locked, the plot of instantaneous relative phase aids in determining the temporal interaction between the signals of these oscillators.

At $x_f = 15.1$ cm [Fig. 4(I-a)], we notice the bounded fluctuations in the relative phases of p' and $\dot{q}'$ indicating the phase locking (or synchronization) of these coupled subsystems during the quasiperiodic state. The bounded fluctuations in the relative phase are also referred to as phase trapping. The fluctuations in the relative phase are periodic with a dominant frequency same as the beating frequency of the quasiperiodic signals. The relative phases between p' and $\dot{q}'$ remain in the region of $-\pi/2$ to $\pi/2$ [Fig. 4(a-I)] leading to the persistence of periodic oscillations. With further increase in x_f [15.3 cm, Fig. 4(b-I)], the instantaneous relative phase between p' and $\dot{q}'$ shows transition to a state of the intermittently phase locked regime. In this state, the instantaneous phases of p' and $\dot{q}'$ are locked for short durations of time and the epochs of the consecutive phase locked oscillations are separated by the phase slips, which occur as integer multiples of 2π radians [inset of Fig. 4(b-I)]. As x_f is further increased [Fig. 4(c-I), x_f=15.6 cm], we notice an increase in the occurrence of phase slips, which finally end up into a continuous drift in the relative phase. A continuous drift in the relative phase between p' and $\dot{q}'$ is an indication of desynchronization of two coupled quasiperiodic oscillations observed in a thermoacoustic system. As the flame location is further increased in the quasiperiodic regime, we notice a transition of the system dynamics to a state of phase-locking [Fig. 4(e-I)] through a state of intermittent phase-locking [Fig. 4(d-I)].

The transition from a state of phase locking to a state of phase drifting happens through intermittent or imperfect phase locking with occasional slips. This happens due to the phase jumps in any one of the signals (individual phase dynamics are provided in the supplementary material, section S-A, Fig. S1). We find phase jumps either in $\dot{q}'$ [for the case of Fig. 4(b-I), see S1] or in p' [for the case of Fig. 4(d-I), not shown]. Therefore, even though phase portraits and first return maps indicate the quasiperiodic nature of oscillations [Figs. 2(b-II), 4(II), and 4(III)], the phase jumps show imperfectness in quasiperiodic oscillations. A similar kind of imperfect phase locking was also observed in quasiperiodically forced coupled chaotic systems.[44]

C. Quantitative characterization of synchronization behaviour

We further utilize different quantitative measures of synchronization such as phase locking value (PLV), correlation coefficient (r), and relative mean frequency ($\Delta\omega$). These measures help in detecting transitions between the different regimes of synchronization of coupled oscillators observed in the thermoacoustic system.

The measure, PLV[45] ($= N^{-1}|\sum_{t=1}^{N} \exp(i\Delta\theta_t)|$, where $\Delta\theta_t$ represents the instantaneous relative phase at the t^{th} time instant) computes the mean variability of instantaneous relative phases between p' and $\dot{q}'$. PLV lies close to 1 for a perfectly synchronized state, whereas, it lies close to 0 for a perfectly desynchronized state of oscillations. In the case of intermittent synchronization, its value appears between 0 and 1. During limit cycle oscillations, PLV stays close to one [Fig. 5(a)] confirming the perfect phase locking of the acoustic pressure and the heat release rate oscillations in the system. As the system dynamics enters into a quasiperiodic regime of oscillations, PLV shows a drop from the value observed during the limit cycle state. The reduction in PLV further continues till 0.3, after which, it shows an increase to a value close to 0.8, observed for the final state of quasiperiodic oscillations in the system. Although the oscillations of p' and $\dot{q}'$ are phase locked during the quasiperiodic state [Fig. 5(a-I)], the value of PLV shows a decrease in this region, which is manifested as an increased amplitude of the bounded oscillations of the relative phase (time series of relative phase with different amplitudes and corresponding PLV are provided in the supplementary material, section S-B, Fig. S2). A further decrease in PLV agrees with the transition of system dynamics from a state of phase-locked oscillations [Fig. 5(a-I)] to a state of phase drift oscillations [Fig. 5(a-III)] that happens through a regime of intermittent phase locking [Fig. 5(a-II)]. Subsequently, the increase in PLV depicts the transition of the system dynamics back to the phase locking state [Fig. 5(a-V)] *via* a realm of intermittent phase locking [Fig. 5(a-IV)]. Further, in the chaotic state of oscillations, PLV drops from a value close to 0.75 to a value close to 0.5 during the transition from a state of intermittent phase locking to a state of phase drifting [Fig. 5(a)].

Figure 5(b) shows the variation of linear correlation coefficient (r) between p' and $\dot{q}'$ with the location of flame inside the duct. During limit cycle oscillations, the value of r remains close to 1 indicating the presence of a strong positive correlation between p' and $\dot{q}'$. However, in the quasiperiodic state of oscillations, we observe a continuous fall in the values of r from the regions shown in Figs. 5(b-I)–5(b-V), in which, r changes its value from positive to negative in the desynchronization region as shown in Fig. 5(b-III). As the values of r approaches zero, the system dynamics during the quasiperiodic state starts regaining its initial state of phase locked oscillations. This change in the sign of r, during the quasiperiodic state, maybe due to the phase shift which happens primarily in the heat release rate oscillations when the flame location crosses the acoustic velocity node of the standing wave. Such a phase shift due to crossing a node is explained by Lieuwen[46] for a single frequency standing wave. As the system dynamics enters into a chaotic state of oscillations, the value of r suddenly becomes more negative and it shows a small change during the chaotic region. Furthermore, a sudden change in the slope of variation of r $\left(\frac{dr}{dx_f}\right)$ from negative to positive is discernible as the dynamics changes from quasiperiodic to chaotic.

Figure 5(c) shows the variation in the relative mean frequency computed for the acoustic pressure and the heat release

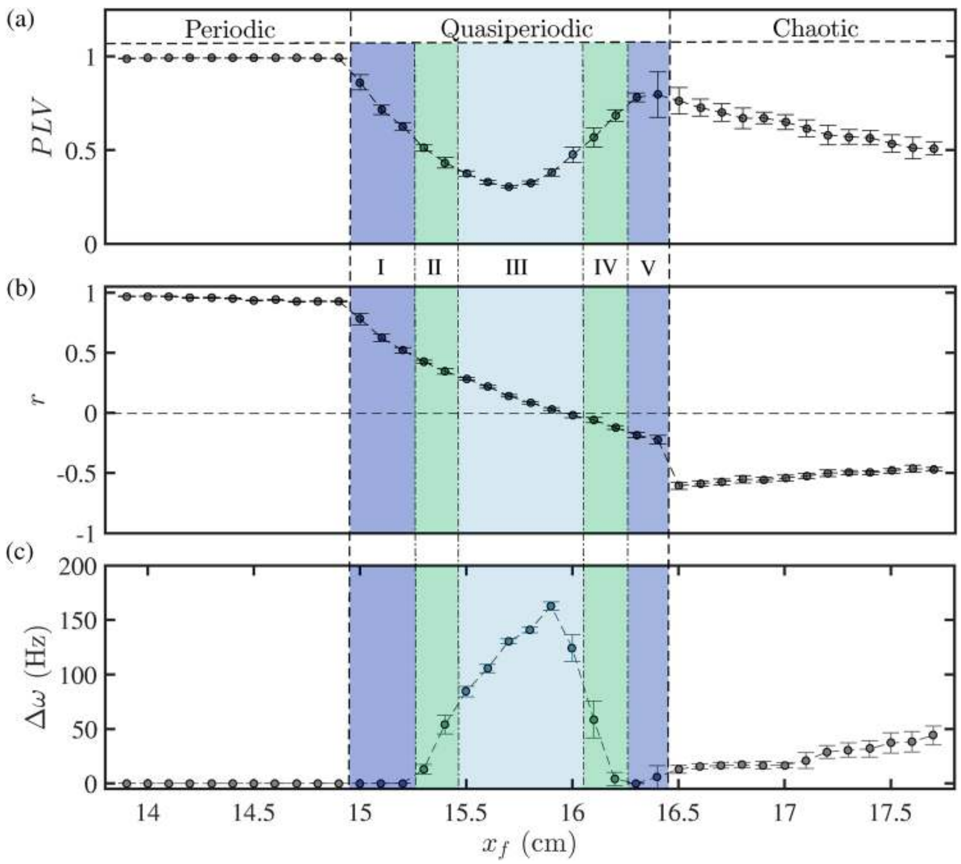

FIG. 5. (a) The variation of the phase locking value (PLV), (b) correlation coefficients (r), and (c) the variation of relative mean frequency ($\Delta\omega$) between p' and $\dot{q}'$ oscillations are plotted for different locations of flame (x_f) inside the duct. In the quasiperiodic regime, the computed values of PLV, r and $\Delta\omega$ for the region of phase locking (I and V), intermittent phase locking (II and IV), and phase drifting (III) are separately shown. The error bars in (a), (b), and (c) show the statistical variation of these parameters around their mean values obtained for 15 different segments (each of 2 s) of the entire signals (30 s). Figs. 4(a)–4(e) show the representative plots of phase dynamics and the first return map of the regions I–V, respectively.

rate oscillations at different locations of the flame inside the duct. The relative mean frequency ($\Delta\omega = |\omega_1 - \omega_2|$) is the difference in the mean frequencies of two signals, where the mean frequency is an average of the first order time derivative of the instantaneous phase of the signal ($\omega = \langle d\phi/dt \rangle$).[47] The advantage of this parameter over the others [shown in Figs. 5(a) and 5(b)] is that $\Delta\omega$ qualitatively demarcates the regions of perfectly phase locked states [Figs. 5(c-I) and 5(c-V)] from the regions of desynchronized [Fig. 5(c-III)] and imperfectly synchronized [Figs. 5(c-II) and 5(c-IV)] states. Such demarcations are not observed in the plots of PLV and r. When the system is in a state of phase locked oscillations, the value of $\Delta\omega$ becomes zero, whereas, during the state of desynchronized oscillations, it shows a value higher than zero. In Fig. 5(c), we see that the values of $\Delta\omega$ stay at zero, for the few values of x_f in the quasiperiodic state. This behaviour of synchronization during quasiperiodic oscillations is not clearly reflected in the plots of PLV and r as shown in Figs. 5(a) and 5(b). The high amplitude oscillations observed in the relative phase during the phase locking state [Fig. 4(a-I)] result in lower values of PLV and r. However, the oscillations in the relative phase get averaged out in the mean frequency ($\omega = \langle d\phi/dt \rangle$). In the regime of quasiperiodic oscillations, as we move from synchronization (phase locking) to desynchronization (phase drifting) through imperfect synchronization (intermittent phase locking), the values of $\Delta\omega$ increase to its maximum. As the system dynamics consequently regains the state of perfectly phase-locked quasiperiodic oscillations at higher values of x_f, the value of $\Delta\omega$ falls to zero. However, phase drift in the chaotic regime is observed with much lower $\Delta\omega$. This might be due to the broadband spectrum in the chaotic regime unlike the quasiperiodic regime.

We further notice that, for all states during the quasiperiodicity route to chaos presented in Fig. 4, the dominant frequencies of both p' and $\dot{q}'$ oscillations always remain locked (shown in the supplementary material, section S-C, Fig. S6). Even if the locked frequencies are the same, the spectral content may lead to different phase dynamics (a treatment with synthetic quasiperiodic signals is given in the supplementary material, S-C). The frequency with a high spectral content might repeat more in time which is reflected in $\Delta\omega$. In such situations, the values of $\Delta\omega$ of the coupled oscillators seem to be a better measure in detecting the synchronization states than the locking of dominant frequencies. We believe that our findings of the present paper may shed light on the studies where two coupled subsystems exhibit a quasiperiodic route to chaos. Further, we note that the results presented in this paper may not be generalized to all other configurations of combustion systems, as all thermoacoustic systems do not show the kind of dynamics presented in our paper. However, the method to perform the instantaneous interactions of two signals is universally applicable and is independent of the configuration of the combustor.

The change of flame location (x_f) results in the onset of different dynamical states such as periodic, quasiperiodic, and chaotic oscillations in the system and also results in different synchronous behaviours. The power transferred to the acoustic field from the flame primarily depends on the value of acoustic admittance ($Y = \hat{u}/\hat{p}$) at the flame location.

Therefore, depending upon the location of the heat release source (or flame) along the combustor length, the acoustic admittance for a particular mode at the flame varies, thus, causing a change in the acoustic driving. Therefore, as x_f changes, different acoustic modes of the systems may get excited. Further, the simultaneous excitation of these modes may lead the system to exhibit various dynamical states that have different frequency contents. In the case of such multi-frequency signals, a mean frequency of the signal is commonly used as a quantifying parameter for synchronization. We observe that with the change in x_f, the difference in mean frequencies of the acoustic pressure and the heat release rate signals changes, which further results in different synchronization behaviours such as phase locking, phase trapping, intermittent phase locking or phase drifting in the system dynamics. This suggests that the location of flame inside the duct might have a direct influence in altering the coupling strength between p' and $\dot{q}'$. Furthermore, the change in the energy content of each acoustic mode can also contribute towards the various synchronization dynamics during the quasiperiodic state of oscillations (as shown in the supplementary material, S-C). However, further investigations are needed for more detailed understanding of the importance of the flame location on different synchronous behaviours observed.

IV. CONCLUSION

In summary, the interaction between the acoustic field and the heat release rate fluctuations from the flames located inside a confinement is investigated in a well-controlled experiment of a thermoacoustic system. With the variation of flame location inside a glass duct, rich dynamical states have been observed in previous studies. The present study deals with the regime of quasiperiodic route to chaos. The periodic (or limit cycle) state corresponds to a region of phase-locked oscillations of acoustic pressure and heat release rate fluctuations. On the other hand, the region of quasiperiodic dynamics depicts the presence of different behaviours of synchronization of these oscillators such as phase locking (or phase trapping as the relative phase becomes bounded but oscillatory), intermittent phase locking, and phase drifting. Different phase dynamics can be attributed to dissimilar spectral content of two locked frequencies. In the chaotic regime of oscillations, we notice the state of intermittent phase locking and subsequent state of phase drifting. Furthermore, we computed some statistical measures such as the phase locking value (*PLV*) and correlation coefficient (r) for quantitative characterization of the synchronization behaviour observed in the dynamics of the relative phase. These measures further aid in detecting the boundaries of different states of synchronization observed during the quasiperiodicity route to chaos. We find the relative mean frequency to be a better measure for detecting the phase locking behaviour of coupled oscillators. This measure would give robust results even for situations where the dominant frequencies of the signals are locked; but their relative phase shows a diverse dynamics.

SUPPLEMENTARY MATERIAL

See supplementary material for: (1) the phase jumps in one of the signals considered and its consequence in the relative phase between the signals (section S-A), (2) the dependence of amplitude of relative phase oscillations on *PLV* (section S-B), and (3) the dependence of phase dynamics on the spectral contents of two synthetic quasiperiodic signals (section S-C).

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