

## Symplectic potentials and resolved Ricci-flat ACG metrics

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### Abstract

We pursue the symplectic description of toric Kähler manifolds. There exists a general local classification of metrics on toric Kähler manifolds equipped with Hamiltonian two-forms due to Apostolov, Calderbank and Gauduchon (ACG). We derive the symplectic potential for these metrics. Using a method due to Abreu, we relate the symplectic potential to the canonical potential written by Guillemin. This enables us to recover the moment polytope associated with metrics and we thus obtain global information about the metric. We illustrate these general considerations by focusing on six-dimensional Ricci-flat metrics and obtain Ricci-flat metrics associated with real cones over  $L^{pqr}$  and  $Y^{pq}$  manifolds. The metrics associated with cones over  $Y^{pq}$  manifolds turn out to be partially resolved with two blow-up parameters taking special (non-zero) values. For a fixed  $Y^{pq}$  manifold, we find explicit metrics for several inequivalent blow-ups parametrised by a natural number  $k$  in the range  $0 < k < p$ . We also show that all known examples of resolved metrics such as the resolved conifold and the resolution of  $\mathbb{C}^3/\mathbb{Z}_3$  also fit the ACG classification.

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# 1 Introduction

The natural target space for  $(2, 2)$  supersymmetric non-linear sigma models in two dimensions is a Kähler manifold,  $X$  [1]. For applications in string theory, one needs the non-linear sigma model to be conformally invariant. To leading order, conformality of the non-linear sigma model requires the Kähler manifold to be Ricci-flat [2]. In 1993, Witten provided a simpler construction of such sigma models by introducing the gauged linear sigma model (GLSM) [3]. Among the many phases of the GLSM, he showed that there is a phase where one recovers the non-linear sigma model. A notable feature of this construction was a simple description of a necessary condition (i.e.,  $c_1(X) = 0$ ) for the manifold to be Ricci-flat. Further, he showed that the GLSM naturally realises a symplectic quotient and that the induced metric in the NLSM limit was a natural generalisation of the Fubini-Study metric associated with complex projective spaces.

Around the same time, Guillemin carried out a systematic treatment of toric Kähler manifolds and wrote a simple formula that generalised the Fubini-Study metric for  $\mathbb{C}\mathbb{P}^n$  [4]. The only data that went into writing the metric was the moment polytope associated with a toric Kähler manifold. The moment polytope is a convex polytope defined by several inequalities of the form

$$\ell_a(P) > 0, \quad a = 1, 2, \dots \quad (1.1)$$

Guillemin wrote the metric in symplectic coordinates rather than (the more commonly used) complex coordinates. The metric in symplectic coordinates is determined by a single function called the *symplectic potential* [5]. The symplectic potential written by Guillemin is given by

$$G_{\text{can}}(P) = \frac{1}{2} \sum_a \ell_a(P) \log \ell_a(P). \quad (1.2)$$

We will refer to this as the *canonical* symplectic potential. For projective spaces, this metric reduces to the Fubini-Study metric. However, while the metric correctly captures the singularities in more general examples, it is not necessarily Einstein (or even extremal) as the Fubini-Study metric. It turns out that the metric given by the GLSM is identical to the one written out by Guillemin<sup>1</sup>.

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<sup>1</sup>This result may be obvious to some and non-obvious to others. However, the GLSM has a

Abreu had a simple suggestion to obtain Einstein/extremal metrics from the canonical one [6]. Adapting a method due to Calabi in the complex context [7], Abreu modified the canonical symplectic potential by adding a ‘function’ to it as follows:

$$G(P) = G_{\text{can}}(P) + h(P) , \tag{1.3}$$

where  $h(P)$  is *non-singular* in the interior as well as the boundary of the polytope. We will refer to  $h(P)$  as the *Abreu* function in this paper. The Abreu function is determined by requiring that the new metric has the required property such as extremality. For instance, the differential equation for  $h(P)$  is the analog of the Monge-Ampère equation that appears when one imposes Ricci-flatness on the Kähler potential [8]. The function  $h(P)$  has been determined in only a small number of examples [6, 9, 10]. However, there have been a recent attempt to obtain the function numerically [11].

This paper focuses on a special sub-class of toric Kähler manifolds, those that admit a Hamiltonian 2-form. For Kähler manifolds that admit such a 2-form (and possibly non-toric), there exists a classification of these metrics due to Apostolov, Calderbank and Gauduchon(ACG) [12]. The main merit of such metrics is that it replaces a PDE in  $m$  variables that one needs to solve to obtain the symplectic potential by an ODE’s in  $m$  functions of one-variable in the best of situations. We obtain the symplectic potential for these metrics and find that it can be easily written in the form given in Eqn.(1.3). Then the associated polytope is easily recovered. We find that all known examples of resolved metrics in six-dimensions admit a Hamiltonian 2-form and add a *new* infinite family of partially resolved spaces to the list of known examples.

Another application of these methods is in the context of the AdS-CFT correspondence which relates four-dimensional conformal field theories with type IIB string theory on  $AdS_5 \times X^5$ , where  $X^5$  is a compact five-dimensional Sasaki-Einstein manifold [13]. Real cones over these spaces turn out to be non-compact Ricci-flat Kähler manifolds. Thus, our examples will focus on six-dimensional Ricci-flat toric Kähler manifolds which are allowed to have a conical singularity at the tip of the cone. Resolutions of these singularities correspond to non-

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wider range of validity than the Guillemin formula. For instance, it is valid even for non-toric examples.

conformal deformations of the conformal field theory and are also of independent interest [14].

The paper is organised as follows. Section 2 is a review of the symplectic quotient as obtained from the gauged linear sigma model. In section 3, we review the local classification of toric Kähler metrics admitting a Hamiltonian 2-form due to Apostolov, Calderbank and Gauduchon. We then discuss the conditions under which their metrics are Einstein and Ricci-flat. In section 4, we construct the symplectic potential for all their metrics. We then carry out a global analysis of the ACG metrics and discuss how one recovers the precise singularity structure by writing the symplectic potential as the sum of the canonical symplectic potential and the Abreu function. Sections 5 and 6 make use of the results of section 4 to generate examples of unresolved and resolved metrics respectively. While the results in section 5 are not new, the methods used are new and have independent merit. In section 6, we obtain a new infinite family of metrics corresponding to partially resolved metrics on cones over  $Y^{pq}$ . We conclude in section 7 with a brief discussion on our results.

**Note added:** While this paper was being readied for publication, a paper by Martelli and Sparks appeared [15]. This paper also deals with ACG metrics and resolved metrics. There is some overlap with this work though the methods differ. The authors also mention a forthcoming paper which discusses the partial resolutions of cones over  $Y^{pq}$  spaces. This also may have some overlap with section 6 of this paper.

## 2 The symplectic quotient in the GLSM

A large family of Kähler manifolds are obtained by means of the Kähler quotient. The construction proceeds as follows [16]:

$$X^{(2m)} = \frac{\mathbb{C}^n - F_\Delta}{(\mathbb{C}^*)^d}, \quad m = n - d. \quad (2.1)$$

The various  $\mathbb{C}^*$  actions are specified by the *charge vectors*  $Q_a^\alpha$  (which we sometimes write as a  $n \times d$  matrix  $\mathcal{Q}$ ):

$$\phi_a \longrightarrow \lambda^{Q_a^\alpha} \phi_a, \quad a = 1, \dots, n \text{ and } \alpha = 1, \dots, d \quad (2.2)$$

$F_\Delta$  corresponds to the set of fixed points under the  $\mathbb{C}^*$  actions. For instance,  $\mathbb{C}\mathbb{P}^{n-1}$  is obtained by the Kähler quotient with one  $\mathbb{C}^*$  action with charge vector  $Q = (1, 1, \dots, 1)^T$  and  $F_\Delta = \{0\}$ .

Writing  $\mathbb{C}^* = \mathbb{R}_+ \times S^1$ , the  $\mathbb{C}^*$  quotient can be carried out as a two-step process. First, carry out  $\mathbb{R}^+$  quotient and then the  $S^1$  action. This is called the *symplectic quotient* and this is the way the GLSM naturally realises the quotient [3].

The symplectic quotient is implemented in the GLSM as follows. The GLSM has  $(2, 2)$  supersymmetry and the field content consists of  $n$  chiral superfields,  $\Phi_a$  ( $a = 1, \dots, n$ ) and  $d$  Abelian vector multiplets  $V_\alpha$  ( $\alpha = 1, \dots, d$ ). (Please see [3] for more details.) The charges of the chiral fields under the  $d$  gauge fields is given by  $d$  charge vectors,  $Q_a^\alpha$ ,  $\alpha = 1, \dots, d$ . The parameters of the GLSM are the gauge coupling constant  $e^2$  (we take all the  $d$  couplings to be identical for simplicity). Each abelian multiplet admits a Fayet-Iliopoulos (FI) term which is represented by a complex coupling,  $\tau_\alpha \equiv r_\alpha + i(\theta_\alpha/2\pi)$ . We will refer to  $r_\alpha$  as FI parameters or blow-up parameters.

In the GLSM, the  $(\mathbb{R}_+)^d$  quotient is imposed by the D-term constraints<sup>2</sup>:

$$\sum_{a=1}^n Q_a^\alpha |\phi_a|^2 = r_\alpha . \quad (2.3)$$

and the  $(S^1)^d \sim U(1)^d$  action is taken care of by the gauging in the GLSM. Not all values of  $|\phi_i|^2$  can satisfy the D-term conditions. The set of allowed values of  $|\phi_i|^2$  are best represented by the interior points of a convex polytope, the **moment polytope**. Writing the complex field  $\phi_a = \sqrt{\ell_a} \exp i\varphi_a$  in polar coordinates, the D-term conditions can be written as

$$\sum_{a=1}^n Q_a^\alpha \ell_a = r_\alpha , \quad (2.4)$$

These linear conditions can be solved for in terms of  $m = n - d$  independent variables that we will call  $P_i$  ( $i = 1, \dots, m$ ). We can then rewrite the  $\ell_a$  as implicit functions of the  $P_i$ ,  $\ell_a(P)$ . The moment polytope is then given by the

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<sup>2</sup>In the strong coupling limit(s), typically of the form  $e^2 r_\alpha \rightarrow \pm\infty$ , the fields in the vector multiplets become Lagrange multipliers imposing various constraints (explicitly given, for instance, in [17]).

conditions

$$\ell_a(P) > 0 . \tag{2.5}$$

The Kähler two-form on the toric manifold  $X^{(2m)}$  is

$$\omega = \sum_{i=1}^m dP_i \wedge dt_i .$$

where  $t_i$  are the angles that remain after the  $U(1)^d$  gauge degrees are removed. The metric on  $X^{2m}$  is determined by a single function,  $G(P)$ , called the symplectic potential<sup>3</sup>

$$ds^2 = G_{ij}dP_i dP_j + G^{ij}dt_i dt_j , \tag{2.6}$$

where  $G_{ij} = \partial^2 G / \partial P_i \partial P_j$  and  $G^{ij}$  is the matrix inverse of  $G_{ij}$ . The metric induced by the symplectic quotient is the canonical symplectic potential  $G_{\text{can}}$  given in Eqn. (1.2).

There is a theorem due to Delzant that states that one can recover a compact toric symplectic Kähler manifold from its polytope provided it satisfies certain conditions such as convexity, simplicity etc [18]. Such polytopes have been called Delzant polytopes. An extension of Delzant's theorem to include toric symplectic orbifolds leads to polytopes with a positive integer attached to each facet [19]. Weighted projective spaces have polytopes of this kind. The formula of Guillemin, given in Eqn. (1.2) though originally written out only for Delzant polytopes is valid for toric symplectic orbifolds as well [20]. The  $P_i$  are thus coordinates on the polytope and the toric manifold is a  $U(1)^m$  fibration with base, the polytope. The boundaries of the polytope correspond to points where the fibration degenerates [21].

The usual toric data associated with toric manifold  $X^{2m}$  is specified given by a set of  $n$  vectors in  $\mathbb{R}^m$ , written out as a  $m \times n$  matrix  $\mathcal{V}$ . These vectors are obtained from the charge vectors  $Q_a^\alpha$  by solving

$$\mathcal{V} \cdot \mathcal{Q} = 0 . \tag{2.7}$$

Thus, while the charge vectors  $\mathcal{Q}$  appear naturally in the GLSM, the toric description is given in terms of  $\mathcal{V}$ . In our examples, we will go back and forth between the two objects.

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<sup>3</sup>The symplectic potential is the analogue of the Kähler potential appearing in complex coordinates. The two are related by a Legendre transformation [5].

## 2.1 Six-dimensional Ricci-flat manifolds

A simple class of six-dimensional manifolds are obtained by considering the symplectic quotient involving the D-term given by the charge vector  $Q = (p_1, p_2, p_3, p_4)^T$ :

$$p_1 \ell_1 + p_2 \ell_2 + p_3 \ell_3 + p_4 \ell_4 = r , \quad (2.8)$$

where the  $p_a$  are taken to be integers. This corresponds to the symplectic quotient  $\mathbb{C}^4/\mathbb{C}^*$ . A necessary condition for the manifold to admit a Ricci-flat metric is the condition  $p_1 + p_2 + p_3 + p_4 = 0$ . There are two inequivalent classes of these four integers: (i)  $p_1, p_2 > 0$  and  $p_3, p_4 < 0$  and (ii)  $p_1, p_2, p_3 > 0$  and  $p_4 < 0$ . All other possibilities can be obtained by suitably relabelling the  $p_a$ . The first choice leads to the conifold and its generalisations corresponding to real cones over  $L^{pqr}$  spaces [22] and the second choice leads to orbifolds of the form  $\mathbb{C}^3/\mathbb{Z}_N$  with  $N = p_4$ .

A blow-up is implemented in the GLSM by adding a new chiral superfield along with an additional abelian vector superfield. This adds a new D-term and leads to the symplectic quotient  $\mathbb{C}^5/(\mathbb{C}^*)^2$ . The FI parameter of the new D-term determines the size of the blown-up manifolds. In this paper, we will consider this situation as well.

## 3 The ACG metrics.

In this section, we will summarize the results from the paper [12] that are relevant for our purposes. The paper [12] concerns Hamiltonian 2-forms and a local classification of Kähler metrics that admit such 2-forms. On a Kähler manifold  $X$  of real dimension  $2m$  with metric  $g_{ij}$ , complex structure  $J^j_i$  and Kähler two-form  $\omega_{ij} = g_{jk} J^k_i$ , a *Hamiltonian 2-form*,  $\phi_{ij}$ , is a  $(1, 1)$  form satisfying the equations

$$\nabla_k \phi_{[ij]} + \frac{1}{2} [\partial_{[i} \text{Tr} \phi \omega_{j]k} + \partial_l \text{Tr} \phi J^l_{[i} g_{j]k}] = 0 . \quad (3.1)$$

where  $\text{Tr}(\phi) = \omega^{kl} \phi_{kl}$  and  $\nabla_k$  is the covariant derivative with respect to the Levi-Civita connection.

The notion of a Hamiltonian 2-form and the special properties of this object first appeared in [23], where the authors were investigating a special class of

four-dimensional Kähler metrics. It turned out that the Ricci form of this class of Kähler metrics was a Hamiltonian 2-form. The nomenclature “Hamiltonian” alludes<sup>4</sup> to the fact that two scalars constructed out of the two-form: the trace,  $s$ , and the Pfaffian,  $p$ , in the four-dimensional context, are Hamiltonian functions for (Hamiltonian) Killing vector fields of the Kähler metric. The scalar functions also arise as the co-efficients of the characteristic polynomial of the  $2 \times 2$  hermitian matrix,  $(\phi - t\omega)$ , constructed out of the Hamiltonian 2-form and the Kähler form:

$$p(t) := t^2 - P_1 t + P_2. \quad (3.2)$$

More importantly, the roots of this polynomial, call them,  $\xi$  and  $\eta$ , so that

$$P_1 = \xi + \eta, \quad P_2 = \xi \eta, \quad (3.3)$$

provide coordinates in which it becomes possible to classify a sub-class of toric Kähler metrics known as *orthotoric* metrics in terms of two polynomials of one variable, one of  $\xi$  and the other of  $\eta$ . An orthotoric metric is one with  $g^{\xi\eta} = 0$  and the most general four-dimensional orthotoric metric is [23]

$$ds_{\text{OT2}}^2 = (\xi - \eta) \left( \frac{d\xi^2}{f(\xi)} - \frac{d\eta^2}{g(\eta)} \right) + \frac{1}{\xi - \eta} (f(\xi)(dt + \eta dz)^2 - g(\eta)(dt + \xi dz)^2). \quad (3.4)$$

In [12], ACG worked in arbitrary dimensions and classified all  $2m$ -dimensional Kähler metrics which admit Hamiltonian 2-forms. The existence of a Hamiltonian 2-form leads to the existence of  $m$  (Hamiltonian) Killing vector fields that commute. The co-efficients of the “momentum polynomial,”  $p(t) = \det(\phi - t\omega)$ , are the Hamiltonian functions for the Killing vector fields. Further the roots of the momentum polynomial provide special coordinates which permit an explicit classification of the metric. In general situations, the  $m$  Killing vector fields may not all be linearly independent. A Hamiltonian two-form of order  $l \leq m$  leads to  $l$  linearly independent Killing vector fields. Thus some (i.e.,  $(m - l)$ ) of the roots of the momentum polynomial are constants and hence can’t provide for coordinates.

ACG have shown that the existence of a Hamiltonian two-form of order  $l$  implies that

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<sup>4</sup>For the various equivalent and more precise definitions, see [12, 23]



- (i) the Kähler metric on  $X$  can locally be written as a fibration (using a construction due to Pedersen and Poon [24]), with a  $2l$ -dimensional toric fibre over a  $(2m - 2l)$  dimensional base,
- (ii) the Kähler structure of the manifold, i.e.,  $(g, J, \omega)$  are completely specified by  $l$  functions of one-variable and the Kähler structure of the base.

When  $l = m$ , the manifold is necessarily toric though not all toric manifolds admit a Hamiltonian 2-form of order  $l = m$ . Thus such manifolds are called *orthotoric* reflecting the extra structure. The results of [23], (3.4) is the special case,  $m = 2, l = 2$ . The other extreme,  $l = 0$  is the situation with no Killing vector fields. Thus, the results of ACG provides a nice classification of Kähler manifolds that takes one from manifolds with no symmetries to orthotoric Kähler manifolds.

In this paper since we are interested mainly in metrics on six-dimensional manifolds, we will focus on the case  $m = 3$ , when the possible values for  $l = 1, 2, 3$ . In [12], the term orthotoric is used for the  $l = m$  case and we shall do the same. We will add a subscript ‘OT $m$ ’ to indicate the  $2m$ -dimensional orthotoric metric. In all other situations, we will indicate the values of  $m$  and  $l$  in the subscript. The momentum polynomial has no constant roots

$$p(t) = (t - \xi)(t - \eta)(t - \chi) \quad (3.5)$$

and the Hamiltonian functions for the Killing vector fields  $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3}$  are

$$P_1 = \xi + \eta + \chi, \quad P_2 = \xi\eta + \eta\chi + \chi\xi, \quad P_3 = \xi\eta\chi \quad (3.6)$$

The most general orthotoric metric admitting a Hamiltonian 2-form is then given in terms of three polynomials of one variable:

$$\begin{aligned} ds_{\text{OT3}}^2 = & -\Delta \left[ \frac{d\xi^2}{(\eta - \chi)f(\xi)} + \frac{d\eta^2}{(\chi - \xi)g(\eta)} + \frac{d\chi^2}{(\xi - \eta)h(\chi)} \right] \\ & - \frac{1}{\Delta} \left[ (\eta - \chi) f(\xi) (dt_1 + (\eta + \chi)dt_2 + \eta\chi dt_3)^2 \right. \\ & \quad + (\chi - \xi) g(\eta) (dt_1 + (\chi + \xi)dt_2 + \chi\xi dt_3)^2 \\ & \quad \left. + (\xi - \eta) h(\chi) (dt_1 + (\xi + \eta)dt_2 + \xi\eta dt_3)^2 \right]. \end{aligned} \quad (3.7)$$

where  $\Delta = (\xi - \eta)(\eta - \chi)(\chi - \xi)$ . The Kähler form, the Hamiltonian 2-form and

the scalar curvature ( $R$ ) for the  $m = 3, l = 3$  ACG metrics are given by

$$\omega_{\text{OT3}} = dP_1 \wedge dt_1 + dP_2 \wedge dt_2 + dP_3 \wedge dt_3, \quad (3.8)$$

$$\phi_{\text{OT3}} = [P_1 dP_1 - dP_2] \wedge dt_1 + [P_2 dP_1 - dP_3] \wedge dt_2 + P_3 dP_1 \wedge dt_3, \quad (3.9)$$

$$R_{\text{OT3}} = -\frac{f''(\xi)}{(\xi - \eta)(\xi - \chi)} - \frac{g''(\eta)}{(\eta - \xi)(\eta - \chi)} - \frac{h''(\chi)}{(\chi - \eta)(\chi - \xi)}. \quad (3.10)$$

We will also need the  $m = 3, l = 2$  case, when the momentum polynomial is

$$p(t) = (t - a)(t - \xi)(t - \eta), \quad a = \text{constant}. \quad (3.11)$$

We then have only two Hamiltonian functions for the Killing vector fields  $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}$  viz.  $P_1 = \xi + \eta, P_2 = \xi \eta$ . Thus the roots of the momentum polynomial will provide two of the coordinates for the local description. We will refer to these Kähler metrics admitting Hamiltonian 2-forms as the  $m = 3, l = 2$  ACG metrics.

The most general  $m = 3, l = 2$  ACG metric is then given by [12]

$$ds_{m=3,l=2}^2 = (a - \xi)(a - \eta) ds_a^2 + (\xi - \eta) \left[ \frac{\xi - a}{f(\xi)} d\xi^2 - \frac{\eta - a}{g(\eta)} d\eta^2 \right] + \frac{1}{\xi - \eta} \left[ \frac{f(\xi)}{\xi - a} (\theta_1 + \eta \theta_2)^2 - \frac{g(\eta)}{\eta - a} (\theta_1 + \xi \theta_2)^2 \right], \quad (3.12)$$

where  $ds_a^2$  is a Kähler metric on a two-dimensional manifold with a Kähler form  $\omega_a$ ,  $\theta_1$  and  $\theta_2$  are one-forms which satisfy the following conditions,

$$d\theta_1 = -a \omega_a, \quad d\theta_2 = \omega_a. \quad (3.13)$$

The Kähler form, Hamiltonian 2-form and the scalar curvature for the  $m = 3, l = 2$  ACG metrics are given by

$$\omega_{m=3,l=2} = (a - \xi)(a - \eta) \omega_a + d(\xi + \eta) \wedge \theta_1 + d(\xi \eta) \wedge \theta_2, \quad (3.14)$$

$$\phi_{m=3,l=2} = a(a - \xi)(a - \eta) \omega_a + [P_1 dP_1 - dP_2] \wedge d\theta_1 + P_2 dP_1 \wedge d\theta_2 \quad (3.15)$$

$$R_{m=3,l=2} = \frac{R(ds_a^2)}{(a - \xi)(a - \eta)} - \frac{f''(\xi)}{(\xi - \eta)(\xi - a)} - \frac{g''(\eta)}{(\eta - \xi)(\eta - a)}, \quad (3.16)$$

where  $R(ds_a^2)$  is the scalar curvature of the metric  $ds_a^2$ .

We will also need the  $m = 3, l = 1$  case, when there are *two* possibilities for the momentum polynomial:

$$p_1(t) = (t - a)^2 (t - \xi), \quad a = \text{constant}, \quad (3.17)$$

$$p_2(t) = (t - a)(t - b)(t - \xi), \quad a, b = \text{constants}, \quad (3.18)$$

with  $a < b$ . We then have only one Hamiltonian function for the Killing vector field  $\frac{\partial}{\partial t_1}$  viz.  $P_1 = \chi$ . We will refer to these Kähler metrics admitting Hamiltonian 2-forms as the  $m = 3, l = 1$  ACG metrics. The most general  $m = 3, l = 1$  ACG metric is of either of two types depending on the momentum polynomial. For (3.17), the most general  $m = 3, l = 1$  ACG metric is given by [12],

$$\left[ ds_{m=3,l=1}^2 \right]_{p_1(t)} = (a - \chi) ds_a^2 + \frac{(\chi - a)^2}{h(\chi)} d\chi^2 + \frac{h(\chi)}{(\chi - a)^2} \theta_1^2, \quad (3.19)$$

where  $ds_a^2$  is a Kähler metric for the four-dimensional base with Kähler form  $\omega_a$  and  $\theta_1$  is a one-form which satisfies

$$d\theta_1 = -\omega_a. \quad (3.20)$$

The Kähler form, Hamiltonian 2-form and scalar curvature for the above  $m = 3, l = 1$  ACG metrics are

$$\left[ \omega_{m=3,l=1} \right]_{p_1(t)} = (a - \chi) \omega_a + d\chi \wedge \theta_1, \quad (3.21)$$

$$\left[ \phi_{m=3,l=1} \right]_{p_1(t)} = a(a - \chi) \omega_a + \chi d\chi \wedge \theta_1, \quad (3.22)$$

$$\left[ R_{m=3,l=1} \right]_{p_1(t)} = \frac{R(ds_a^2)}{(a - \chi)} - \frac{h''(\chi)}{(\chi - a)^2}, \quad (3.23)$$

where  $R(ds_a^2)$  is the scalar curvature of the metric  $ds_a^2$ .

For the momentum polynomial (3.18), the most general  $m = 3, l = 1$  ACG metric is given by [12],

$$\left[ ds_{m=3,l=1}^2 \right]_{p_2(t)} = (a - \chi) ds_a^2 + (b - \chi) ds_b^2 + \frac{(\chi - a)(\chi - b)}{h(\chi)} d\chi^2 + \frac{h(\chi)}{(\chi - a)(\chi - b)} \theta_1^2, \quad (3.24)$$

where  $ds_a^2$  and  $ds_b^2$  are two Kähler metrics with Kähler forms  $\omega_a$  and  $\omega_b$  with the one-form  $\theta_1$  satisfying

$$d\theta_1 = -\omega_a - \omega_b. \quad (3.25)$$

The Kähler form, Hamiltonian 2-form and the scalar curvature for the above  $m = 3, l = 1$  ACG metrics are given by

$$\left[ \omega_{m=3,l=1} \right]_{p_2(t)} = (a - \chi) \omega_a + (b - \chi) \omega_b + d\chi \wedge \theta_1, \quad (3.26)$$

$$\left[ \phi_{m=3,l=1} \right]_{p_2(t)} = a(a - \chi) \omega_a + b(b - \chi) \omega_b + \chi d\chi \wedge \theta_1, \quad (3.27)$$

$$\left[ R_{m=3,l=1} \right]_{p_2(t)} = \frac{R(ds_a^2)}{(a - \chi)} + \frac{R(ds_b^2)}{(b - \chi)} - \frac{h''(\chi)}{(\chi - a)(\chi - b)}. \quad (3.28)$$

### 3.1 Imposing extra conditions on the ACG metrics

In the sequel, we will find examples for resolutions of metric cones in the  $m = 3$ ,  $l = 2$  ACG case (3.12) and the  $m = 3$ ,  $l = 1$  ACG cases (3.19) and (3.24). We will therefore gather some more facts about these cases, mainly restrictions imposed on the polynomials  $f(\xi)$ ,  $g(\eta)$  and  $h(\chi)$  by conditions such as Ricci-flatness, Einstein, etc.

#### Extremality

A Kähler metric is said to be *extremal* when the scalar curvature is the Hamiltonian function for a Killing vector field. In our examples, this implies that the scalar curvature is an affine function of the Hamiltonian function,  $P_1$ . For instance for the OT3 metric, this occurs when  $f''(x) = g''(x) = h''(x)$  and the functions  $f/g/h$  have degree four. For all other metrics, additional conditions arise and are described below.

If further, one requires a slightly stronger condition (as ACG do) than the one required by extremality, i.e.,  $f'(x) = g'(x) = h'(x)$ , then the normalized Ricci form,

$$\tilde{\rho} \equiv \rho - \frac{R\omega}{8},$$

is a Hamiltonian 2-form and can be written as a linear combination of the Hamiltonian 2-form,  $\phi$  and the Kähler form.

#### The Einstein condition

The ACG metrics are Einstein metrics when the following three conditions are satisfied:

- (i)  $f'(x) = g'(x) = h'(x)$  and furthermore they should factorize in the following way:

$$\begin{aligned} m = 3, l = 3, \quad f'(x) &= b_{-1}x^4 + b_0x^3 + b_1x^2 + b_2x + b_3 \\ m = 3, l = 2, \quad f'(x) &= (x - a)(b_{-1}x^3 + b_0x^2 + b_1x + b_2) \\ m = 3, l = 1 \text{ with } p_1(t), \quad h'(x) &= (x - a)^2(b_{-1}x^2 + b_0x + b_1) \quad (3.29) \\ m = 3, l = 1 \text{ with } p_2(t), \quad h'(x) &= (x - a)(x - b)(b_{-1}x^2 + b_0x + b_1) \end{aligned}$$

for some constants  $b_i$ .

- (ii) the smaller Kähler metrics,  $ds_a^2$ ,  $ds_b^2$  should be Kähler-Einstein with their scalar curvatures satisfying the following relations:

$$\begin{aligned}
m = 3, l = 2, & \quad -R(ds_a^2) = b_{-1} a^3 + b_0 a^2 + b_1 a + b_2 \\
m = 3, l = 1 \text{ with } p_1(t), & \quad -\frac{R(ds_a^2)}{2} = b_{-1} a^2 + b_0 a + b_1 \\
m = 3, l = 1 \text{ with } p_2(t), & \quad -R(ds_a^2) = b_{-1} a^2 + b_0 a + b_1 \text{ and} \\
& \quad -R(ds_b^2) = b_{-1} b^2 + b_0 b + b_1 . \quad (3.30)
\end{aligned}$$

- (iii) The Ricci-form is then given by

$$\rho = -\frac{1}{2} [b_{-1}(\phi + P_1\omega) + b_0\omega] . \quad (3.31)$$

which clearly leads to an Einstein metric when

$$b_{-1} = 0 . \quad (3.32)$$

The scalar curvature for these Einstein manifolds then is equal to  $-3b_0$ .

### The Ricci-flatness condition

For the ACG metrics to be Ricci-flat as well, one needs

$$b_0 = 0 . \quad (3.33)$$

As we will be interested in Ricci-flat metrics, we note that we will end up with the functions  $f/g/h$  being cubic polynomials. This is all we need for orthotoric metrics. For the ACG metrics with  $l < m = 3$ , it is useful to explicitly write out the conditions that are imposed on the smaller Kähler metrics.

1. When  $m = 3, l = 2$ , One needs  $R(ds_a^2) = -b_1 a - b_2$ .
2. When  $m = 3, l = 1$  with polynomial  $p_1(t)$ , one needs  $R(ds_a^2) = -2b_1$ .
3. When  $m = 3, l = 1$  with polynomial  $p_2(t)$ , one needs  $R(ds_a^2) = R(ds_b^2) = -b_1$ .

## 4 Symplectic potentials for the ACG metrics

In this section, we compute the symplectic potentials for the ACG metrics<sup>5</sup>. We first start with the orthotoric case with  $l = m = 2$  since this is the first non-trivial orthotoric metric. The symplectic potential for higher-dimensional orthotoric metrics and other ACG metrics follows from this case.

### 4.1 The $m = 2$ orthotoric symplectic potential

The coordinate transformation that gives  $(\xi, \eta)$  as a function of  $(P_1, P_2)$  is obtained by identifying  $(\xi, \eta)$  with the roots of the quadratic equation

$$\lambda^2 - P_1\lambda + P_2 = 0 . \quad (4.1)$$

The  $m = 2$  orthotoric metric in the coordinates  $(P_1, P_2)$  is given by

$$ds_{\text{OT2}}^2 = g_{ij}dP_i dP_j + g^{ij}dt_i dt_j \quad (4.2)$$

where

$$g_{ij} = \frac{1}{\xi - \eta} \begin{pmatrix} \left[ \frac{\xi^2}{f(\xi)} - \frac{\eta^2}{g(\eta)} \right] & -\left[ \frac{\xi}{f(\xi)} - \frac{\eta}{g(\eta)} \right] \\ -\left[ \frac{\xi}{f(\xi)} - \frac{\eta}{g(\eta)} \right] & \left[ \frac{1}{f(\xi)} - \frac{1}{g(\eta)} \right] \end{pmatrix}$$

We can check that this metric can indeed be written as the Hessian of a symplectic potential. The integrability condition is given by

$$\partial_i(g_{jk}) = \partial_j(g_{ik}) ,$$

which holds in our case.

We will write the symplectic potential as an explicit function of  $(\xi, \eta)$  and hence as an implicit function of  $(P_1, P_2)$ . Let  $G(\xi, \eta)$  be the symplectic potential for the above metric. Then, one has

$$\begin{aligned} g_{ij} &= \partial_i \partial_j G(\xi, \eta) \\ &= \left[ \frac{\partial^2 G}{\partial \xi^2} \partial_i \xi \partial_j \xi + \frac{\partial^2 G}{\partial \xi \partial \eta} (\partial_i \xi \partial_j \eta + \partial_j \xi \partial_i \eta) + \frac{\partial^2 G}{\partial \eta^2} \partial_i \eta \partial_j \eta \right] + \left[ \frac{\partial G}{\partial \xi} \partial_i \partial_j \xi + \frac{\partial G}{\partial \eta} \partial_i \partial_j \eta \right] . \end{aligned} \quad (4.3)$$

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<sup>5</sup>The symplectic potential for the  $Y^{pq}$  and  $L^{pq}$  metrics has been obtained in ref. [25].

The easiest of the three partial differential equations turns out to be the one for  $g_{22}$  which reads

$$\left[ \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^2 - \frac{2}{\xi - \eta} \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \right] G = (\xi - \eta) \left( \frac{1}{f(\xi)} - \frac{1}{g(\eta)} \right). \quad (4.4)$$

Let us assume that  $f$  and  $g$  are polynomial functions with distinct roots  $(\xi_1, \dots, \xi_N)$  and  $(\eta_1, \dots, \eta_{\tilde{N}})$  respectively. We will need the inverses of  $f$  and  $g$ , which we write as  $(f_0$  and  $g_0$  are constants that turn out to be proportional to the scalar curvature. We deliberately include a minus sign so that the constants are positive in our examples.)

$$\frac{1}{f(\xi)} = \frac{1}{-f_0 \prod_{a=1}^N (\xi - \xi_a)} = \sum_{a=1}^N \frac{A_a}{(\xi - \xi_a)}, \quad (4.5)$$

$$\frac{1}{g(\eta)} = \frac{1}{-g_0 \prod_{a=1}^{\tilde{N}} (\eta - \eta_a)} = \sum_{a=1}^{\tilde{N}} \frac{B_a}{(\eta - \eta_a)}, \quad (4.6)$$

where

$$A_a \equiv -f_0^{-1} \prod_{b \neq a} (\xi_a - \xi_b)^{-1}, \quad \text{and} \quad B_a \equiv -g_0^{-1} \prod_{b \neq a} (\eta_a - \eta_b)^{-1}. \quad (4.7)$$

Since the partial differential equations are linear, we can use superposition. So, all we really need to do is to solve for the simple case when  $f = (\xi - \xi_1)$  and dropping the term involving  $g(\eta)$ . It turns out that this is solved by the function  $\mathcal{S}$  which we define as follows:

$$\mathcal{S}(\xi, \eta, a) = \frac{1}{2}(\xi - \eta)(\xi - a) - (\eta - a)(\xi - a) \log(\xi - a). \quad (4.8)$$

This is the solution to the differential equation

$$\left[ \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^2 - \frac{2}{\xi - \eta} \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \right] \mathcal{S}(\xi, \eta, \xi_1) = \frac{\xi - \eta}{\xi - \xi_1}. \quad (4.9)$$

One can verify that the other partial differential equations for  $g_{11}$  and  $g_{12}$  are also satisfied. Thus, we can now write the symplectic potential explicitly as follows:

$$G(\xi, \eta) = \sum_{a=1}^N A_a \mathcal{S}(\xi, \eta, \xi_a) + \sum_{a=1}^{\tilde{N}} B_a \mathcal{S}(\eta, \xi, \eta_a). \quad (4.10)$$

When  $N$  and  $\tilde{N}$  are both greater than two, a slightly simpler form follows on using the identities  $\sum_a A_a = \sum_a A_a \xi_a = 0$  and a similar one for the  $B$ 's. In this case, we can write,

$$G_{\text{OT2}}(\xi, \eta) = - \sum_{a=1}^N A_a (\eta - \xi_a)(\xi - \xi_a) \log(\xi - \xi_a) - \sum_{a=1}^{\tilde{N}} B_a (\xi - \eta_a)(\eta - \eta_a) \log(\eta - \eta_a) \quad (4.11)$$

where  $\xi_a$  (resp.  $\eta_a$ ) are the *distinct* roots of  $f(\xi)$  (resp.  $g(\eta)$ ). Note that the coefficient of each of the logarithms can be rewritten in terms of linear functions of  $(P_1, P_2)$ . For instance,

$$(\xi - \xi_1)(\eta - \xi_1) = P_2 - P_1 \xi_1 + \xi_1^2 = p(\xi_1) , \quad (4.12)$$

where the last term is the momentum polynomial for this case, i.e.,  $p(t) = (t - \xi)(t - \eta)$ . This observation enables us to analyse global properties of the orthotoric as well as ACG metrics.

We conclude this discussion with a comment on the symplectic potential for the other cases such as the  $m = 3$  orthotoric metric. This involves adding a piece corresponding to the roots of the third function,  $h(\chi)$  and pre-multiplying the argument of all logarithms so that they can be written as the linear function of  $(P_1, P_2, P_3)$  given by the momentum polynomial,  $p(t)$ .

## 4.2 The $m = 3, l = 2$ symplectic potential

In order to be more concrete, we choose the two-dimensional metric  $ds_a^2$  to be the Fubini-Study metric for a  $\mathbb{CP}^1$  with symplectic potential

$$G_{FS}(x) = \frac{1}{2}(1 - x) \log(1 - x) + \frac{1}{2}(1 + x) \log(1 + x) . \quad (4.13)$$

The scalar curvature for the above metric is 2. The natural choice for the symplectic coordinates which follows from the Kähler form is

$$P_1 = (\xi + \eta) , \quad P_2 = \xi\eta \text{ and } P_3 = (a - \xi)(a - \eta)x . \quad (4.14)$$

A calculation similar to the one used to derive the symplectic potential for the  $m = 2$  orthotoric case leads to the following symplectic potential for the  $m = 3$ ,



$l = 2$  ACG metric. We obtain (assuming  $N > 2$  and  $\tilde{N} > 2$ )

$$\begin{aligned}
G_{m=3,l=2}(x, \xi, \eta) &= (a - \xi)(a - \eta)G_{FS}(x) \\
&\quad - \sum_{a=1}^N A_a (a - \xi_a)(\eta - \xi_a)(\xi - \xi_a) \log(\xi - \xi_a) \\
&\quad - \sum_{a=1}^{\tilde{N}} B_a (a - \xi_a)(\xi - \eta_a)(\eta - \eta_a) \log(\eta - \eta_a) ,
\end{aligned} \tag{4.15}$$

where  $A_a$  and  $B_a$  are as defined in (4.7). Again, note the appearance of  $p(t)$  in the coefficient of the logarithms and the coefficient of the Fubini-Study metric is simply  $p(t)$  without the constant root – this is called  $p_{nc}(t)$  in [12].

### 4.3 The $m = 3$ , $l = 1$ symplectic potentials

**For momentum polynomial**  $p_1(t) = (t - a)^2(t - \chi)$

Let  $p_{nc}(t) = (t - \chi)$  be the part of the momentum polynomial involving the non-constant root  $\chi$ . Further, let us assume that  $h$  is a polynomial of degree  $N$  with distinct roots  $\chi_1, \dots, \chi_N$ . Then we can write  $h(\chi) = -h_0 \prod_{r=1}^N (\chi - \chi_r)$ . Then, the symplectic potential takes the form

$$\left[ G_{m=3,l=1} \right]_{p_1(t)} = - \sum_{r=1}^N C_r p_1(\chi_r) \log(\chi - \chi_r) + p_{nc}(a) G_a , \tag{4.16}$$

where  $C_r \equiv -h_0^{-1} \prod_{s \neq r} (\chi_r - \chi_s)^{-1}$  and  $G_a$  is the symplectic potential for the small metric  $ds_a^2$ .

**For momentum polynomial**  $p_2(t) = (t - a)(t - b)(t - \chi)$

The symplectic potential takes the form ( $p_{nc}(t) = (t - \chi)$ )

$$\left[ G_{m=3,l=1} \right]_{p_2(t)} = - \sum_{r=1}^N C_r p_2(\chi_r) \log(\chi - \chi_r) + p_{nc}(a) G_a + p_{nc}(b) G_b , \tag{4.17}$$

where we have again assumed that  $h(\chi)$  is a polynomial of degree  $N$ . Further  $G_a$  (resp.  $G_b$ ) is the symplectic potential for the small metric  $ds_a^2$  (resp.  $ds_b^2$ ).

## 4.4 Global Analysis of the ACG metrics

While most of our analysis will hold in generality, we will restrict all our considerations to the situation when the functions  $f$  and  $g$  are cubic functions. Let

$$f(\xi) = -f_0(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3) \quad , \quad g(\eta) = -g_0(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3) \quad , \quad (4.18)$$

with the ordering  $\xi_1 < \xi_2 < \xi_3$  and  $\eta_1 < \eta_2 < \eta_3$  when the roots are all real. If  $f$  has complex roots, we choose them to be  $\xi_2$  and  $\xi_3$  and similarly for the function  $g$ . We also will assume  $f_0$  and  $g_0$  are real and positive. We will now consider the various ACG metrics and require that the metric be positive-definite.

$m = 2$  **orthotoric metrics:** The following conditions are needed

$$\xi > \eta \quad , \quad \xi < \xi_1 \text{ or } \xi_2 < \xi < \xi_3 \quad , \quad \eta_1 < \eta < \eta_2 \text{ or } \eta > \eta_3 \quad ,$$

and a similar set of conditions if we require  $\xi < \eta$ . If we require that the four-dimensional space be compact, clearly, we obtain the condition that  $\xi_2 > \eta_2$  thus satisfying  $\xi > \eta$  everywhere in the interior. The metric is singular on the boundary of a rectangle in the  $\xi - \eta$  plane. These metrics lead to  $\mathbb{CP}^2$  when the functions  $f$  and  $g$  are identical [26].

$m = 3, l = 1$  **ACG metrics:** Let us assume that the non-compact coordinate is  $\chi$ . Then, positivity of the metric is assured when  $\chi < \chi_1 < a$ , where  $\chi_1$  is the smallest root of  $h(\chi)$ . We choose the four-dimensional metric to be the one given by the  $m = 2$  orthotoric metric. Again, the singularities are given by those of the  $m = 2$  orthotoric metric that we just considered and at  $\chi = \chi_1$ . These metrics will be shown to lead to complex cones over  $L^{pq}$  spaces when  $f \neq g$  for specific choices of the polynomials.

$m = 3, l = 2$  **ACG metrics:** The relevant conditions are

$$\xi < a, \quad \eta < a, \quad \xi > \eta, \quad \xi_1 < \xi < \xi_2 \text{ or } \xi > \xi_3, \quad \eta < \eta_1 \text{ or } \eta_2 < \eta < \eta_3,$$

and of course,  $-1 < x < 1$ . In this example, we will be interested in the situation when we have non-compact domain in the  $\xi - \eta$  plane given by  $\eta < \eta_1 < \xi_1 < \xi < \xi_2 < a$ . The singularities of the metric occur at  $x = \pm 1$  (these are the singularities of the FS metric),  $\eta = \eta_1, \xi = \xi_1, \xi_2$ . This choice leads to partially resolved cones over  $Y^{pq}$  as we will show in the sequel.

## 4.5 Recovering the canonical potential

Consider the simple situation of the  $m = 2$  orthotoric symplectic potential corresponding to  $f = A_1/(\xi - \xi_1)$ . Ignoring the non-logarithmic pieces, the symplectic potential given in (4.8) can be re-written as

$$\frac{1}{2}\ell(P) \log \ell(P) + A_1(\xi - \xi_1)(\eta - \xi_1) \log(\eta - \xi_1) \quad (4.19)$$

where  $\ell(P) = -2A_1(\xi_1 P_1 - P_2 - \xi_1^2)$ . The singularity associated with  $\ell(P) = 0$  is split into two separate singularities in the  $\xi - \eta$  space i.e.,  $\xi = \xi_1$  and  $\eta = \xi_1$ . The ACG metric *subtracts* out one of the two singularities and hence has the form  $G_{\text{can}} + h(P)$ . This simple example shows how one can re-write all the symplectic potentials that we have obtained into the form

$$G_{\text{can}} + h(P) ,$$

where  $G_{\text{can}}$  contains only the  $\ell \log \ell$  pieces associated with the singularities that we obtained from our global analysis. All the remaining pieces are grouped together into the Abreu function  $h(P)$ . We can then use the canonical potential to figure out the moment polytope.

## 5 Unresolved Ricci-flat metrics: examples

We consider the  $m = 3, l = 1$  ACG metrics with momentum polynomial  $p_1(t)$ . We define  $P = (a - \chi)$  and choose the cubic polynomial to be  $h(\chi) = 2(a - \chi)^3$ . Then the metric in Eqn. (3.19) is the metric for the complex cone over a four-dimensional Kähler manifold. The symplectic potential then takes the form

$$G_{m=3,l=1} = \frac{1}{2}P \log P + P G_a(\tilde{P}_1, \tilde{P}_2) , \quad (5.1)$$

where  $\tilde{P}_i \equiv P_i/P$  and  $G_a$  is the symplectic potential for a four-dimensional manifold which we take to be  $m = 2$  orthotoric manifold. Thus, we have  $G_a = G_{\text{OT}2}$  as defined in Eqn. (4.11). The Ricci-flatness of the above metric requires  $G_a(\tilde{P}_1, \tilde{P}_2)$  to be Kähler-Einstein with scalar curvature equal to 4 among other things. This is achieved if we choose  $f'(x) = g'(x)$  and  $f_0 = g_0 = \frac{2}{3}$ .

Focusing on Einstein spaces amongst the ACG metrics in four dimensions, we thus need to consider cubic function  $f$  and  $g$  such that

$$f(x) = -\frac{2}{3}x^3 + f_1x^2 + f_2x + f_3, \quad g(x) = f(x) + \mu, \quad (5.2)$$

with  $\mu \neq 0$  and  $(f_1, f_2, f_3)$  are constants. Let  $\xi_1 < \xi_2 < \xi_3$  be the distinct real roots of  $f$  and  $\eta_1 < \eta_2 < \eta_3$  be distinct real roots of  $g$ . As discussed earlier, we choose the values of  $\xi$  and  $\eta$  are such that  $\eta_1 < \eta < \eta_2 < \xi_2 < \xi < \xi_3$ . This implies that the singularities occur on the boundary of a rectangle in the  $\xi - \eta$  plane. In the  $(\tilde{P}_1 = \xi + \eta, \tilde{P}_2 = \xi\eta)$  plane, the rectangle is given by the conditions  $\ell_a = 0$  where

$$\begin{aligned} \ell_1(\tilde{P}_1, \tilde{P}_2) &= \frac{-2}{f_0(\eta_1 - \eta_2)(\eta_1 - \eta_3)}(\eta_1\tilde{P}_1 - \tilde{P}_2 - \eta_1^2), \\ \ell_2(\tilde{P}_1, \tilde{P}_2) &= \frac{-2}{f_0(\eta_2 - \eta_1)(\eta_2 - \eta_3)}(\eta_2\tilde{P}_1 - \tilde{P}_2 - \eta_2^2), \\ \ell_3(\tilde{P}_1, \tilde{P}_2) &= \frac{-2}{f_0(\xi_2 - \xi_1)(\xi_2 - \xi_3)}(\xi_2\tilde{P}_1 - \tilde{P}_2 - \xi_2^2), \\ \ell_4(\tilde{P}_1, \tilde{P}_2) &= \frac{-2}{f_0(\xi_3 - \xi_1)(\xi_3 - \xi_1)}(\xi_3\tilde{P}_1 - \tilde{P}_2 - \xi_3^2). \end{aligned} \quad (5.3)$$

The four functions are linearly-dependent. We assume that the dependence is given by four positive integers  $(p, q, r, s)$  such that (assuming  $q > p$  and  $s > r$ )

$$p \ell_1(\tilde{P}_1, \tilde{P}_2) + q \ell_2(\tilde{P}_1, \tilde{P}_2) - s \ell_3(\tilde{P}_1, \tilde{P}_2) - r \ell_4(\tilde{P}_1, \tilde{P}_2) = 0,$$

The condition  $p + q = r + s$  may be assumed at this point but it can be obtained as a consistency condition. For simplicity, we assume that this is true. These spaces turn out to be real cones over five-dimensional spaces called  $L^{pqr}$  [22]. The explicit map relating this  $m = 3, l = 1$  ACG metric to metrics given in [22] has been obtained by Martelli and Sparks (in [27]) and we shall not present them here. We instead pursue our analysis to completion. The four integers should determine the functions  $f$  and  $g$ . Using the above condition we obtain the three equations after setting  $\eta_1 = 0$  by a simultaneous translation in  $\xi$  and  $\eta$  and choosing  $\eta_2 = 1$  for simplicity as the  $\eta_2$ -dependence can be easily recovered:

$$\begin{aligned} pB_1 + qB_2 - sA_2 - rA_3 &= 0, \\ qB_2 - sA_2\xi_2 - rA_3\xi_3 &= 0, \\ qB_2 - sA_2\xi_2^2 - rA_3\xi_3^2 &= 0, \end{aligned} \quad (5.4)$$

where  $B_1 = 1/\eta_3$ ,  $B_2 = 1/(1 - \eta_3)$ ,  $A_2 = 1/(\xi_2 - \xi_1)(\xi_2 - \xi_3)$  and  $A_3 = 1/(\xi_3 - \xi_1)(\xi_3 - \xi_2)$ . Note that it seems that we have four variables to determine,  $\eta_3$  and the three roots of  $g$ . However, the two functions  $f$  and  $g$  are such that their roots satisfy

$$\xi_1 + \xi_2 + \xi_3 = \eta_1 + \eta_2 + \eta_3, \text{ and } \xi_1\xi_2 + \xi_3\xi_1 + \xi_2\xi_3 = \eta_1\eta_2 + \eta_3\eta_1 + \eta_2\eta_3.$$

This enables us to solve for, say,  $\eta_3$  and  $\xi_1$  in terms of  $\xi_2$  and  $\xi_3$  to obtain:

$$\eta_3 = 1 - \frac{\xi_2\xi_3}{\xi_2 + \xi_3 - 1} \quad \text{and} \quad \xi_1 = (\xi_2 + \xi_3) - \frac{\xi_2\xi_3}{\xi_2 + \xi_3 - 1}. \quad (5.5)$$

Thus, Eqn. (5.4) now becomes three equations for two variables,  $\xi_2$  and  $\xi_3$  given four integers  $(p, q, r, s)$  such that  $p + q - r - s = 0$ .

One can also view Eqn. (5.4) as an equation for three rational numbers  $\frac{1}{p}(q, r, s)$  given  $\xi_2$  and  $\xi_3$ . One can show that the solution is such that  $p + q - r - s = 0$  and one has

$$\begin{aligned} \frac{q - p}{q + p} &= \frac{(u + v)(u^2 + v^2 - 2)}{uv(u^2 + uv + v^2 - 2) - 1} > 0, \\ \frac{s - r}{p + q} &= \frac{(v - u)(u + v)(1 + uv)}{uv(u^2 + uv + v^2 - 2) - 1} > 0, \end{aligned} \quad (5.6)$$

where we have defined

$$\xi_2 = \frac{1 + u}{2}, \quad \xi_3 = \frac{1 + v}{2} \quad \text{with } v > u > 1.$$

The above range of  $(u, v)$  is consistent with the condition that  $q > p$  and  $s > r$  that we assumed at the beginning. It is easy to see that when  $u$  and  $v$  are rational, one is guaranteed to obtain integers for  $(p, q, r, s)$ . This solution is similar to the one considered in [28].

Consider the example when  $(p, q, r, s) = (1, 4, 2, 3)$ . We solve for  $(u, v)$  numerically as the explicit answers are unilluminating. We obtain that  $(u, v) = (1.8933, 2.3258)$  – this is the only solution that satisfies  $v > u > 1$ . This implies that  $\eta_3 = 1.96867$  and  $(\xi_1, \xi_2, \xi_3) = (-0.14065, 1.44665, 1.66229)$ . Again this is consistent with the ordering of the roots that we assumed.

It is of interest to ask what happens when  $r = s$  in (5.6). It is not hard to see that this is achieved when  $u = v$ . In other words, one has  $\xi_2 = \xi_3$  and one side of the rectangle shrinks to zero size. The root  $\xi$  now becomes a constant root. The

singularity may be ‘resolved’ by associating a  $\mathbb{CP}^1$  with the constant root. This provides an intuitive understanding of our next attempt to find metrics for  $Y^{pq}$  from the  $(m = 3, l = 2)$  ACG metrics.

## 6 Partially resolved Ricci-flat metrics: examples

### 6.1 Cones over $Y^{pq}$ spaces

The toric data for general  $Y^{p,q}$  (with  $p > q$  and  $\gcd(p, q) = 1$ ) is given by the four vectors [29]

$$\mathcal{V} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & p - q - 1 & p & 1 \\ 0 & p - q & p & 0 \end{pmatrix} .$$

One can verify that the most general internal point is of the form  $(1, k, k)$  with  $k \in \{1, \dots, (p-1)\}$ . Internal points correspond to blowing up four-cycles and we intend to add one internal point and obtain the Ricci-flat metric on the resulting space. Now, with one internal point added, the toric data is

$$\mathcal{V}_{+1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & p - q - 1 & p & 1 & k \\ 0 & p - q & p & 0 & k \end{pmatrix} .$$

The general D-terms for  $Y^{p,q}$  spaces with one internal point added can be computed by considering the null space to  $\mathcal{V}_{+1}$  and turns out to be

$$\begin{aligned} (p - q)\ell_1 + (p + q)\ell_3 - p(\ell_2 + \ell_4) &= r_1 , \\ (-k + p)\ell_1 + k\ell_3 - p\ell_5 &= r_2 , \end{aligned} \tag{6.1}$$

where we have also turned on the blow-up (F-I) parameters which we call  $r_1$  and  $r_2$ . We thus have the five  $\ell_a$  being subject to these two conditions. This effectively leaves us with three independent fields. We choose these independent fields to be  $(P_1, P_2, P_3)$ .

The metrics for  $Y^{pq}$  spaces were first obtained in [30, 31]. Real cones over these spaces have a conical singularity at the tip of the cone. Resolved metrics for these spaces have been not been found except for the conifold (and its  $\mathbb{Z}_2$

orbifold). The conifold is obtained as a real cone over  $Y^{1,0} = T^{1,1}$ . An intriguing result was obtained in [32] where they obtained a resolved metric for the cone over  $Y^{2,1}$ . What was different about this result was the fact that the blow-up parameters were set to fixed values. We realised that the metric looked like the  $m = 3, l = 2$  ACG metric and verified that it was indeed true. This was our inspiration to look more closely at this class of ACG metrics and see if we could achieve similar results for general  $Y^{p,q}$ . Further, the defining D-term for cones over  $Y^{p,q}$  spaces clearly has a  $\mathbb{CP}^1$  corresponding to  $\ell_2 + \ell_4 = \text{constant}$  in Eqn. (6.1). We now systematically fit the  $m = 3, l = 2$  ACG metrics to the two D-terms that appear in Eqn. (6.1). Higher dimensional generalisations of the result of [32] have appeared in [33]. Our result provides examples in six dimensions that appear to be new.

## 6.2 Fitting to the $m = 3, l = 2$ ACG metrics

We now attempt to fit these metrics in to the  $m = 3, l = 2$  class of ACG metrics. We first set  $ds_a^2$  to the Fubini-Study metric normalised such that the scalar curvature equals 2. As discussed earlier, Ricci-flatness requires  $f$  and  $g$  to be cubic functions such that

$$f'(x) = g'(x) = (x - a)(b_1x + b_2) , \quad (6.2)$$

with  $b_1a + b_2 = -R(ds_a^2) = -2$ . A simultaneous shift in  $\xi$  and  $\eta$  can be done to eliminate the term linear in  $x$  that appears in the functions  $f$  and  $g$ . This is achieved, for instance, by setting  $b_2 = 0$ . We also set  $a = 1$  to match results in the literature. This fixes  $b_1 = -2$ . Thus we obtain

$$f'(x) = -2x(x - 1) \implies f(x) = -\frac{2}{3}x^3 + x^2 + \text{constant} , \quad (6.3)$$

and  $g(x) - f(x)$  is a constant.

We identify  $(P_1, \hat{P}_2, P_3)$  with an  $SL(3, \mathbb{Z})$  transform of the coordinates given in (4.14). The  $SL(3, \mathbb{Z})$  transform is such that  $P_2 = (\hat{P}_2 + P_1 - 1)$  leaving the other two coordinates unchanged. To carry out the fit to the  $Y^{pq}$  D-terms, we identify the five singularities of the  $m = 3, l = 2$  ACG metric with boundary of the  $Y^{pq}$  polytope. The singularities of the  $\mathbb{CP}^1$  are naturally identified with  $\ell_2$

and  $\ell_4$ . We find that the  $\xi = \xi_1$  and  $\xi = \xi_2$  singularities get identified with the  $\ell_1$  and  $\ell_3$  singularities. If the fit has to work, the last singularity  $\eta = \eta_1$  *must* be identified with  $\ell_5 = 0$  singularity. With these inputs, we obtain

$$\begin{aligned}
\ell_1 &= -(1 - \xi_1)A_1 \left[ P_1(\xi_1 - 1) - \hat{P}_2 + 1 - \xi_1^2 \right], \\
\ell_2 &= (\hat{P}_2 + P_3), \\
\ell_3 &= -(1 - \xi_2)A_2 \left[ P_1(\xi_2 - 1) - \hat{P}_2 + 1 - \xi_2^2 \right], \\
\ell_4 &= (\hat{P}_2 - P_3), \\
\ell_5 &= -(1 - \eta_1)B_1 \left[ P_1(\eta_1 - 1) - \hat{P}_2 + 1 - \eta_1^2 \right],
\end{aligned} \tag{6.4}$$

where  $\xi_i$  and  $\eta_i$  are respectively the roots of cubic equations  $f(\xi) = 0$  and  $g(\eta) = 0$ . The roots are taken to have the following ordering:  $\eta < \eta_1 < \xi_1 < \xi < \xi_2$ . The constants  $A_1$ ,  $A_2$  and  $B_1$  are given by

$$A_1 = \frac{-3}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}, \quad A_2 = \frac{-3}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)}, \quad B_1 = \frac{-3}{(\eta_1 - \eta_2)(\eta_1 - \eta_3)}. \tag{6.5}$$

We now need to impose the conditions that the  $\ell_a$  as given above from the  $m = 3, l = 2$  ACG metric satisfies the D-term conditions given in Eqn. (6.1). In the first D-term, one sees that the  $P_3$  drops out and thus leads to two equations corresponding to the vanishing of the coefficients of  $P_1$  and  $\hat{P}_2$ . Further, this does not involve the roots of  $g$  since they appear only in  $\ell_5$ . Here  $f$  is such that its roots satisfy

$$\xi_1 + \xi_2 + \xi_3 = \frac{3}{2}, \quad \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 = 0.$$

Thus, the first D-term is an over-determined system – we have two equations and one unknown – the undetermined constant in  $f$ . It turns out there is indeed a solution.

$$\xi_1 = \frac{2p - 3q - \sqrt{\Delta}}{4p}, \quad \xi_2 = \frac{2p + 3q - \sqrt{\Delta}}{4p}, \quad \xi_3 = \frac{p + \sqrt{\Delta}}{2p}, \tag{6.6}$$

where  $\Delta \equiv (4p^2 - 3q^2)$ . It is easy to verify that the inequality  $\xi_1 < \xi_2 < 1$  is satisfied when  $p > q$ . Note that  $\xi_i$  are *independent* of  $k$ , i.e., the interior point that is blown-up. This is obvious since the second D-term was not used in



determining the roots of  $f$ . The FI parameter  $r_1$  is non-vanishing and is given by

$$r_1 = -\frac{1}{2} \left( \frac{2p^2 - 3q^2}{p} + \sqrt{\Delta} \right) . \quad (6.7)$$

It turns out that  $r_1$  is always negative. This is consistent with our identification of the  $\mathbb{C}\mathbb{P}^1$  arising with from the  $\ell_2 + \ell_4$ .

We now impose the second D-term equation involving  $\ell_5$  and use it to determine the roots of  $g$ . Again, we know that the three roots of  $f$  must satisfy

$$\eta_1 + \eta_2 + \eta_3 = \frac{3}{2} \quad , \quad \eta_1\eta_2 + \eta_2\eta_3 + \eta_3\eta_1 = 0 . \quad (6.8)$$

We can use these two equations to solve for  $\eta_2$  and  $\eta_3$  in terms of  $\eta_1$ . Imposing the D-term leads to the solution

$$\eta_1 = \frac{p(2p - 3q)(p + q) - 2k(2p^2 - 3q^2) - \sqrt{\Delta}(-2kp + p(p + q))}{4(3k^2q + p^2(p + q) - kp(2p + 3q))} \quad (6.9)$$

The second FI parameter is given by

$$r_2 = -(p - k)(1 - \xi_1)A_1(1 - \xi_1^2) - k(1 - \xi_2)A_2(1 - \xi_2^2) + p(1 - \eta_1)B_1(1 - \eta_1^2) \quad (6.10)$$

We do not list the explicit expressions for  $\eta_2, \eta_3$  as we don't really need them. Instead, we just note the value of their sum and product since they appear directly in  $B_1$  which appears in  $\ell_5$ .

$$\eta_2 + \eta_3 = \frac{3}{2} - \eta_1 \quad , \quad \eta_2\eta_3 = -\eta_1\left(\frac{3}{2} - \eta_1\right) . \quad (6.11)$$

The constants that appear in  $f$  and  $g$  can be obtained directly from the roots and we do not give expressions for them.

An important point to note here is that we have not verified that  $\eta_1 < \xi_1$  as that is required by the positivity of the ACG metric. While our expressions seem to be valid for any  $k \in (1, 2, \dots, p - 1)$ , it turns out that in all the examples that we have considered, the inequality is violated when  $k = p - 1$  and  $p > 2$ . Experimentally, we find that for all values of  $k$  that are greater than  $p/2$  and thereabouts, the inequality is violated and we do *not* obtain a resolved metric for those values of  $k$ . For instance, for  $Y^{3,1}$ , we obtain a resolved metric for  $k = 1$  but the one for  $k = 2$  violates the inequality and we do not have a positive definite metric.

The Abreu function may be extracted using the formula

$$G_{m=3,l=2} = \frac{1}{2} \sum_{a=1}^5 \ell_a \log[\ell_a] + h(P_i) . \quad (6.12)$$

We do not write an explicit formula for the Abreu function. We now work out details for some specific values of  $(p, q, k)$ . The polynomials are taken to be<sup>6</sup>

$$f(\xi) = -\frac{2}{3}\xi^3 + \xi^2 + a , \quad g(\eta) = -\frac{2}{3}\eta^3 + \eta^2 + b , \quad (6.13)$$

where we define the constants to be  $a$  and  $b$  (this is not to be confused with our earlier use of the same to indicate constant roots in the momentum polynomial).

### $Y^{2,1}$

There is only one point in the interior of the polytope corresponding to setting  $k = 1$ . The FI parameters are given by

$$r_1 = -\frac{1}{4}(5 + 2\sqrt{13}) , \quad r_2 = \frac{3}{4}(4 + \sqrt{13}) . \quad (6.14)$$

The form of the resolved  $Y^{2,1}$  metric in the [32] can be obtained by carrying out

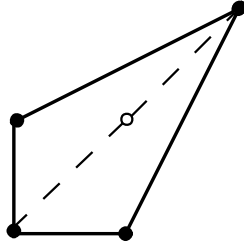


Figure 1: Toric diagram for  $Y^{2,1}$

an  $SL(3, \mathbb{Z})$  transformation such that the new  $P_i$  are given by

$$P_1 = \xi + \eta , \quad P_2 = (1 - \xi)(1 - \eta) , \quad P_3 = (1 - \xi)(1 - \eta)x , \quad (6.15)$$

---

<sup>6</sup>Our metrics differ from the ones usually written for  $Y^{pq}$  [30–32] by a factor of 3 due to our choice of normalisation for the scalar curvature of  $\mathbb{CP}^1$ . Our  $f$  will have to be multiplied by  $-3$  to match with the corresponding cubic function in those papers.

and setting  $x = \cos \theta$ . The roots of the polynomials turn out to be

$$\begin{aligned} \xi_1 &= \frac{1}{8}(1 - \sqrt{13}), \quad \xi_2 = \frac{1}{8}(7 - \sqrt{13}), \quad \xi_3 = \frac{1}{4}(2 + \sqrt{13}), \\ \eta_1 &= -\frac{1}{2}(2 + \sqrt{13}), \quad \eta_2 = \bar{\eta}_3 = \frac{5 + \sqrt{13} - i\sqrt{54 + 18\sqrt{13}}}{4}. \end{aligned} \quad (6.16)$$

The constants  $a$  and  $b$  appearing in the functions  $f$  and  $g$  are

$$a = -\frac{1}{96}(16 - \sqrt{13}), \quad b = -\frac{1}{12}(137 + 37\sqrt{13}). \quad (6.17)$$

**$Y^{3,1}$  with  $k = 1$**

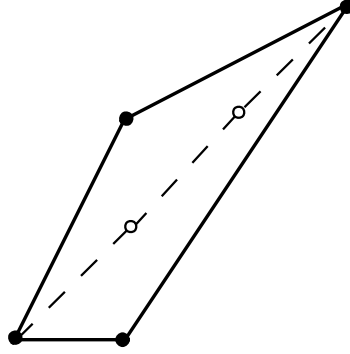


Figure 2: Toric diagram for  $Y^{3,1}$

For  $Y^{3,1}$ , we have two internal points. We add the point corresponding to the vector  $(1, 1, 1)$ . The other point  $(1, 2, 2)$  does *not* give a positive definite metric and hence is not considered. The roots when  $k = 1$  turn out to be

$$\xi_1 = \frac{3 - \sqrt{33}}{12}, \quad \xi_2 = \frac{9 - \sqrt{33}}{12}, \quad \xi_3 = \frac{3 + \sqrt{33}}{6}, \quad \eta_1 = \frac{1 - \sqrt{33}}{8}. \quad (6.18)$$

We note that the ordering  $\eta_1 < \xi_1 < \xi_2 < \xi_3$  is respected. The constants in the two polynomials are

$$a = -\frac{9 - \sqrt{33}}{54}, \quad b = -\frac{77 + 3\sqrt{33}}{192}. \quad (6.19)$$

The FI parameter  $r_2 = \frac{1}{8}(7 + \sqrt{33})$ .

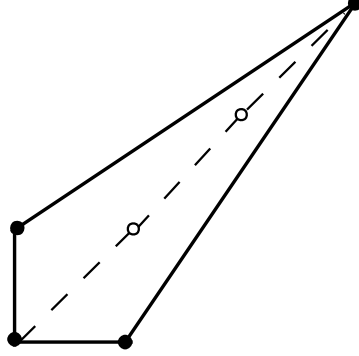


Figure 3: Toric diagram for  $Y^{3,2}$

### $Y^{3,2}$ with $k = 1$

This example is similar to  $Y^{3,1}$ . The  $k = 2$  solution is not valid but the  $k = 1$  is and hence we present the results for that metric. The roots turn out to be

$$\xi_1 = -\frac{1}{\sqrt{6}}, \quad \xi_2 = \frac{6 - \sqrt{6}}{6}, \quad \xi_3 = \frac{3 + 2\sqrt{6}}{6}, \quad \eta_1 = -\frac{2 + 3\sqrt{6}}{10}. \quad (6.20)$$

We note that the ordering of the roots is as expected. The constants that appear in the polynomials are

$$a = -\frac{9 + \sqrt{6}}{54}, \quad b = -\frac{601 + 189\sqrt{6}}{750}. \quad (6.21)$$

The FI parameter  $r_2 = \frac{2}{5}(4 + \sqrt{6})$ .

### $Y^{5,3}$ with $k = 1, 2$

This is the first example where we obtain inequivalent resolutions corresponding to adding internal points for  $k = 1$  and  $k = 2$ . Since the roots of  $f$  are independent of  $k$ , we will quote them once and write out the root  $\eta_1$  separately. We obtain

$$\begin{aligned} \xi_1 &= \frac{1 - \sqrt{73}}{20}, \quad \xi_2 = \frac{19 - \sqrt{73}}{20}, \quad \xi_3 = \frac{5 + \sqrt{73}}{10}, \\ \eta_1(k=1) &= -\frac{1 + 5\sqrt{73}}{76}, \quad \eta_1(k=2) = -\frac{13 + 5\sqrt{73}}{46}. \end{aligned} \quad (6.22)$$

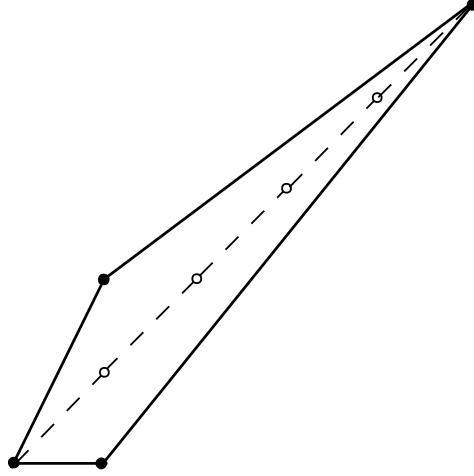


Figure 4: Toric diagram for  $Y^{5,3}$

We note that the ordering of the roots is as expected. The constants that appear in the polynomials are

$$a = -\frac{125+\sqrt{73}}{750},$$

$$b(k=1) = -\frac{26705+1285\sqrt{73}}{82308}, \quad b(k=2) = -\frac{105479+10315\sqrt{73}}{73002}. \quad (6.23)$$

The FI parameter  $r_2(k=1) = \frac{3}{190}(77 + 5\sqrt{73})$  and  $r_2(k=2) = \frac{27}{460}(59 + 5\sqrt{73})$ .

One can ask what happens to our formulae when  $q = 0$ . The roots  $\xi_1$  and  $\xi_2$  coincide. This implies that the  $m = 3, l = 2$  ACG metric is singular.  $\xi$  becomes a constant root. This is similar to what happened in the  $L^{pqr}$  metric earlier. Again, we need to add a  $\mathbb{CP}^1$  to resolve this singularity. So it naturally leads us to the conifold and its orbifolds. We thus move on to the  $m = 3, l = 1$  ACG metrics with momentum polynomial  $p_2(t)$ .

### 6.3 The $m = 3, l = 1$ ACG metric and the resolved conifold

The metric for the resolved conifold as well its  $\mathbb{Z}_2$  orbifold has been obtained by in [34] and [35]. Following these papers, both the metrics can be written as<sup>7</sup>

$$ds_6^2 = \kappa^{-1}(\rho)d\rho^2 + \frac{\rho^2}{9}\kappa(\rho)(d\psi - A_a - A_b)^2 + \frac{\rho^2}{6}ds_{\mathbb{CP}_a^1}^2 + \left(\frac{\rho^2}{6} + \hat{a}^2\right)ds_{\mathbb{CP}_b^1}^2 \quad (6.24)$$

<sup>7</sup>Below  $ds_{\mathbb{CP}^1}^2$  is the metric  $(d\theta^2 + \sin^2\theta d\varphi^2)$  and  $\omega = \cos\theta d\theta \wedge d\varphi$  is the Kähler form for  $\mathbb{CP}^1$ . The indices  $a$  and  $b$  distinguish the two  $\mathbb{CP}^1$ 's that appear.

where

$$\kappa(\rho) = (1 + \frac{9\hat{a}^2}{\rho^2} - \frac{\hat{b}^6}{\rho^6}) / (1 + \frac{6\hat{a}^2}{\rho^2}) , \quad dA_a = \omega_a \text{ and } dA_b = \omega_b .$$

The metric of the resolved conifold is obtained after setting  $\hat{b} = 0$  and choosing the periodicity the angle  $\psi$  to be  $4\pi$ . The metric of the resolution of the  $\mathbb{Z}_2$  orbifold of the conifold is obtained by simply choosing the periodicity of  $\psi$  to be  $2\pi$ . The periodicity of the angles are determined by requiring the metrics to be non-singular at  $\rho = 0$ . The parameter  $\hat{a}$  is the size of the blown-up  $\mathbb{CP}^1$ . We will now show how these two metrics are indeed  $m = 3, l = 1$  ACG metrics with momentum polynomial  $p_2(t)$ .

Hence consider the  $m = 3, l = 1$  ACG metrics and choose  $ds_a^2$  and  $ds_b^2$  to be the Fubini-Study metric on  $\mathbb{CP}^1$ . Both are taken to have scalar curvature equal to 2. Ricci-flatness of the metric implies that the function  $h(\chi)$  must be of the form

$$h'(x) = -2(x - a)(x - b) , \quad (6.25)$$

With no loss of generality, assume  $a = 0$  and  $b > 0$ . Then, we obtain

$$h(x) = -\frac{2}{3}x^3 + bx^2 - c , \quad c = \text{a positive constant} . \quad (6.26)$$

The resolved conifold is obtained when the constant  $c = 0$ . Positivity of the metric requires  $\chi < 0$ . The metric that we obtain here becomes the metric on the resolved conifold given in [34] after the change of variable  $\chi = -\rho^2/6$ , identifying  $\theta_1 = (d\psi - A_a - A_b)$  and finally setting  $b = \hat{a}^2$ .

The metric for the  $\mathbb{Z}_2$  orbifold is obtained if we choose the constant  $c = \hat{b}^6/324$  and choose the periodicity of  $\psi$  to be  $2\pi$  as follows from the analysis in [35].

#### 6.4 The resolution of $\mathbb{C}^3/\mathbb{Z}_3$ as a $l = 3, m = 1$ ACG metric

The metric for the resolution of  $\mathbb{C}^3/\mathbb{Z}_3$  when written as the resolution of a cone is(after a rescaling) [36]

$$ds^2 = 3 \left[ 1 - \frac{b^6}{r^6} \right]^{-1} dr^2 + \frac{r^2}{3} \left[ 1 - \frac{b^6}{r^6} \right] \left( d\gamma + A \right)^2 + \frac{r^2}{2} ds_{\mathbb{P}^2}^2 . \quad (6.27)$$

where  $ds_{\mathbb{P}^2}^2$  is the Fubini-Study metric on  $\mathbb{CP}^2$  with scalar curvature equal to 8 and the Kähler form is  $\omega_{\mathbb{P}^2} \equiv -dA$ .

We wish to show that this is an example of the  $m = 3$   $l = 1$  with momentum polynomial  $p_1(t)$ . We take the small Kähler metric to be the Fubini-Study metric on  $\mathbb{C}\mathbb{P}^2$  with scalar curvature equal to 8. The Ricci-flatness condition requires  $h(\chi)$  such that  $h'(x) = b_1(x - a)^2$  with  $b_1 = -4$ . Setting the constant  $a$  to zero with no loss of generality, we obtain

$$h(\chi) = -\frac{4}{3}\chi^3 - \frac{b^6}{6},$$

where we have chosen the constant suitably. One further has the condition that  $d\theta_1 = -\omega_{\mathbb{P}^2}$ . Identifying  $\chi = -r^2/2$  and  $\theta_1 = d\gamma + A$ , we recover the resolved metric given above.

## 7 Conclusion and Outlook

In this paper, we have constructed symplectic potentials for a large family of metrics due to Apostolov, Calderbank and Gauduchon. We carry out a global analysis of these metrics, largely focusing on non-compact six-dimensional examples, by relating the symplectic potential to the canonical one due to Guillemin. We then systematically worked out the situations where we recover D-terms associated with known manifolds such as cones over  $L^{pqr}$  and  $Y^{pq}$  manifolds. We find among these metrics, an infinite family of partially resolved metrics for cones over  $Y^{pq}$  for non-zero blow-up parameters. Interestingly, we also recover the resolved conifold (and its orbifold) and the resolution of  $\mathbb{C}^3/\mathbb{Z}_3$  among the ACG metrics. Thus, all known examples of resolved metrics appear in this classification.

The  $m = 3$  orthotoric metrics seem the natural place to look for metrics corresponding to partial resolutions of  $L^{pqr}$ . In specific examples, we have found that there are no such solutions even though the blown down metric is recovered in a limit. Nevertheless, we feel that our analysis in this particular situation is incomplete and we hope to report on this in the future.

The paper has largely dealt with symplectic coordinates. One may wish to know if this is always a good approach. As a test case, we have attempted to work out the symplectic potential for resolutions of  $\mathbb{C}^2/\mathbb{Z}_N$  using the symplectic quotient rather than the hyper-Kähler quotient that is natural in this setting. The symplectic method works only for  $N = 2$  but does *not* work for  $N > 2$

[36]. However, it is known that a partial Legendre transform of the symplectic potential can be exactly determined in these examples [37] and an explicit map to the Gibbons-Hawking metrics worked out. In carrying out the inverse Legendre transform to recover the symplectic potential, one needs to find the roots of polynomials of degree greater than four to come up with a closed-form expression for the symplectic potential. Since no formulae exist for roots of polynomials with degree  $> 4$ , one does not obtain an algebraic expression the symplectic potential.

Our results clearly have implications in the context of the AdS-CFT correspondence. For instance, it is known that resolutions associated with two-cycles and four-cycles lead to different kinds of corrections to the radial part of the metric, i.e.,  $g_{rr}$  [14]. These metrics provide an arena where this can be verified. The Abreu function that we have obtained in this paper may be used to verify the prediction of Martelli, Sparks and Yau on its behaviour [29]. Finally, the gravity dual of the (marginal) Leigh-Strassler deformations of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory is not yet known. The gravity dual is expected to have a  $U(1)$  isometry implying that it may arise from a  $m = 3, l = 1$  ACG metric whose four-dimensional base is a *non-toric* Kähler-Einstein manifold. The CFT implies that the four-dimensional manifold must arise as a two-parameter deformation of  $\mathbb{CP}^2$ .

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