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Stochastic sensitivity analysis using HDMR and score function

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Abstract. Probabilistic sensitivities provide an important insight in reliability analysis and often crucial towards understanding the physical behaviour underlying failure and modifying the design to mitigate and manage risk. This article presents a new computational approach for calculating stochastic sensitivities of mechanical systems with respect to distribution parameters of random variables. The method involves high dimensional model representation and score functions associated with probability distribution of a random input. The proposed approach facilitates first- and second-order approximation of stochastic sensitivity measures and statistical simulation. The formulation is general such that any simulation method can be used for the computation such as Monte Carlo, importance sampling, Latin hypercube, etc. Both the probabilistic response and its sensitivities can be estimated from a single probabilistic analysis, without requiring gradients of performance function. Numerical results indicate that the proposed method provides accurate and computationally efficient estimates of sensitivities of statistical moments or reliability of structural system.

Keywords. Stochastic sensitivity; structural reliability; high dimensional model representation; score function; statistical moment.

1. Introduction

Sensitivity analysis provides an important insight towards understanding the physical mechanisms underlying failure and modifying the design towards mitigating risk. Significant advancements have been made over the past few decades in developing methods such that the sensitivity information is provided as a by-product of the analysis or at a significantly reduced cost with improved accuracy. Studies on sensitivities of the random variables are carried out in the literature (Melchers & Ahammed 2004; Au 2005) in order to establish a systematic framework for identifying relative importance of parameters that merit

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descriptions through random variables or optimizing a system's performance with an acceptable risk. For estimating the sensitivity of a general probabilistic response, there are three broad approaches, namely, finite-difference method (L'Ecuyer & Perron 1994), perturbation analysis (Ho & Cao 1991) and score function/likelihood method (Rubinstein & Shapiro 1993). The finite-difference method involves repeated stochastic analyses for nominal and perturbed values of system parameters, and then invoking differentiation schemes to approximate their partial derivatives. This method is often expensive because evaluating probabilistic response for each system parameter, which constitutes a complete stochastic analysis, is a computationally challenging task. Remaining methods are mostly viewed as competing methods, where both performance and sensitivities can be obtained from a single stochastic simulation. For perturbation analysis (Glasserman 1991), the probability measure is fixed, and the gradient of a performance function is taken, assuming that the differential and integral operators are interchangeable. Score function method (Rubinstein & Shapiro 1993), which involves probability measure that continuously varies with respect to a design parameter, also requires a somewhat similar interchange of differentiation and integration, but in many practical examples, interchange in the score function method holds in a much wider range than that in perturbation analysis. However, both methods are typically employed in conjunction with the direct Monte Carlo simulation (MCS), a premise well-suited to stochastic optimization of discrete event systems. Unfortunately, gradient estimation of a stochastic system, where stochastic response and sensitivity analyses are required at each realization, even a single cycle of MCS is impractical, as each deterministic trial of the simulation may require expensive finite element (FE) or other numerical calculations. This is the principal reason why neither the perturbation analysis nor the score function method have found their way in to the stochastic optimization of mechanical systems.

The direct differentiation method, commonly used in deterministic sensitivity analysis (Haug *et al* 1986), provides an attractive alternative to the finite-difference method for calculating stochastic sensitivities. In conjunction with first-order reliability method (FORM), Liu & Der Kiureghian (1991) have significantly contributed to the development of such methods for obtaining reliability sensitivities. The direct differentiation method, also capable of generating both reliability and its sensitivities from a single stochastic analysis, is particularly effective in solving FE-based reliability problems, when (a) the most probable point can be efficiently located and (b) a linear approximation of the performance function at that point is adequate. In contrast, the three sensitivity methods described in the preceding are independent of underlying stochastic analysis. In this article, high dimensional model representation (HDMR) is used in conjunction with score function for calculating stochastic sensitivities of structural/mechanical system with respect to probability distribution parameters.

The paper is organized as follows. Section 2 presents a brief overview of statistical moments. Section 3 describes the concept of sensitivity analysis using score function. Section 4 presents a brief overview of HDMR and its applicability to reliability analysis. Section 5 presents approximation of the original implicit limit state/performance function using HDMR. Section 6 presents the estimation of failure probability, statistical moments and sensitivities by MCS using the approximate limit state/performance function generated by HDMR. Numerical examples involving elementary mathematical functions and structural problems are presented in Section 7 to illustrate the proposed method.

2. Statistical moments

Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$ denote a family of $\Re^N$-valued input random vector, describing uncertainties in loads, material properties and geometry of structural/mechanical system. The probability law of the random variables is completely defined by joint density function $\{f_{\boldsymbol{x}}(\boldsymbol{x};\boldsymbol{\theta}), \boldsymbol{x} \in \Re^N, \boldsymbol{\theta} \in \Re^N\}$ that is associated with probability measure $\{P_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \Re^N\}$. It is assumed that the performance/limit state function $g(\boldsymbol{x})$ is not an explicit function of $\boldsymbol{\theta}$, although $g(\boldsymbol{x})$ implicitly depends on $\boldsymbol{\theta}$ through the probability law of $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$. The objective of probabilistic sensitivity analysis is to obtain the partial derivatives of a probabilistic characteristic of $g(\boldsymbol{x})$ with respect to a parameter $\theta_i, i = 1, 2, \ldots, M$, given a reasonably arbitrary probability law of $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$.

The q^{th} moment of $g(\boldsymbol{x})$ can be defined by

$$m_q(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}}[g^q(\boldsymbol{x})] = \int_{\Re^N} g^q(\boldsymbol{x}) f_X(\boldsymbol{x};\boldsymbol{\theta}) d\boldsymbol{x}; \quad q = 1, 2, \ldots. \tag{1}$$

A similar integral appears in time-invariant reliability analysis, which entails calculating the failure probability

$$P_F(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}[\boldsymbol{x} \in \Omega_F] = \int_{\Re^N} \mathcal{J}_{\Omega_F}(\boldsymbol{x}) f_X(\boldsymbol{x};\boldsymbol{\theta}) d\boldsymbol{x}, \tag{2}$$

where Ω_F is the failure set for component reliability analysis and

$$\mathcal{J}_{\Omega_F}(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega_F \\ 0, & \boldsymbol{x} \in \Omega \backslash \Omega_F \end{cases}; \quad \boldsymbol{x} \in \Re^N, \tag{3}$$

is an indicator function. Therefore, both the expressions in (2) and (3) can be consolidated into a generic probabilistic response

$$h(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}}[g(\boldsymbol{x})] = \int_{\Re^N} g(\boldsymbol{x}) f_X(\boldsymbol{x};\boldsymbol{\theta}) d\boldsymbol{x}. \tag{4}$$

3. Sensitivity analysis using score function

The score functions depend only on the probability distribution of random input $\boldsymbol{x} = \{x_1, x_2, \ldots, x_N\}$. When the distribution of x_i is either independent or dependent, the expressions of the score functions simplify slightly. Since a major application of sensitivity analysis is design optimization, where the second-moment properties of random input play the role of design parameters, attention is confined to the score functions associated with the mean and standard deviations of input.

3.1 *Independent distribution*

Consider a distribution parameter $\theta_i, i = 1, 2, \ldots, M$, and suppose that the gradient of a generic probabilistic response $h(\boldsymbol{\theta})$, which is either statistical moment or reliability of a structural/mechanical system, with respect to θ_i is sought. According to Rubinstein & Shapiro (1993), pioneer of the score function method, few assumptions are necessary for such sensitivity analysis. Details of these assumptions can be found in Rubinstein & Shapiro (1993). Taking the partial derivative of both sides of (4) with respect to θ_i yields

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \int_{\Re^N} g(\boldsymbol{x}) f_X(\boldsymbol{x}; \boldsymbol{\theta}) dx. \tag{5}$$

By invoking Lebesgue dominated convergence theorem, the differential and integral operators can be interchanged, which yields

$$\begin{aligned} \frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} &= \int_{\Re^N} g(\boldsymbol{x}) \frac{\partial f_X(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i} d\boldsymbol{x} = \int_{\Re^N} g(\boldsymbol{x}) \frac{\partial \ln f_X(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i} f_X(\boldsymbol{x}; \boldsymbol{\theta}) d\boldsymbol{x} \\ &= E_\theta \left[g(\boldsymbol{x}) \frac{\partial \ln f_X(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i} \right]; \quad i = 1, \dots, M. \end{aligned} \tag{6}$$

Define

$$K_\theta^{(1)}(\boldsymbol{x}; \boldsymbol{\theta}) = \frac{\partial \ln f_{\boldsymbol{x}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i}, \tag{7}$$

which is known as the first-order score function for the parameter θ_i.

3.2 *Dependent distribution*

When the probability distribution of random input $\boldsymbol{x} = \{x_1, x_2, \dots, x_N\}$ is dependent, the derivation of score functions is generally tedious. For example, when $\boldsymbol{x} = \{x_1, x_2, \dots, x_N\}$ is following Gaussian distribution with mean $\boldsymbol{\mu} = \{\mu_1, \mu_2, \dots, \mu_N\}$ and covariance $\Sigma = E_{\boldsymbol{\theta}}[(\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})^T] = [\rho_{ij}\sigma_i\sigma_j]$ and joint density $f_{\boldsymbol{x}}(\boldsymbol{x}; \boldsymbol{\theta}) = [(2\pi)^{N/2}|\Sigma|^{0\cdot 5}]^{-1} \exp[-(\boldsymbol{x} - \boldsymbol{\mu})\Sigma^{-1}(\boldsymbol{x} - \boldsymbol{\mu})^T]$, where μ_i and σ_i are the mean and standard deviation, respectively, of x_i and ρ_{ij} is the correlation coefficient between x_i and x_j, the first-order score functions are

$$K_{\mu_i}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta}) = \{0, \dots, 1, \dots, 0\}\Sigma^{-1}(\boldsymbol{x} - \boldsymbol{\theta}), \tag{8}$$

and

$$K_{\sigma_i}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta}) = -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\theta})^T \frac{\partial \Sigma^{-1}}{\partial \sigma_i}(\boldsymbol{x} - \boldsymbol{\theta}) - \frac{1}{2}\frac{\partial \ln |\Sigma|}{\partial \sigma_i}. \tag{9}$$

Therefore, the first-order sensitivity of $h(\boldsymbol{\theta})$ can be expressed by

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} = E_{\boldsymbol{\theta}}[g(\boldsymbol{x}) K_\theta^{(1)}(\boldsymbol{x}; \boldsymbol{\theta})]; \quad i = 1, \dots, M. \tag{10}$$

The score function method requires differentiating only the probability density function. Also in most cases, resulting score functions can be easily determined analytically. In contrast, the perturbation analysis requires derivatives or perturbation of the limit state/performance function, which is always expensive in stochastic mechanics applications. Furthermore, if the performance function is not differentiable, interchangeability of differential and integral operators is violated and the direct differentiation based approaches will not work. In the score function method, $g(\boldsymbol{x})$ can be discontinuous – for example, the indicator function $\mathcal{J}_{\Omega_F}(\boldsymbol{x})$ that comes from reliability analysis – but the method still allows evaluation of the sensitivity if the density function is differentiable. Due to these facts the score function method is chosen in this paper, as a tool for efficient computation of probabilistic sensitivity.

4. High dimensional model representation

The fundamental principle underlying the HDMR (Rabitz & Alis 1999; Rabitz *et al* 1999; Wang *et al* 1999; Alis & Rabitz, 2001; Li *et al* 2001a; Li *et al* 2001b; Sobol, 2003) is that, from the perspective of the output/response, the order of cooperative effects between the independent variables will die off rapidly. This assertion does not eliminate strong variable dependence or even the possibility that all the variables are important. Various sources (Rabitz & Alis 1999; Wang *et al* 1999; Alis & Rabitz 2001) of information support this point of there being limited high-order correlations. First, the variables in most systems are chosen to enter as independent entities. Second, traditional statistical analyses of system behavior have revealed that a variance and covariance analysis of the output in relation to the input variables often adequately describes the physics of the problem. These general observations lead to a dramatically reduced computational scaling when one seeks to map input-output relationships of complex systems.

Evaluating the input-output mapping of the system generates a HDMR. This is achieved by expressing system response as a hierarchical, correlated function expansion of a mathematical structure and evaluating each term of the expansion independently. One may show that system response that is a function of N input variables, $g(\boldsymbol{x}) = g(x_1, x_2, \ldots, x_N)$, can be expressed as summands of different dimensions:

$$g(\boldsymbol{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{1 \le i_1 < i_2 \le N} g_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{1 \le i_1 < \cdots < i_l \le N} g_{i_1 i_2 \ldots i_l}(x_{i_1}, x_{i_2}, \ldots, x_{i_l}) + \cdots + g_{12\ldots N}(x_1, x_2, \ldots, x_N), \tag{11}$$

where g_0 is a constant term representing the mean response of $g(\boldsymbol{x})$. The function $g_i(x_i)$ describes the independent effect of variable x_i acting alone, although generally nonlinearly, upon the output $g(\boldsymbol{x})$. The function $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ gives pair correlated effect of the variables x_{i_1} and x_{i_2} upon the output $g(\boldsymbol{x})$. The last term $g_{12\ldots N}(x_1, x_2, \ldots, x_N)$ contains any residual correlated behaviour over all of the system variables. Usually the higher order terms in (11) are negligible (Rabitz & Alis 1999; Rabitz *et al* 1999) such that HDMR with only low order correlations to second-order (Li *et al* 2001a), amongst the input variables are typically adequate in describing the output behaviour.

The expansion functions are determined by evaluating the input-output responses of the system relative to the defined reference point $\boldsymbol{c} = \{c_1, c_2, \ldots, c_N\}$ along associated lines, surfaces, subvolumes, etc. (i.e. cuts) in the input variable space. This process reduces to the following relationship for the component functions in (11)

$$g_0 = g(\boldsymbol{c}), \tag{12}$$

$$g_i(x_i) = g(x_i, \boldsymbol{c}^i) - g_0, \tag{13}$$

$$g_{i_1 i_2}(x_{i_1}, x_{i_2}) = g_{i_1 i_2}(x_{i_1}, x_{i_2}, \boldsymbol{c}^{i_1 i_2}) - g_{i_1}(x_{i_1}) - g_{i_2}(x_{i_2}) - g_0, \tag{14}$$

where the notation $g(x_i, \boldsymbol{c}^i) = g(c_1, c_2, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N)$ denotes that all the input variables are at their reference point values except x_i. The g_0 term is the output response of the system evaluated at the reference point $\boldsymbol{c}$. The higher order terms are evaluated as cuts in the input variable space through the reference point. Therefore, each first-order term $g_i(x_i)$ is evaluated along its variable axis through the reference point. Each second-order term $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ is evaluated in a plane defined by the binary set of input variables x_{i_1}, x_{i_2} through the reference point, etc. The process of subtracting off the lower order expansion functions removes their dependence to assure a unique contribution from the new expansion function.

Considering terms up to first- and second-order in (11) yields first- and second-order HDMR approximation of $g(\boldsymbol{x})$ as

$$\tilde{g}(\boldsymbol{x}) = \sum_{i=1}^{N} g(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) - (N-1)g(\boldsymbol{c}), \tag{15}$$

and

$$\begin{aligned}\tilde{g}(\boldsymbol{x}) = &\sum_{\substack{i_1=1, i_2=1 \\ i_1<i_2}}^{N} g(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}, c_{i_2+1}, \ldots, c_N) \\ &- (N-2)\sum_{i=1}^{N} g(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) + \frac{(N-1)(N-2)}{2} g(\boldsymbol{c})\end{aligned} \tag{16}$$

respectively. It can also be noted that, compared with FORM and SORM which retains only linear and quadratic terms, respectively, first-order HDMR provides more accurate approximation $\tilde{g}(\boldsymbol{x})$ of the original implicit limit state/performance function $g(\boldsymbol{x})$ (Chowdhury *et al* 2009). If first-order HDMR approximation is not sufficient the second-order HDMR approximation may be adopted at the expense of additional computational cost.

5. Generation of HDMR approximation

HDMR in (11) is exact along any of the cuts and the output response $g(\boldsymbol{x})$ at a point $\boldsymbol{x}$ off of the cuts can be obtained by following the procedure in step 1 and step 2 below:

Step 1: Interpolate each of the low dimensional HDMR expansion terms with respect to the input values of the point $\boldsymbol{x}$. For example, consider the first-order component function $g(x_i, \boldsymbol{c}^i) = g(c_1, c_2, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N)$. If for $x_i = x_i^j$, n function values

$$g(x_i^j, \boldsymbol{c}^i) = g(c_1, \ldots, c_{i-1}, x_i^j, c_{i+1}, \ldots, c_N); \quad j = 1, 2, \ldots, n, \tag{17}$$

are given at $n(= 3, 5, 7$ or $9)$ equally spaced sample points $\mu_i - (n-1)\sigma_i/2$, $\mu_i - (n-3)\sigma_i/2, \ldots, \mu_i, \ldots, \mu_i + (n-3)\sigma_i/2$, $\mu_i + (n-1)\sigma_i/2$ along the variable axis x_i with mean μ_i and standard deviation σ_i, the function value for arbitrary x_i can be obtained by the moving least square (MLS) interpolation (Lancaster & Salkauskas, 1986) as

$$g(x_i, \boldsymbol{c}^i) = \sum_{j=1}^{n} \phi_j(x_i) g'(c_1, \ldots, c_{i-1}, x_i^j, c_{i+1}, \ldots, c_N), \tag{18}$$

where

$$\begin{Bmatrix} g'(x_i^1, \boldsymbol{c}^i) \\ \vdots \\ \vdots \\ g'(x_i^n, \boldsymbol{c}^i) \end{Bmatrix} = \begin{bmatrix} \phi_1(x_i^1) & \phi_2(x_i^1) & \cdots & \phi_n(x_i^1) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1(x_i^n) & \phi_2(x_i^n) & \cdots & \phi_n(x_i^n) \end{bmatrix}^{-1} \begin{Bmatrix} g(x_i^1, \boldsymbol{c}^i) \\ \vdots \\ \vdots \\ g(x_i^n, \boldsymbol{c}^i) \end{Bmatrix}. \tag{19}$$

Similarly, consider the second-order component function $g(x_{i_1}, x_{i_2}, \boldsymbol{c}^{i_1 i_2}) = g(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}, c_{i_2+1}, \ldots, c_N)$. If for $x_{i_1} = x_{i_1}^{j_1}$, and $x_{i_2} = x_{i_2}^{j_2}$, n^2 function values

$$\begin{aligned} &g(x_{i_1}^{j_1}, x_{i_2}^{j_2}, \boldsymbol{c}^{i_1 i_2}) = g(c_1, \ldots, c_{i_1-1}, x_{i_1}^{j_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}^{j_2}, c_{i_2+1}, \ldots, c_N); \\ &\quad j_1 = 1, 2, \ldots, n, \qquad j_2 = 1, 2, \ldots, n \end{aligned} \tag{20}$$

are given on a grid formed by taking $n(= 3, 5, 7 \text{ or } 9)$ equally spaced sample points $\mu_{i_1} - (n-1)\sigma_{i_1}/2, \mu_{i_1} - (n-3)\sigma_{i_1}/2, \ldots, \mu_{i_1}, \ldots, \mu_{i_1} + (n-3)\sigma_{i_1}/2, \mu_{i_1} + (n-1)\sigma_{i_1}/2$ along x_{i_1} axis with mean μ_{i_1} and standard deviation σ_{i_1}, and $n(= 3, 5, 7 \text{ or } 9)$ equally spaced sample points $\mu_{i_2}-(n-1)\sigma_{i_2}/2, \mu_{i_2}-(n-3)\sigma_{i_2}/2, \ldots, \mu_{i_2}, \ldots, \mu_{i_2}+(n-3)\sigma_{i_2}/2, \mu_{i_2}+(n-1)\sigma_{i_2}/2$ along x_{i_2} axis with mean μ_{i_2} and standard deviation σ_{i_2}, the function value for arbitrary (x_{i_1}, x_{i_2}) can be obtained by the MLS interpolation as

$$\begin{aligned} g(x_{i_1}, x_{i_2}, \boldsymbol{c}^{i_1 i_2}) = &\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \phi_{j_1 j_2}(x_{i_1}, x_{i_2}) \\ &\times g'(c_1, \ldots, c_{i_1-1}, x_{i_1}^{j_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}^{j_2}, c_{i_2+1}, \ldots, c_N), \end{aligned} \tag{21}$$

where

$$\begin{aligned} &\begin{Bmatrix} g'(x_{i_1}^1, x_{i_2}^1, \boldsymbol{c}^{i_1 i_2}) \\ \vdots \\ g'(x_{i_1}^1, x_{i_2}^n, \boldsymbol{c}^{i_1 i_2}) \\ \vdots \\ g'(x_{i_1}^n, x_{i_2}^n, \boldsymbol{c}^{i_1 i_2}) \end{Bmatrix} \\ &= \begin{bmatrix} \phi_{11}(x_{i_1}^1, x_{i_2}^1) & \cdots & \phi_{1n}(x_{i_1}^1, x_{i_2}^1) & \cdots & \phi_{nn}(x_{i_1}^1, x_{i_2}^1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{11}(x_{i_1}^1, x_{i_2}^n) & \cdots & \phi_{1n}(x_{i_1}^1, x_{i_2}^n) & \cdots & \phi_{nn}(x_{i_1}^1, x_{i_2}^n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{11}(x_{i_1}^n, x_{i_2}^n) & \cdots & \phi_{1n}(x_{i_1}^n, x_{i_2}^n) & \cdots & \phi_{nn}(x_{i_1}^n, x_{i_2}^n) \end{bmatrix}^{-1} \begin{Bmatrix} g(x_{i_1}^1, x_{i_2}^1, \boldsymbol{c}^{i_1 i_2}) \\ \vdots \\ g(x_{i_1}^1, x_{i_2}^n, \boldsymbol{c}^{i_1 i_2}) \\ \vdots \\ g(x_{i_1}^n, x_{i_2}^n, \boldsymbol{c}^{i_1 i_2}) \end{Bmatrix}. \end{aligned} \tag{22}$$

The interpolation functions $\phi_j(x_i)$ and $\phi_{j_1 j_2}(x_{i_1}, x_{i_2})$ can be obtained using the MLS interpolation scheme.

By using (18), $g_i(x_i)$ can be generated if n function values are given at corresponding sample points. Similarly, by using (21), $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ can be generated if n^2 function values at corresponding sample points are given. The same procedure shall be repeated for all the first-order component functions, i.e. $g_i(x_i)$; $i = 1, 2, \ldots, N$ and the second-order component functions, i.e. $g_{i_1 i_2}(x_{i_1}, x_{i_2})$; $i_1, i_2 = 1, 2, \ldots, N$.

Step 2: Sum the interpolated values of HDMR expansion terms from zeroth-order to the highest order retained in keeping with the desired accuracy. This leads to first- and second-order HDMR approximation of the function $g(\boldsymbol{x})$ as

$$\tilde{g}(\boldsymbol{x}) = \sum_{i=1}^{N}\sum_{j=1}^{n}\phi_j(x_i)g'(c_1, \ldots, c_{i-1}, x_i^j, c_{i+1}, \ldots, c_N) - (N-1)g_0, \tag{23}$$

and

$$\begin{aligned}\tilde{g}(\boldsymbol{x}) = & \sum_{\substack{i_1=1, i_2=1\\ i_1<i_2}}^{N}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\phi_{j_1 j_2}(x_{i_1}, x_{i_2})\\ & \times g'(c_1, \ldots, c_{i_1-1}, x_{i_1}^{j_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}^{j_2}, c_{i_2+1}, \ldots, c_N)\\ & - (N-2)\sum_{i=1}^{N}\sum_{j=1}^{n}\phi_j(x_i)g'(c_1, \ldots, c_{i-1}, x_i^j, c_{i+1}, \ldots, c_N) + \frac{(N-1)(N-2)}{2}g_0\end{aligned} \tag{24}$$

respectively.

6. Estimation of failure probability & sensitivity

Recall that $\tilde{g}(\boldsymbol{x})$ is the approximate function of the original limit state/performance function. Based on this approximation, let $\hat{\Omega}_{FS} = \{\boldsymbol{x} : \tilde{g}(\boldsymbol{x}) < 0\}$ define approximate failure set in reliability analysis. Therefore, Monte Carlo method estimates of the failure probability $P_F(\boldsymbol{\theta})$ and its sensitivity $\partial P_F(\boldsymbol{\theta})/\partial\theta_i$, employing HDMR approximation are

$$P_F(\boldsymbol{\theta}) \cong E_\theta[\mathcal{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x})] = \lim_{N_s\to\infty}\frac{1}{N_S}\sum_{i=1}^{N_S}\mathcal{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i), \tag{25}$$

and

$$\frac{\partial P_F(\boldsymbol{\theta})}{\partial\theta_i} \cong E_\theta[\mathcal{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x})K_\theta^{(1)}(\boldsymbol{x};\boldsymbol{\theta})] = \lim_{N_s\to\infty}\frac{1}{N_S}\sum_{i=1}^{N_S}\mathcal{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i)K_\theta^{(1)}(\boldsymbol{x}^i;\boldsymbol{\theta}), \tag{26}$$

respectively, where $\boldsymbol{x}^i$ is i^{th} realization of $\boldsymbol{x}$, N_S is the sample size, $\mathcal{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i)$ is an indicator of fail or safe state such that

$$\mathcal{J}_{\hat{\Omega}_{FS}}(\boldsymbol{x}^i) = \begin{cases}1, & \boldsymbol{x}^i \in \hat{\Omega}_{FS}\\ 0, & \boldsymbol{x}^i \in \Omega\backslash\hat{\Omega}_{FS}\end{cases}. \tag{27}$$

Monte Carlo estimate of moment of HDMR approximation can be found as

$$m_q(\boldsymbol{\theta}) \cong E_\theta[\tilde{g}^q(\boldsymbol{x})] = \lim_{N_s \to \infty} \frac{1}{N_S} \sum_{i=1}^{N_S} \tilde{g}^q(\boldsymbol{x}^i), \tag{28}$$

and its sensitivity

$$\frac{\partial m_q(\boldsymbol{\theta})}{\partial \theta_i} \cong E_\theta[\tilde{g}^q(\boldsymbol{x}) K_\theta^{(1)}(\boldsymbol{x};\boldsymbol{\theta})] = \lim_{N_s \to \infty} \frac{1}{N_S} \sum_{i=1}^{N_S} \tilde{g}^q(\boldsymbol{x}^i) K_\theta^{(1)}(\boldsymbol{x}^i;\boldsymbol{\theta}). \tag{29}$$

Since first- and second-order HDMR approximation leads to explicit representation of the original implicit limit state/performance function, the MCS can be conducted for any sampling size. The total cost of original function evaluation entails a maximum of $(n-1) \times N + 1$ and $(n-1)^2(N-1)N/2 + (n-1)N + 1$ by the present method using first- and second-order HDMR approximation, respectively.

7. Numerical examples

Two numerical examples are presented to illustrate the proposed approach, for obtaining first-order sensitivity of the moment or reliability. Whenever possible, finite-difference method and the direct MCS, is employed to evaluate the accuracy and computational efficiency of the present method. The sample sizes for the direct MCS and the MCS in conjunction with HDMR approximation vary from 10^4 to 10^6, depending on the examples, but they are identical for a specific problem. For first-order HDMR, n equally spaced sample points are deployed along the variable axis through the reference point. Sampling scheme for HDMR approximation of a function that is having one variable (x) and two variables (x_1 and x_2) using first-order HDMR is shown in figures 1(a) and 1(b) respectively. For second-order HDMR, n equally spaced sample points are deployed along each of the variable axis to form a regular grid. Sampling scheme for HDMR approximation of a function having two variables (x_1 and x_2) using second-order HDMR is shown in figure 2. In all numerical examples presented, the reference point $\boldsymbol{c}$ is taken as mean values of the random variables.

7.1 *Example 1: Cubic function with two variables*

Consider a cubic limit state/performance function of the following form:

$$g(\boldsymbol{x}) = 2{\cdot}2257 - \frac{0{\cdot}025\sqrt{2}}{27}(x_1 + x_2 - 20)^3 + \frac{33}{140}(x_1 - x_2), \tag{30}$$

with two independent normal variables. Mean and standard deviation of the random variables are 10 and 3, respectively. For evaluating the failure probability P_F, HDMR approximation is constructed by deploying five equally spaced sample points ($n = 5$) along each of the variable axis. Table 1 compares the results obtained by the present method using HDMR approximation with direct MCS. A sampling size $N_S = 10^6$ is considered in direct MCS to evaluate the failure probability P_F. Table 1 also contains the computational effort in terms of number of function evaluations, associated with each of the methods. Compared with the failure probability obtained using direct MCS ($P_F = 0{\cdot}01907$), first- and second-order

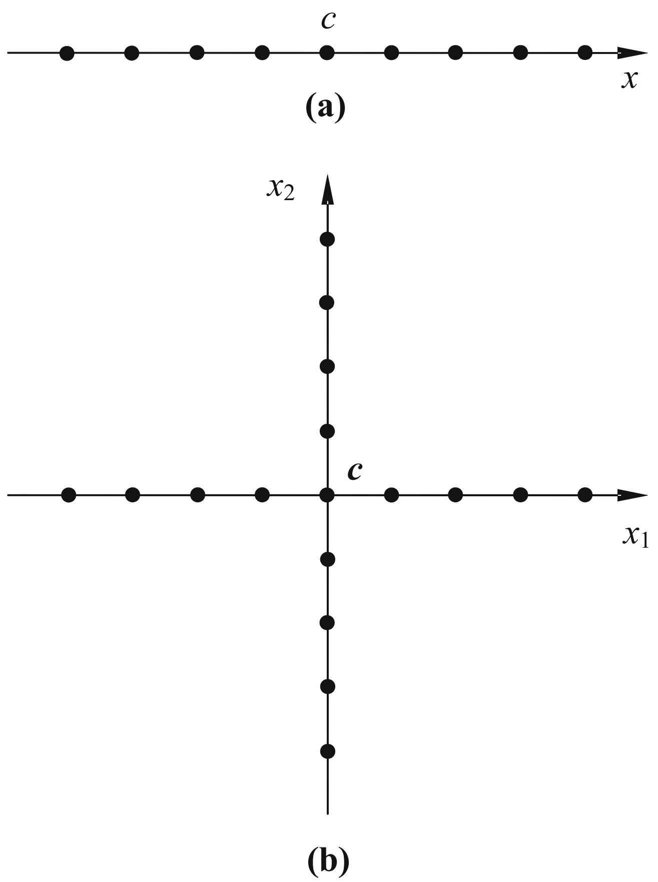


Figure 1. Sampling scheme for first-order HDMR; **(a)** for a function having one variable (x); and **(b)** For a function having two variables (x_1 and x_2).

HDMR approximation underestimates the failure probability by 16·99% ($P_F = 0{\cdot}01583$) and 0·94% ($P_F = 0{\cdot}01889$), respectively. Table 2 lists the first-order sensitivities of failure probability with respect to mean and standard deviation of random variables $\partial P_F/\partial \mu_i$ and $\partial P_F/\partial \sigma_i$ for $i = 1, 2$. Similarly, table 3 presents first three moments m_q and their first-order sensitivities $\partial m_q(\boldsymbol{\theta})/\partial \mu$ and $\partial m_q(\boldsymbol{\theta})/\partial \sigma$ for $q = 1, 2, 3$. Alternative sensitivity estimates from the finite difference method involving $N_S = 10^6$ samples for each direct simulation run are also calculated, and can be found in the last column of tables 2 and 3. The results of the proposed approach and the finite difference method agree well.

The effect of number of sample points on first- and second-order HDMR approximation for reliability and sensitivity estimation is studied by carrying out a similar analysis varying n from 3 to 9. Using first-order HDMR approximation, P_F ranges from 0·01314(+31·11%) (at $n = 3$) to 0·01569(+17·70%) (at $n = 9$), whereas P_F ranges from 0·01672(+12·32%) (at $n = 3$) to 0·01892(+0·79%) (at $n = 9$) using second-order HDMR approximation. Compared with direct MCS, error in the estimated failure probability using different methods is tabulated in table 4. Similarly, errors in the estimated moments and sensitivities are tabulated in tables 5 and 6. It can be observed that, compared with first-order HDMR, second-order approximation resulted in drastic reduction of the approximation error of the estimated failure probability as well as resulted sensitivities. The computational effort in terms of number of function evaluations for second-order HDMR approximation increased from 5 to 9 for $n = 3$,

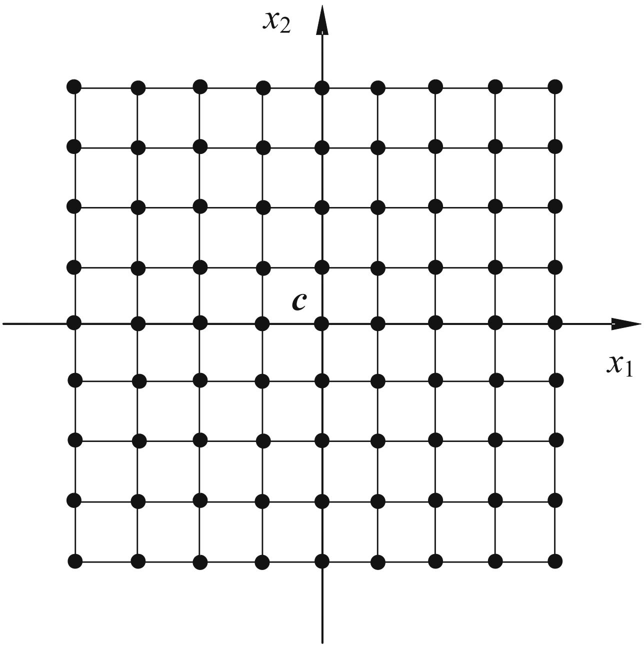


Figure 2. Sampling scheme for second-order HDMR for a function having two variables (x_1 and x_2).

Table 1. Estimation of error in failure probability using different methods for Example 1.

Method	Failure probability	Number of function evaluation[a]
FORM	0·01302	21
SORM (Hohenbichler *et al* 1987)	0·01302	204
First-order HDMR	0·01583	9[b]
Second-order HDMR	0·01889	25[c]
Direct MCS	0·01907	10^6

[a]Total number of times the original performance function is calculated.
[b]$(n-1) \times N + 1 = (5-1) \times 2 + 1 = 9$
[c]$(n-1)^2(N-1)N/2 + (n-1)N + 1 = (5-1)^2(2-1)2/2 + (5-1)2 + 1 = 25$

Table 2. Estimation of error in sensitivities of failure probability using different methods for Example 1.

	First-order HDMR	Second-order HDMR	Direct MCS[a]
$\partial P_F/\partial\mu_1$	$-5{\cdot}56 \times 10^{-3}$	$-5{\cdot}06 \times 10^{-3}$	$-5{\cdot}07 \times 10^{-3}$
$\partial P_F/\partial\mu_2$	$4{\cdot}38 \times 10^{-3}$	$1{\cdot}24 \times 10^{-2}$	$1{\cdot}36 \times 10^{-2}$
$\partial P_F/\partial\sigma_1$	$1{\cdot}19 \times 10^{-2}$	$2{\cdot}63 \times 10^{-2}$	$2{\cdot}67 \times 10^{-2}$
$\partial P_F/\partial\sigma_2$	$2{\cdot}32 \times 10^{-2}$	$2{\cdot}46 \times 10^{-2}$	$2{\cdot}48 \times 10^{-2}$

[a]For sensitivity estimation, finite difference with 1% perturbation is used

Table 3. Moments and sensitivities of moments (Example 1).

	First-order HDMR	Second-order HDMR	Direct MCS[a]
m_1	2·2270	2·2245	2·2245
m_2	6·0002	6·1106	6·1088
m_3	17·9941	18·7491	18·7502
$\partial m_1/\partial \mu_1$	0·2008	0·1633	0·1632
$\partial m_2/\partial \mu_1$	0·8918	0·7211	0·7215
$\partial m_3/\partial \mu_1$	3·5453	2·7445	2·7448
$\partial m_1/\partial \mu_2$	−0·2725	−0·3082	−0·3082
$\partial m_2/\partial \mu_2$	−1·2125	−1·3826	−1·3827
$\partial m_3/\partial \mu_2$	−5·0903	−5·9408	−5·9410
$\partial m_1/\partial \sigma_1$	0·0004	0·0002	0·0002
$\partial m_2/\partial \sigma_1$	0·1713	−0·2911	−0·2912
$\partial m_3/\partial \sigma_1$	1·1388	−1·9266	−1·9267
$\partial m_1/\partial \sigma_2$	$-8{\cdot}3321 \times 10^{-4}$	0·0002	0·0002
$\partial m_2/\partial \sigma_2$	0·5792	−0·7211	−0·7210
$\partial m_3/\partial \sigma_2$	3·8612	−4·8139	−4·8140

[a] For sensitivity estimation, finite difference with 1% perturbation is used

Table 4. Estimation of error in failure probability using different methods for Example 1.

	Number of sample points, n			
Method	3	5	7	9
First-order HDMR	31·11%	16·99%	16·66%	17·70%
Second-order HDMR	12·32%	0·94%	0·52%	0·79%

Table 5. Estimation of error in sensitivities of failure probability using different methods for Example 1.

		Number of sample points, n			
Method		3	5	7	9
First-order	$\partial P_F/\partial \mu_1$	−49·57	−9·66	−11·44	−10·26
	$\partial P_F/\partial \mu_2$	16·29	67·79	66·82	67·06
	$\partial P_F/\partial \sigma_1$	68·74	55·43	55·18	55·81
	$\partial P_F/\partial \sigma_2$	44·55	6·45	6·98	7·74
Second-order	$\partial P_F/\partial \mu_1$	−5·72	0·20	−0·20	−1·18
	$\partial P_F/\partial \mu_2$	15·15	8·82	5·15	6·62
	$\partial P_F/\partial \sigma_1$	7·12	1·50	−0·75	0·37
	$\partial P_F/\partial \sigma_2$	6·45	0·81	0·00	2·82

Table 6. Estimation of error in moments and sensitivities of moments using different methods for Example 1.

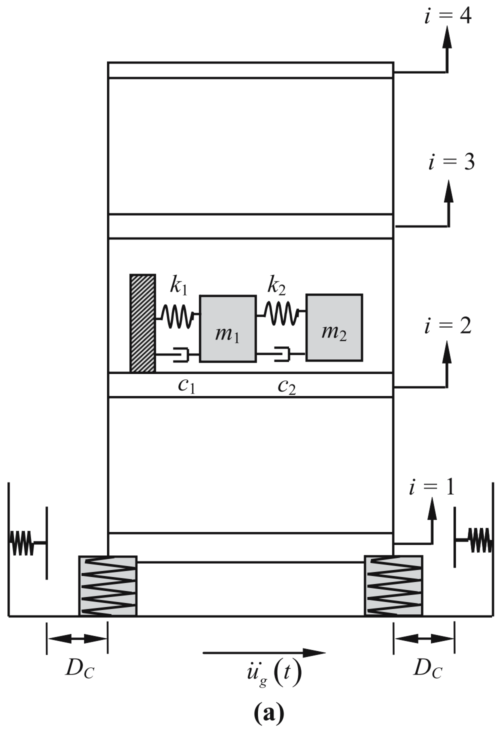
Method		Number of sample points, n			
		3	5	7	9
First-order	m_1	−0·11	−0·11	−0·11	−0·11
	m_2	2·38	1·78	1·76	1·81
	m_3	5·33	4·03	3·99	4·11
	$\partial m_1/\partial \mu_1$	−37·93	−23·04	−23·65	−23·65
	$\partial m_2/\partial \mu_1$	−39·02	−23·60	−24·21	−24·23
	$\partial m_3/\partial \mu_1$	−46·24	−29·16	−29·87	−29·79
	$\partial m_1/\partial \mu_2$	19·44	11·58	11·68	11·91
	$\partial m_2/\partial \mu_2$	20·11	12·31	12·40	12·63
	$\partial m_3/\partial \mu_2$	25·33	14·32	14·56	14·78
	$\partial m_1/\partial \sigma_1$	−125·29	−100·00	−92·70	−92·94
	$\partial m_2/\partial \sigma_1$	−4·36	41·17	40·01	40·01
	$\partial m_3/\partial \sigma_1$	−5·06	40·89	39·69	39·69
	$\partial m_1/\partial \sigma_2$	−334·36	−316·61	−306·92	−317·58
	$\partial m_2/\partial \sigma_2$	48·96	19·67	20·57	20·75
	$\partial m_3/\partial \sigma_2$	48·77	19·79	20·59	20·86
Second-order	m_1	−0·01	0·00	0·00	0·00
	m_2	−0·04	−0·03	−0·01	−0·02
	m_3	0·01	0·01	0·00	0·00
	$\partial m_1/\partial \mu_1$	−0·43	−0·06	−0·06	−0·18
	$\partial m_2/\partial \mu_1$	−0·04	0·06	0·01	0·04
	$\partial m_3/\partial \mu_1$	0·02	0·01	0·00	0·00
	$\partial m_1/\partial \mu_2$	0·26	0·00	−0·06	0·00
	$\partial m_2/\partial \mu_2$	−0·04	0·01	−0·01	0·01
	$\partial m_3/\partial \mu_2$	−0·01	0·00	0·00	0·00
	$\partial m_1/\partial \sigma_1$	−50·00	0·00	5·00	0·00
	$\partial m_2/\partial \sigma_1$	−0·14	0·03	0·10	0·03
	$\partial m_3/\partial \sigma_1$	0·03	0·01	0·01	0·01
	$\partial m_1/\partial \sigma_2$	10·00	0·00	0·00	0·00
	$\partial m_2/\partial \sigma_2$	−0·10	−0·01	−0·01	−0·01
	$\partial m_3/\partial \sigma_2$	0·02	0·00	0·01	0·00

from 9 to 25 for $n = 5$, from 13 to 49 for $n = 7$, and from 17 to 81 for $n = 9$, when compared with first-order HDMR approximation.

7.2 *Example 2: Vibration of 6-DOF system*

This example considers a four-story building excited by a single period sinusoidal pulse of ground motion, studied by Gavin & Yau (2008). Figure 3a shows the four-storey building with isolation systems and figure 3b presents the acceleration history. The building contains isolated equipment resting on the second floor. The motion of the ground floor is resisted mainly by base isolation bearings (Wen 1976) and if its displacement exceeds $D_c (= 0{\cdot}50\,\text{m})$ then an additional stiffness force contributes to the resistance. Mass, stiffness and damping coefficient m_f, k_f and c_f, respectively at each floor are assumed to be same.

There are two isolated masses, representing isolated, shock-sensitive equipment resting on the second floor. The larger mass $m_1 (= 500\,\text{kg})$ is connected to the floor by a relatively



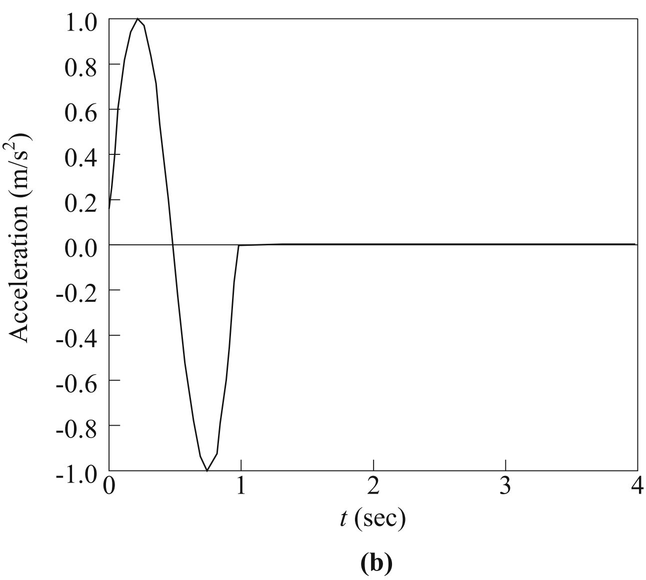


Figure 3. Problem statement (Example 2); **(a)** Base isolated structure with an equipment isolation system on the second floor; and **(b)** Acceleration history.

flexible spring, $k_1 (= 2500\,\text{N/m})$, and a damper, $c_1 (= 350\,\text{N/m/s})$, representing the isolation system. The smaller mass ($m_2 = 100\,\text{kg}$) is connected to the larger mass by a relatively stiff spring, $k_2 (= 10^5\,\text{N/m})$, and a damper, $c_2 (= 200\,\text{N/m/s})$, representing the equipment itself. All variables are assumed to be lognormal and independent. Probabilistic descriptions of the random variables are listed in table 7. The limit state/performance function is defined by the

Table 7. Properties of the random variables for Example 2.

Random variable	Distribution	Units	Description	Mean	COV
m_f	Lognormal	kg	Floor mass	6000	0·10
k_f	Lognormal	N/m	Floor stiffness	3×10^7	0·10
c_f	Lognormal	N/m/s	Floor damping coefficient	6×10^4	0·20
f_y	Lognormal	N	Isolation yield force	2×10^4	0·20
d_y	Lognormal	m	Isolation yield displacement	0·05	0·20
k_c	Lognormal	N/m	Isolation contact stiffness	3×10^7	0·30
T	Lognormal	s	Force period	1·0	0·20
A	Lognormal	m/m/s	Force amplitude	1·0	0·50

combination of three failure modes leading to system failure and is the following form

$$\begin{aligned} g(\boldsymbol{x}) = {} & 12{\cdot}50\left(0{\cdot}04 - \max_t |x_{f_i}(t) - x_{f_{i-1}}(t)|\right)_{i=2,3,4} \\ & + \left(0{\cdot}50 - \max_t |\ddot{u}_g(t) + \ddot{x}_{m_2}(t)|\right) \\ & + 2{\cdot}0\left(0{\cdot}25 - \max_t |x_{f_2}(t) - x_{m_1}(t)|\right), \end{aligned} \tag{31}$$

where $x_{f_i}(t)$ refers to the displacement of i^{th} floor and $(x_{f_i}(t) - x_{f_{i-1}}(t))$ is the inter storey drift. $\ddot{u}_g(t)$ is the ground acceleration and $\ddot{x}_{m_2}(t)$ is the acceleration smaller mass block. The displacement $x_{m_1}(t)$ is of the larger mass block, and represents the displacement of the equipment isolation system. The limit state/performance function in (31) is the overall representation of three failure modes. The first term describes the damage to the structural system due to excessive deformation. The second term represents the damage to equipment caused by excessive acceleration. The last term represents the damage of the isolation system. The weighing factors, multiplied with each term in (31), are mainly to emphasize the equal contribution of the individual failure modes to the overall failure of system. It is desirable that (a) inter storey drift is limited to 0·04 m, (b) the peak acceleration of the equipment is less than 0·5 m/s^2, and (c) the displacement across the equipment isolation system is less than 0·25 m. Equation (31) signifies overall system failure, which does not necessarily occur when above mentioned one or two failure criteria satisfies. For evaluating the failure probability P_F, first-order HDMR approximation is constructed by deploying five equally spaced sample points ($n = 5$) along each of the variable axis. The reference point is taken as mean values of the random variables. Table 8 compares the results obtained using first- and second-order HDMR approximation with FORM, SORM (Hohenbichler *et al* 1987), and direct MCS and also presents the computational effort in terms of number of function evaluations, associated with each of the methods. The benchmark solution of the failure probability is obtained by direct MCS with $N_S = 10^5$. Compared with the benchmark solution ($P_F = 0{\cdot}19599$), FORM and SORM overestimates the failure probability by around 15·05% ($P_F = 0{\cdot}22549$) and 8·46% ($P_F = 0{\cdot}21410$), respectively. First-order HDMR approximation overestimates the failure probability by about 1·88% ($P_F = 0{\cdot}19968$) and it needs only 33 function evaluations, while FORM, SORM and direct MCS requires 86, 356 and 10^5 number of original function evaluations, respectively. This shows the accuracy and the efficiency (in terms of original

Table 8. Estimation of failure probability for Example 2.

Method	Failure probability	Number of function evaluation[a]
FORM	0·22549	86
SORM (Hohenbichler *et al* 1987)	0·21410	357
First-order HDMR	0·19968	33[b]
Second-order HDMR	0·19246	481[c]
Direct MCS	0·19599	10^5

[a]Total number of times the original performance function is calculated.
[b]$(n-1)\times N+1=(5-1)\times 8+1=33$
[c]$(n-1)^2(N-1)N/2+(n-1)N+1=(5-1)^2(8-1)8/2+(5-1)8+1=481$

function calculations) of the first-order HDMR approximation, over FORM, SORM and direct MCS. Compared to the first-order HDMR, the error in the estimated failure probability reduces from +1·88% to −1·80%, and the number of function evaluations increases from 33 to 481, using second-order HDMR approximation.

Tables 9 and 10 present the first-order sensitivities of failure probability with respect to mean and standard deviation of random variables $\partial P_F/\partial\mu_i$ and $\partial P_F/\partial\sigma_i$ for $i=1,\ldots,8$,

Table 9. Sensitivities of failure probability with respect to mean (Example 2).

	First-order HDMR	Second-order HDMR	Direct MCS[a]
$\partial P_F/\partial\mu_1$	$-7{\cdot}8376\times10^{-6}$	$-8{\cdot}0783\times10^{-6}$	$-8{\cdot}2371\times10^{-6}$
$\partial P_F/\partial\mu_2$	$-5{\cdot}9548\times10^{-10}$	$-6{\cdot}1156\times10^{-10}$	$-6{\cdot}1242\times10^{-10}$
$\partial P_F/\partial\mu_3$	$9{\cdot}1605\times10^{-8}$	$8{\cdot}9812\times10^{-8}$	$8{\cdot}9905\times10^{-8}$
$\partial P_F/\partial\mu_4$	5·4109	5·4219	5·4309
$\partial P_F/\partial\mu_5$	$5{\cdot}2208\times10^{-6}$	$5{\cdot}6898\times10^{-6}$	$5{\cdot}7218\times10^{-6}$
$\partial P_F/\partial\mu_6$	$6{\cdot}5064\times10^{-11}$	$6{\cdot}9064\times10^{-11}$	$6{\cdot}9064\times10^{-11}$
$\partial P_F/\partial\mu_7$	0·9655	0·9788	0·9834
$\partial P_F/\partial\mu_8$	0·3705	0·4186	0·4215

[a]For sensitivity estimation, finite difference with 1% perturbation is used

Table 10. Sensitivities of failure probability with respect to standard deviation (Example 2).

	First-order HDMR	Second-order HDMR	Direct MCS[a]
$\partial P_F/\partial\sigma_1$	$-5{\cdot}0739\times10^{-6}$	$-5{\cdot}1682\times10^{-6}$	$-5{\cdot}1702\times10^{-6}$
$\partial P_F/\partial\sigma_2$	$-1{\cdot}9825\times10^{-10}$	$-1{\cdot}9961\times10^{-10}$	$-1{\cdot}9965\times10^{-10}$
$\partial P_F/\partial\sigma_3$	$1{\cdot}2518\times10^{-7}$	$1{\cdot}3112\times10^{-7}$	$1{\cdot}3117\times10^{-7}$
$\partial P_F/\partial\sigma_4$	−0·7237	−0·7338	−0·7342
$\partial P_F/\partial\sigma_5$	$-4{\cdot}4779\times10^{-6}$	$-4{\cdot}4887\times10^{-6}$	$-4{\cdot}4878\times10^{-6}$
$\partial P_F/\partial\sigma_6$	$1{\cdot}3931\times10^{-10}$	$1{\cdot}3941\times10^{-10}$	$1{\cdot}3942\times10^{-10}$
$\partial P_F/\partial\sigma_7$	0·4060	0·4106	0·4106
$\partial P_F/\partial\sigma_8$	$-6{\cdot}4636\times10^{-3}$	$-6{\cdot}4736\times10^{-3}$	$-6{\cdot}4737\times10^{-3}$

[a]For sensitivity estimation, finite difference with 1% perturbation is used

Table 11. Moments and sensitivities of moments (Example 1).

	First-order HDMR	Second-order HDMR	Direct MCS[a]
m_1	0·3741	0·3717	0·3722
m_2	0·3358	0·3347	0·3354
m_3	0·2615	0·2602	0·2613
$\partial m_1/\partial \mu_1$	$1{\cdot}3873 \times 10^{-5}$	$1{\cdot}4615 \times 10^{-5}$	$1{\cdot}4621 \times 10^{-5}$
$\partial m_1/\partial \mu_2$	$2{\cdot}9342 \times 10^{-10}$	$1{\cdot}3468 \times 10^{-9}$	$1{\cdot}3476 \times 10^{-9}$
$\partial m_1/\partial \mu_3$	$-6{\cdot}0250 \times 10^{-8}$	$-1{\cdot}2104 \times 10^{-7}$	$-1{\cdot}2107 \times 10^{-7}$
$\partial m_1/\partial \mu_4$	−9·4181	−9·7674	−9·7682
$\partial m_1/\partial \mu_5$	$-9{\cdot}1614 \times 10^{-6}$	$-8{\cdot}9295 \times 10^{-6}$	$-8{\cdot}9308 \times 10^{-6}$
$\partial m_1/\partial \mu_6$	$-2{\cdot}2457 \times 10^{-11}$	$-1{\cdot}7345 \times 10^{-10}$	$-1{\cdot}7385 \times 10^{-10}$
$\partial m_1/\partial \mu_7$	−1·5725	−1·5691	−1·5708
$\partial m_1/\partial \mu_8$	−0·7712	−0·7722	−0·7735
$\partial m_1/\partial \sigma_1$	$8{\cdot}9684 \times 10^{-6}$	$9{\cdot}5177 \times 10^{-6}$	$9{\cdot}5198 \times 10^{-6}$
$\partial m_1/\partial \sigma_2$	$-4{\cdot}9191 \times 10^{-11}$	$-1{\cdot}3686 \times 10^{-9}$	$-1{\cdot}3716 \times 10^{-9}$
$\partial m_1/\partial \sigma_3$	$-1{\cdot}1677 \times 10^{-7}$	$1{\cdot}3160 \times 10^{-7}$	$1{\cdot}3271 \times 10^{-7}$
$\partial m_1/\partial \sigma_4$	2·5100	2·9250	2·9269
$\partial m_1/\partial \sigma_5$	$8{\cdot}1566 \times 10^{-6}$	$8{\cdot}8671 \times 10^{-6}$	$8{\cdot}8689 \times 10^{-6}$
$\partial m_1/\partial \sigma_6$	$-1{\cdot}7508 \times 10^{-10}$	$-4{\cdot}1060 \times 10^{-10}$	$-4{\cdot}1108 \times 10^{-10}$
$\partial m_1/\partial \sigma_7$	0·1116	0·1019	0·1027
$\partial m_1/\partial \sigma_8$	0·2909	0·2966	0·2974

[a]For sensitivity estimation, finite difference with 1% perturbation is used

respectively. Table 11 lists the first three moments m_q and their first-order sensitivities $\partial m_1(\boldsymbol{\theta})/\partial \mu_i$ and $\partial m_1(\boldsymbol{\theta})/\partial \sigma_i$ for $i = 1, \ldots, 8$. The agreement between the results of proposed approach and the exact solution demonstrates that the proposed approach can indeed

Table 12. Estimation of error in failure probability and sensitivities of failure probability using different methods for Example 2.

	First-order	Second-order
P_F	−1·883	1·801
$\partial P_F/\partial \mu_1$	4·850	1·928
$\partial P_F/\partial \mu_2$	2·766	0·140
$\partial P_F/\partial \mu_3$	−1·891	0·103
$\partial P_F/\partial \mu_4$	0·368	0·166
$\partial P_F/\partial \mu_5$	8·756	0·559
$\partial P_F/\partial \mu_6$	5·792	0·000
$\partial P_F/\partial \mu_7$	1·820	0·468
$\partial P_F/\partial \mu_8$	12·100	0·688
$\partial P_F/\partial \sigma_1$	1·863	0·039
$\partial P_F/\partial \sigma_2$	0·701	0·020
$\partial P_F/\partial \sigma_3$	4·567	0·038
$\partial P_F/\partial \sigma_4$	1·430	0·054
$\partial P_F/\partial \sigma_5$	0·221	−0·020
$\partial P_F/\partial \sigma_6$	0·079	0·007
$\partial P_F/\partial \sigma_7$	1·120	0·000
$\partial P_F/\partial \sigma_8$	0·156	0·002

Table 13. Estimation of error in moments and sensitivities of moments using different methods for Example 2.

	First-order	Second-order
m_1	−0·510	0·134
m_2	−0·119	0·209
m_3	−0·077	0·421
$\partial m_1/\partial \mu_1$	5·116	0·041
$\partial m_1/\partial \mu_2$	78·226	0·059
$\partial m_1/\partial \mu_3$	50·235	0·025
$\partial m_1/\partial \mu_4$	3·584	0·008
$\partial m_1/\partial \mu_5$	−2·582	0·015
$\partial m_1/\partial \mu_6$	87·083	0·230
$\partial m_1/\partial \mu_7$	−0·108	0·108
$\partial m_1/\partial \mu_8$	0·297	0·168
$\partial m_1/\partial \sigma_1$	5·792	0·022
$\partial m_1/\partial \sigma_2$	96·414	0·219
$\partial m_1/\partial \sigma_3$	187·989	0·836
$\partial m_1/\partial \sigma_4$	14·244	0·065
$\partial m_1/\partial \sigma_5$	8·031	0·020
$\partial m_1/\partial \sigma_6$	57·410	0·117
$\partial m_1/\partial \sigma_7$	−8·666	0·779
$\partial m_1/\partial \sigma_8$	2·186	0·269

account high nonlinearity and large input uncertainties. Similar to the Example 1, alternative sensitivity estimates from the finite difference method are also calculated, and can be found in the last column of tables 9–11. The agreement between the results of the proposed method and the finite difference method is good. Compared with direct MCS, errors in the estimated failure probability and sensitivities of failure probability using different methods are tabulated in table 12. Similarly, errors in the estimated moments and sensitivities are tabulated in table 13. It can be observed that, compared with first-order HDMR, second-order approximation resulted in drastic reduction of the approximation error of the estimated failure probability as well as resulted sensitivities.

8. Conclusions

This paper presented a new computational method based on HDMR and score functions associated with the probability distribution of a random input, for estimating stochastic sensitivities of structural/mechanical systems with respect to probability distribution parameters. Both the probabilistic response and its sensitivities can be estimated from a single stochastic analysis, without requiring limit state/performance function gradients. The effort in obtaining probabilistic sensitivities can be viewed as calculating the conditional response at a selected deterministic input, defined by sample points. Therefore, the proposed method can be easily adapted for solving stochastic problems involving third-party, commercial FE codes. First- and second-order HDMR approximation are employed to solve two numerical problems, where the performance functions are linear or nonlinear, include Gaussian and/or non-Gaussian random variables, and are described by simple mathematical functions or mechanical responses from finite-element analysis. The results indicate that HDMR approximation, in particular the second-order, provide very accurate estimates of sensitivities of statistical moments or reliability. The computational effort by the first-order HDMR approximation varies linearly with respect to the number of random variables, and therefore first-order HDMR is economical. In contrast, second-order HDMR approximation, which generally outperforms first-order HDMR approximation, demands a quadratic cost scaling, making it more expensive than first-order approximation. Nonetheless, both approaches are far less expensive than the finite-difference method or the existing score function method entailing direct MCS.

It can be argued from the discussion of the formulation that, the proposed methodology can be applied to solve any multi-physics problems. Some of the work, in the field of stochastic electromagnetic and flow-structure interaction, is underway.

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Nomenclature

$E_{\boldsymbol{\theta}}$	Expectation operator
Ω_F	Failure set
$\hat{\Omega}_{FS} = \{\boldsymbol{x} : \tilde{g}(\boldsymbol{x}) < 0\}$	Approximate failure set
$\mathcal{J}_{\Omega_F}$	Indicator of fail or safe set
$K_\theta^{(1)}(\boldsymbol{x}; \boldsymbol{\theta})$	First-order score function for the parameter θ
μ_i	Mean of input random variable, i
$\varphi_j(x_i), \varphi_{j_1 j_2}(x_{i_1}, x_{i_2})$	Moving least square interpolation function
ρ_{ij}	Correlation coefficient between x_i and x_j
σ_i	Standard deviation of input random variable, i
$\Re$	Real coordinate space
Σ	Covariance matrix
$f_{\boldsymbol{x}}(\boldsymbol{x}; \boldsymbol{\theta})$	Distribution function of $\boldsymbol{x}$ with parameters $\boldsymbol{\theta}$
$g(\boldsymbol{x})$	Limit state/performance function
$\tilde{g}(\boldsymbol{x})$	Approximation of original limit state/performance function, $g(\boldsymbol{x})$
$g_i(x_i)$	First-order HDMR component function
$g_{i_1 i_2}(x_{i_1}, x_{i_2})$	Second-order HDMR component function
$h(\boldsymbol{\theta})$	Generic probabilistic response
$P_F(\boldsymbol{\theta})$	Failure probability
$\partial P_F(\boldsymbol{\theta})/\partial\theta_i$	Sensitivity of failure probability with respect to the parameter θ_i
$\partial m_q(\boldsymbol{\theta})/\partial\theta_i$	Sensitivity of moment with respect to the parameter θ_i
$x_{f_i}(t)$	Displacement of i^{th} floor
$(x_{f_i}(t) - x_{f_{i-1}}(t))$	Inter storey drift
$\ddot{u}_g(t)$	Ground acceleration
D_c	Critical displacement of ground floor
m_f, k_f and, c_f	Mass, stiffness and damping coefficient at each floor
m_q	q^{th} moment
g_0	Mean response of $g(\boldsymbol{x})$
N_S	Simulation size in Monte Carlo simulation
$P_{\boldsymbol{\theta}}$	Probability measure

N	Number of variables
$\mathbf{c}$	Reference point
n	Number of sample points

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