Available online at www.sciencedirect.com

ScienceDirect

Procedia Engineering 144 (2016) 998 – 1006

Procedia Engineering

www.elsevier.com/locate/procedia

12th International Conference on Vibration Problems, ICOVP 2015

Stochastic Bifurcation Analysis of a Duffing Oscillator With Coulomb Friction Excited by Poisson White Noise

Pankaj Kumar[a,*], S. Narayanan[a], Sayan Gupta[b]

[a]*Department of Mechanical Engineering, Indian Institute of Technology Madras, Chennai, 600036, India*
[b]*Department of Applied Mechanics, Indian Institute of Technology Madras, Chennai, 600036, India*

Abstract

The stochastic analysis of the response of frictionally damped Duffing oscillator subjected to Poisson white noise (PWN) and its stochastic bifurcation analysis are considered. The behaviour of the stochastic attractors is examined through the stationary solution of the corresponding generalized Fokker-Planck-Kolmogorov (FPK) or Kolmogorov-Feller (KF) equation. A finite element (FE) scheme has been used for the solution of the FPK equation, using C^1 continuity shape functions. Parametric studies are carried out to gain insights into the effects of the Coulomb friction, and arrival rates of the underlying Poisson process of PWN. The results of FE solution are shown to be in good agreement with the results of Monte Carlo simulation (MCS).

© 2016 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

Peer-review under responsibility of the organizing committee of ICOVP 2015

Keywords: Poisson noise; Fokker - Planck equation; Finite element; Duffing oscillator; Friction damper

1. Introduction

In the study of the dynamics of many engineering systems, the systems and the excitations are usually considered to be continuous. However, in many cases of vibrating systems, discontinuities in the system equations arise due to presence of dry friction, impact and backlash. Also, the excitations can be discontinuous having discrete non-Gaussian characteristics. Excitations due to earthquakes and their aftershocks [1], traffic load [2], wave action on ship [3], wind buffeting of airplane tail [4], can be modeled as a sum of discrete train of random pulses with random amplitudes occurring at random times. The random pulses have been modeled as Poisson white noise (PWN) in the literature [5]. The response and bifurcation analyses of nonlinear systems subjected to PWN have been carried out in the literature using equivalent linearization, cumulant neglect closure and other approximate methods.

However, the results from these methods are not very accurate for strong nonlinearities and/or low arrival rates of the PWN. As in the case of white noise excitation, for the PWN excitation the joint probability density function (pdf) of response of the nonlinear oscillators is governed by a corresponding Fokker-Planck-Kolmogorov (FPK) equation [3, 6]. The solution of the FPK equation will give directly the joint pdf of the response from which the statistics of the response can be obtained. However, closed form solution of the FPK equation is available only for a limited

* Corresponding author. Tel.: +91-966-667-0722.
E-mail address: pankajiit1@yahoo.co.in

1877-7058 © 2016 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
Peer-review under responsibility of the organizing committee of ICOVP 2015
doi:10.1016/j.proeng.2016.05.032

class of problems when the joint pdf can be expressed as a separable product of marginal pdfs. Approximate methods like the path integral (PI) method [7], finite element (FE) method [8] have been developed and used to numerically obtain the solution of the FPK equation subjected to white noise and colored noise excitations. The FPK equation corresponding to the Duffing oscillator subjected to PWN excitation and its solution has been studied in the literature by the perturbation [3, 6], finite difference [10], and exponential closure [11] methods. Muscolino *et al.*[12] have used Monte Carlo simulations (MCS) for the stochastic analysis of a nonlinear system under PWN excitation.

The presence of friction modeled by the Coulomb (dry friction) model with constant or variable friction coefficients in the dynamics of a linear or nonlinear system introduces yet another nonlinearity which is discontinuous. The stochastic response analysis of nonlinear systems with friction subjected to PWN excitation is an important study which has not been treated adequately in the literature. In the present work, a recently developed FE method [8], with C^1 continuity shape functions is used for the solution of the FPK equation corresponding to the Duffing oscillator with single well and double well potentials including Coulomb friction, subjected to PWN excitation. The effects of the Poisson arrival rate and friction coefficient on the response are investigated. In this study the Coulomb friction force is approximated using arc tangent function and the effect of the approximation representing the discontinuity is verified by comparing the results obtained by the FE method with MCS results. The mean up-crossing rate of the response is estimated using the Rice's formula [13] and the joint pdf of response states is obtained by the FE method. It is observed that the increase in the Coulomb damping coefficient has the effect of reducing the probability of large excursion leading to improved reliability of the system.

2. Problem Formulation

Consider the Duffing oscillator with Coulomb friction (Fig. 1) subjected to PWN excitation

$$\ddot{X} + c\dot{X} - \alpha X + \beta X^3 + \mu g \,\mathrm{sgn}(\dot{X}) = \gamma W_P(t), \tag{1}$$

where c, α, β, γ and g (acceleration due to gravity) are constants, μ is the coefficient of friction, and sgn is the signum function. $W_P(t)$ is the PWN excitation. The number of over dots represents the order of differentiation with respect to the time parameter t. The signum function during the stick condition leads to convergence difficulties as its derivative involves the Dirac-delta function. The problem is circumvented in this work by using an approximate expression of the form [14]

$$\mathrm{sgn}(\dot{X}) = \frac{2}{\pi}\,\mathrm{atan}(\Theta\dot{X}), \qquad \Theta >> 1, \tag{2}$$

where Θ is a large number. $\Theta = 10^4$ seems to be an adequate approximation for the problem considered in this paper.

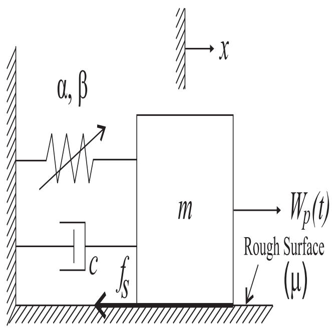

Fig. 1. Schematic of Duffing oscillator with Coulomb friction.

The PWN process can be assumed to be of the form

$$W_P(t) = \sum_{k=1}^{N(t)} Y_k \delta(t - t_k), \tag{3}$$

where Y_k is the amplitude of the k^{th} pulse arriving randomly at time t_k with assigned pdf $P_Y(y)$, $\delta(\cdot)$ is Dirac-delta function and $N(t)$ is a Poisson process. Y_ks are identically distributed independent random variables which are also independent of the time t_k.

Using the state space approach Eq. (1) can be expressed in the form of the following Ito stochastic differential equations

$$\begin{aligned} \mathrm{d}X_1 &= X_2\,\mathrm{d}t, \\ \mathrm{d}X_2 &= (-cX_2 + \alpha X_1 - \beta X_1^3 - \mu g\,\mathrm{sgn}(X_2))\,\mathrm{d}t + \gamma \mathrm{d}C(t), \end{aligned} \tag{4}$$

where $\mathrm{d}C(t)$ is an increment of the compound Poisson process $C(t)$ defined by

$$C(t) = \sum_{k=1}^{N(t)} Y_k U(t - t_k), \tag{5}$$

and $U(t)$ is the unit step function. The response $(X_1, X_2) \in \Re^2$, is a Markov vector and the corresponding transitional joint pdf $p(\mathbf{X}, t|\mathbf{X}_0, t_0)$ is governed by the following FPK equation [6]

$$\frac{\partial p}{\partial t} = -X_2 \frac{\partial p}{\partial X_1} - \frac{\partial\{-cX_2 + \alpha X_1 - \beta X_1^3 - \mu g\,\mathrm{sgn}(X_2)\}p}{\partial X_2} + \frac{\gamma^2 \lambda\, E[Y^2]}{2!}\frac{\partial^2 p}{\partial X_2^2} - \frac{\gamma^3 \lambda\, E[Y^3]}{3!}\frac{\partial^3 p}{\partial X_2^3} + \frac{\gamma^4 \lambda\, E[Y^4]}{4!}\frac{\partial^4 p}{\partial X_2^4} \tag{6}$$

where $p = p(\mathbf{X}, t|\mathbf{X}_0, t_0)$, the joint transition pdf of the state variables is used for notational convenience. The stationary solution of the generalized FPK equation is obtained by letting $\partial p(\mathbf{X}, t|\mathbf{X}_0, t_0)/\partial t = 0$.

3. Finite Element Method

The Galerkin weighted residual approach, leads to weak form of the generalized FPK Eq. (6) of the form

$$\mathbf{M}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{0}, \tag{7}$$

where the elements of $\mathbf{M}$ and $\mathbf{K}$ are given by

$$M_{rs} = \int_\Omega \psi_r \psi_s \mathrm{d}X_1 \mathrm{d}X_2 \tag{8}$$

and

$$\begin{aligned} K_{rs} = \int_\Omega \Big[&-\psi_r X_2 \frac{\partial[\psi_s]}{\partial X_1} - \psi_r \frac{\partial[-cX_2 + \alpha X_1 - \beta X_1^3 - \mu g\,\mathrm{sgn}(X_2)]\psi_s}{\partial X_2} \\ &- \frac{\gamma^2 \lambda E[Y^2]}{2!}\frac{\partial[\psi_r]}{\partial X_2}\frac{\partial[\psi_s]}{\partial X_2} + \frac{\gamma^3 \lambda E[Y^3]}{3!}\frac{\partial[\psi_r]}{\partial X_2}\frac{\partial^2[\psi_s]}{\partial X_2^2} + \frac{\gamma^4 \lambda E[Y^4]}{4!}\frac{\partial^2[\psi_r]}{\partial X_2^2}\frac{\partial^2[\psi_s]}{\partial X_2^2}\Big]\mathrm{d}X_1 \mathrm{d}X_2 \end{aligned} \tag{9}$$

subjected to the initial condition $\mathbf{p}(0) = \mathbf{p}$, where $\mathbf{p}$ is a vector containing the values of the joint pdf at the nodal points. $\{\psi_i\}_{i=1}^9$ are nine shape functions corresponding to the nine nodes of the quadratic element shown in Fig. 2. Since the generalized FPK equation is of fourth order, hence its weak form requires the interpolation functions to be twice differentiable. The nine-noded quadratic element shown in Fig. 2 is chosen for the FE discretization. In terms of normalized coordinates ξ, η, the shape functions are polynomials of degree of 4 and are of the form

$$\begin{aligned} \psi_1 &= \frac{1}{4}(\xi^2 - \xi)(\eta^2 - \eta), & \psi_2 &= \frac{1}{4}(\xi^2 + \xi)(\eta^2 - \eta), \\ \psi_3 &= \frac{1}{4}(\xi^2 + \xi)(\eta^2 + \eta), & \psi_4 &= \frac{1}{4}(\xi^2 - \xi)(\eta^2 + \eta), \\ \psi_5 &= \frac{1}{2}(1 - \xi^2)(\eta^2 - \eta), & \psi_6 &= \frac{1}{2}(\xi^2 + \xi)(1 - \eta^2), \\ \psi_7 &= \frac{1}{2}(1 - \xi^2)(\eta^2 + \eta), & \psi_8 &= \frac{1}{2}(\xi^2 - \xi)(1 - \eta^2), \\ & & \psi_9 &= (1 - \xi^2)(1 - \eta^2), \end{aligned} \tag{10}$$

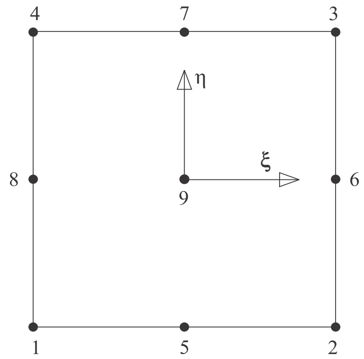

Fig. 2. Nine-node quadratic element.

The phase space domain for the FE solution is discretized into a mesh of 200 × 200 elements. More details on the numerical implementation and validation of the FE method is available in [8] and is not repeated here for the sake of brevity.

The computed joint pdf $p_{x_1,x_2}(X_1, X_2)$, is used to estimate the expected up-crossings of the process, given by Rice's formula [13]

$$E[N^+(\tilde{\alpha}, T)] = \int_0^T \int_0^\infty \dot{x} p_{X\dot{X}}(\tilde{\alpha}, \dot{x}, t)\, \mathrm{d}\dot{x}\, \mathrm{d}t. \tag{11}$$

Here, $N^+(\tilde{\alpha}, T)$ is the number of up-crossings of $X(t)$ of the level $\tilde{\alpha}$ in duration T and $E[\cdot]$ is the expectation operator. If $X(t)$ and $\dot{X}(t)$ are jointly stationary in the weak sense Eq. (11) can be simplified to

$$E[N^+(\tilde{\alpha}, T)] = T\nu^+(\tilde{\alpha}) = T \int_0^\infty \dot{x} p_{X\dot{X}}(\tilde{\alpha}, \dot{x})\, \mathrm{d}\dot{x}\, . \tag{12}$$

Here, $\nu^+(\tilde{\alpha})$ is the mean upcrossing intensity across level $\tilde{\alpha}$.

In order to check the accuracy of the FE solution, the estimated pdf obtained from solving the FP equations are compared with those obtained from MCS which are treated as the benchmark. The MCS is a direct numerical method which requires the generation of a family of sample functions of the excitation consistent with its stochastic nature. Corresponding sample functions of the response are simulated by numerical integration of the equation of motion (Eq. (1)). The response statistics and their probability structure are estimated from the simulated sample functions. In this work, the sample path of the PWN is obtained by generating a sequence of impulse magnitudes modeled as Dirac-delta functions. The impulse inter-arrival times $\delta t_j = \frac{1}{\lambda}\nu$ are assumed to be exponentially distributed with mean $\frac{1}{\lambda}$ with ν assumed to be a uniformly distributed random variable in the interval [0, 1].

4. Illustrative Examples

4.1. Duffing Oscillator with Friction (Mono-Stable Oscillator)

The FE method described in section 3 is used to solve the FPK equation of the Duffing oscillator with friction and subjected to PWN excitation. First the case of the mono-stable (uni-modal) Duffing oscillator is considered where the parameter α has a negative value. The parameters in Eq. (1) are taken as $c = 0.1, \alpha = -1, \beta = 0.5, \gamma = 1$ and $\lambda = 0.1$. Two values of frictional force are considered namely $\mu g = 0.05$ and 1, the former representing a very low amount of friction. The undamped and unforced Duffing oscillator with negative α values implies positive linear stiffness and has a single stable equilibrium point at the origin of the phase space leading to mono-stable behaviour. The intensity of pulse amplitude Y is assumed to be a zero mean Gaussian random variable with $E[Y_1^2] = \sigma_Y^2$, $E[Y_1^4] = \sigma_Y^4$.

Cai and Lin [6] and Muscolino *et al.* [12] have investigated the response of Duffing oscillator without friction to PWN excitation, the former using a perturbation method and the latter using MCS. Cai and Lin [6] have derived an

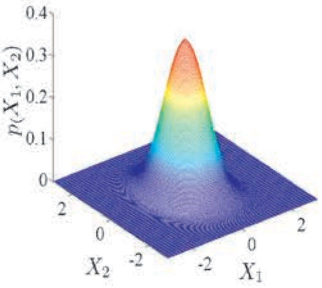

Fig. 3. (a)

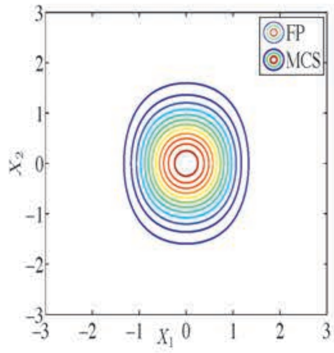

Fig. 4. (b)

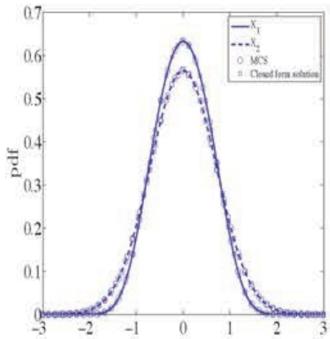

Fig. 5. (c)

Fig. 6. Stationary solution of Duffing oscillator with $\mu g = 0$ (a) Joint-pdf (b) contour plot (c) marginal pdf of response

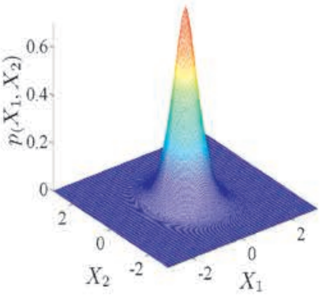

Fig. 7. (a)

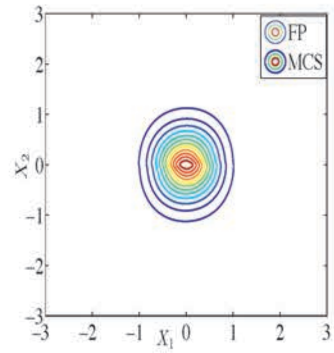

Fig. 8. (b)

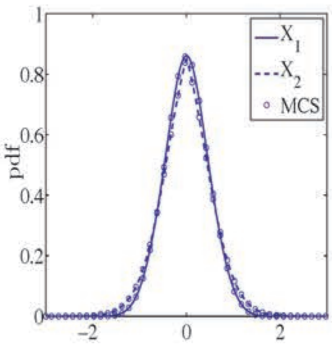

Fig. 9. (c)

Fig. 10. Stationary solution of Duffing oscillator with $\mu g = 0.05 m/sec^2$ (a) Joint-pdf (b) contour plot (c) marginal pdf of response

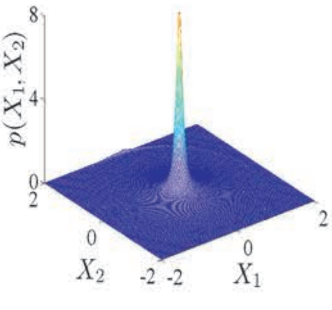

Fig. 11. (a)

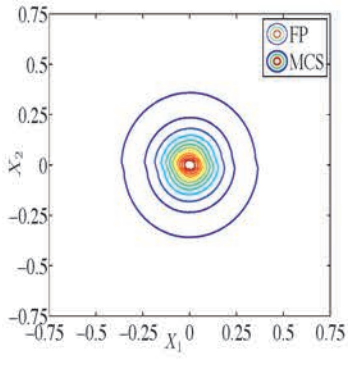

Fig. 12. (b)

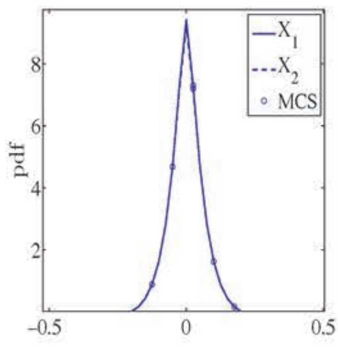

Fig. 13. (c)

Fig. 14. Stationary solution of Duffing oscillator with $\mu g = 1 m/sec^2$ (a) Joint-pdf (b) contour plot (c) marginal pdf of response

approximate closed form expression for the stationary joint pdf using a second order perturbation analysis which is given by

$$p(X_1, X_2) = N\{1 - \frac{2\lambda E[Y^4]c^2}{I_0^3}[AX_1^2 + cX_1X_2 + X_1^2 - \frac{c}{6I_0}(3A^2X_1^4 + 6AX_1^2X_2^2 + 8cX_1X_2^3 + 3X_2^4)]\} \exp\left[-\frac{2c}{I_0}(X_2^2 - \alpha X_1^2 + 0.5\beta X_1^2)\right] \tag{13}$$

where N is a normalization constant, $I_0 = \lambda E[Y^2]$, $A = -\alpha + \beta\eta$ and $\eta = \frac{E[X_1^4]}{E[X_1^2]}$.

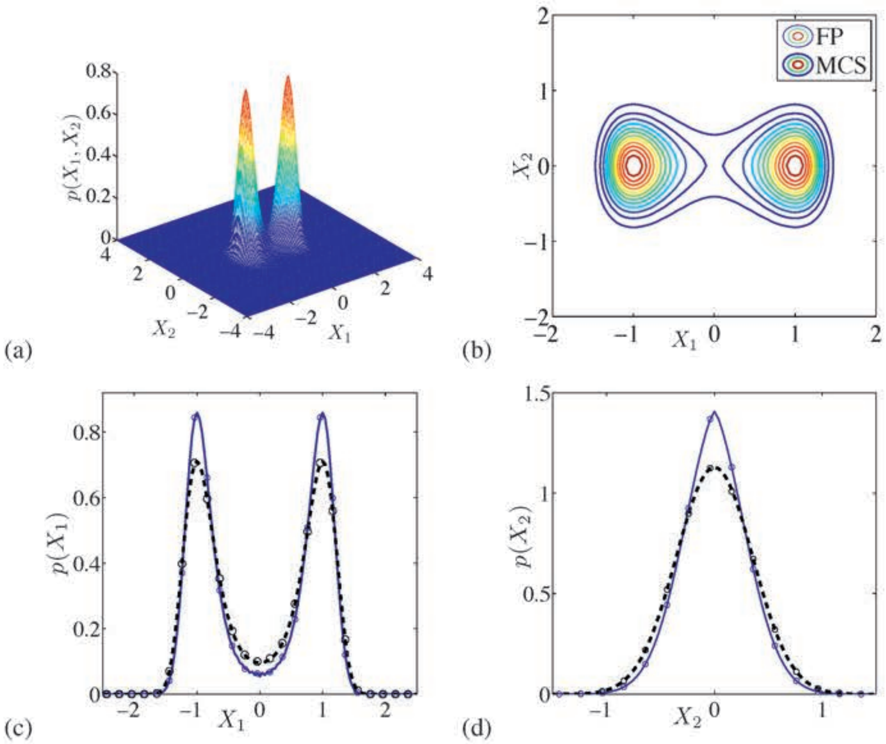

Fig. 15. Stationary solution of Duffing oscillator with $\lambda = 0.1$ (a) Joint-pdf (b) contour plot (c) marginal pdf of displacement (d) marginal pdf of velocity; Frictionally Damped (______); Friction less (- - - - - -), MCS(∘, ∘, ∘).

Figures 3, 4 and 5 (a,b,c) show the joint pdf, contours of joint pdf and the marginal pdfs respectively for $\mu g = 0, 0.05$ and 1 obtained by the FE method. $\mu g = 0$ corresponds to the case of no friction. The closed form solution of Cai and Lin [6] for $\mu g = 0$ is also shown in Fig. 6, showing the close agreement between the two results. From Figs.6-14 we can conclude that the FE method gives the joint pdf and marginal pdfs very accurately as the results match very closely with MCS results even in the tail regions of very low probabilities. From Figs.6-14, it is clear that friction has the effect of narrowing the joint pdf towards the stable equilibrium point and increasing the peak value namely at the origin of the phase space. The joint pdf and the marginal pdfs even for the lower friction value show the tendency of the joint pdf having a larger and sharper peak at the origin. Increased friction has also the effect of making the response non-Gaussian with the peak at the origin becoming sharper (almost a cusp) for larger μg values. As the joint pdf and the marginal pdfs reduce for larger values of the response with increasing friction it can also be concluded that the probability of large excursion of the response reduces implying increased reliability.

The MCS results are obtained based on 2×10^5 sample realizations with the computational time an order of magnitude greater (nearly 10 times) on the same platform compared to the time required for the FE method.

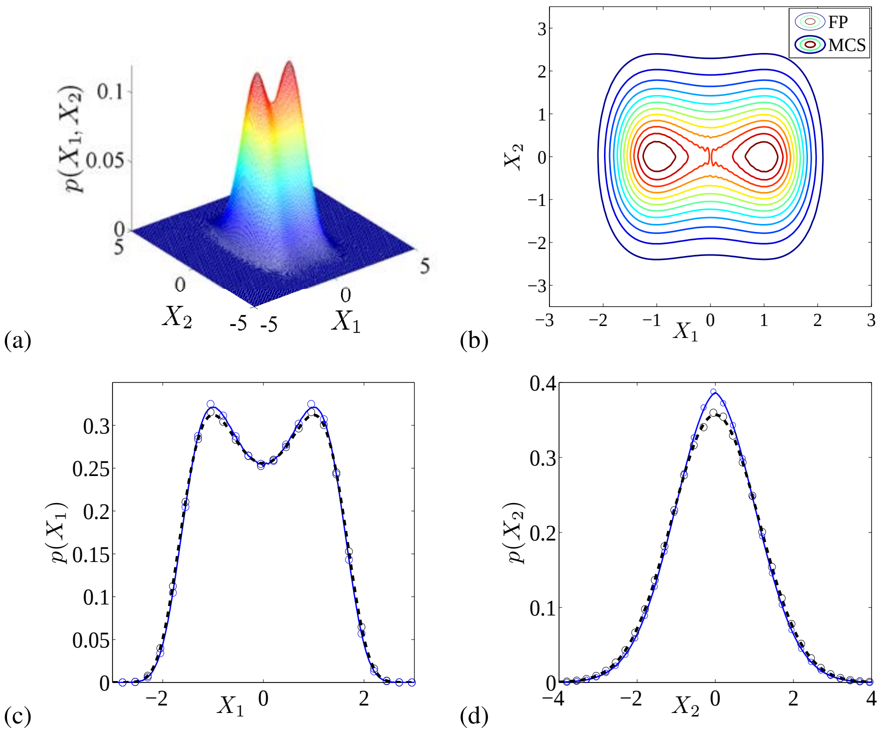

Fig. 16. Stationary solution of Duffing oscillator with $\lambda = 1$ (a) Joint-pdf (b) contour plot (c) marginal pdf of displacement (d) marginal pdf of velocity; key as in Fig. 15.

4.2. Duffing Oscillator with Friction (Bi-Stable Oscillator)

Next, a double well Duffing oscillator is considered. For the unexcited and undamped case and for positive value of α, the Duffing oscillator has three equilibrium points

$$O_{-}\left\{-\sqrt{\frac{\alpha}{\beta}},0\right\},\quad O\left\{0,0\right\},\quad O_{+}\left\{\sqrt{\frac{\alpha}{\beta}},0\right\}$$

The first and third are stable equilibrium points symmetrically placed on either side of the unstable equilibrium point at the origin. The potential energy function has double wells corresponding to the stable equilibrium points.

For the parameter values $c = 0.4, \alpha = \beta = 1, \gamma = 1$ and $\mu g = 0.05$ the joint pdf, its contour plot and the marginal pdfs obtained by the FE method and by MCS for three values of arrival rate λ, ($\lambda = 0.1, 1$ and 5) are shown in Figs.15-17. For the FE method, the phase space domain $X_1 - X_2$ is discretized into 200×200 elements.

Increase in the arrival rate of the Poisson process changes the bimodal nature of the joint pdf into a unimodal characteristic with the two peaks in the joint pdf trying to merge. This indicates a possible P-bifurcation for a critical value of λ. For low values of λ, the response essentially stays in the neighborhood of either of the two stable equilibrium points as the joint pdf has distinct peaks centered around these points (Fig. 15). As λ increases the response has a tendency to wander from one attractor to the other equilibrium point indicated by the near unimodal joint pdf around the origin for $\lambda = 5$. In the case of the bi-modal Duffing oscillator also the FE results agree closely with the MCS results. The FE method takes an order of magnitude less time on the computer than the MCS method.

The stationary mean up crossing rates for the Duffing oscillator with double well potential for three different values of arrival rate with and without friction under PWN excitation calculated using Eq.(12) are shown in Fig. 18. The crossing rate is higher for the oscillator without friction. This is more evident for the lower arrival rate. In this example only a very low value of $\mu g = 0.05$ is considered. For higher value of μg it is expected that the level crossing rates

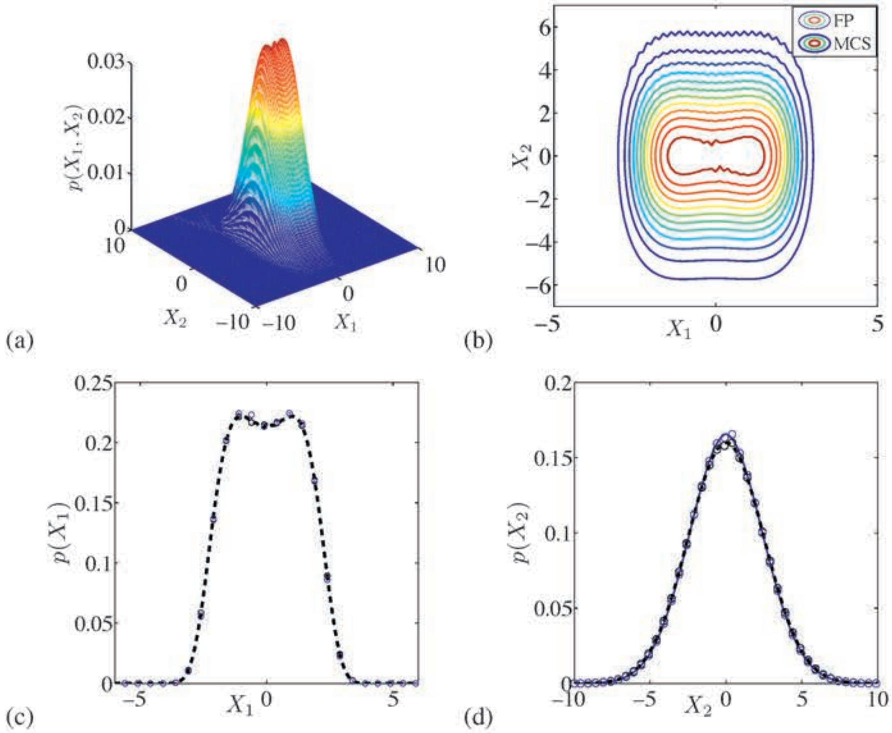

Fig. 17. Stationary solution of Duffing oscillator with $\lambda = 5$ (a) Joint-pdf (b) contour plot (c) marginal pdf of displacement (d) marginal pdf of velocity; key as in Fig. 15.

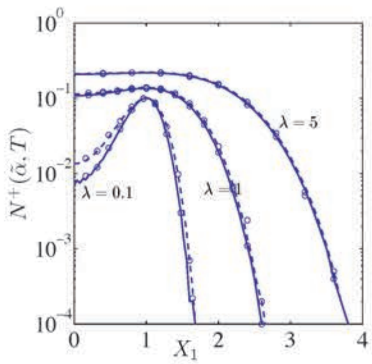

Fig. 18. Mean up-crossing intensity for Duffing oscillator for different arrival rate;key as in Fig. 15.

will be still lower. Lower mean crossing rates imply higher reliability in terms of first excursion failure and fatigue damage based on rain- flow counting [15].

5. Conclusions

The FE [8] is used for the solution of the FPK equation of mono-stable and bi-stable Duffing oscillator with Coulomb friction to PWN excitation. In the case of mono-stable Duffing oscillator increased coefficient of friction narrows the joint pdf of the response towards the origin of the phase space and makes the response non-Gaussian.

Thus the probability of large response is reduced with increase in reliability. In the case of the bi-modal Duffing oscillator with increase in Poisson arrival rate the bimodal nature of the joint pdf tends to become unimodal indicative of a P-bifurcation. In this case also with increase in friction the reliability of the system increases as shown by the decreased values of the expected arrival rate.

References

[1] Y.K. Lin, Application of non-stationary shot noise of system response to non stationary excitations, J. of Appl.Mech.ASME. 30 (1963) 555-558.
[2] C.C. Tung, Random response of highway bridges to vehicle load, Journal of Engrg Mech. ASCE. 93 (1967) 79-94.
[3] J.B. Robert, System response to random impulses, Journal of Sound and Vibration. 24 (1972) 23-34.
[4] H.W. Liepman, On the Application of statistical concepts to the buffeting problems, Journal of Aeronaut.Sci. 19 (1952) 793-800.
[5] M. Grigoriu, Response of dynamic systems to Poisson white noise, Prentice Hall, Englewood Cliffs,NJ, (1995).
[6] G.Q. Cai, Y.K. Lin, Response distribution of non-linear system excited by non-Gaussian impulsive noise, Int. J. Non-Linear Mech. 27 (1992) 955-967.
[7] A. Naess, V. Moe, Efficient path integration method for nonlinear dynamics system, Probab. Engrg. Mech. 15 (2000) 221-231.
[8] Pankaj Kumar, S. Narayanan, S. Gupta. Finite Element solution of Fokker-Planck equation of nonlinear oscillators subjected to colored non-Gaussian noise, Probabilistic Engineering Mechanics. 38 (2015) 143-155.
[9] M.Di. Paola, R. Santoro, Path integral solution for non-linear system enforced by Poisson white noise, Journal of Probabilistic Engg.Mech. 23 (2008) 164-169.
[10] S.F. Wojtkiewicz, E.A. Johnson, L.A. Bergman, M. Grigoriu, Response of stochastic dynamical systems driven by additive Gaussian and Poisson white noise:Solution of a forward Kolmogorov equation by a spectral finite difference method, Computer methods in applied Mech. and Engg.168 (1999) 73-89.
[11] H.T. Zhu, G.K. Er, V.P. Lu, K.P. Kou, Probabilistic solution of nonlinear oscillators excited by combined Gaussian and Poisson white noise excitation, Journal of Sound and Vibration. 330 (2011) 2900-2909.
[12] G. Muscolino, G. Ricciardi, P. Cacciola, Monte carlo simulation in the stochastic analysis of nonlinear system under external stationary Poisson white noise input, Int. J Non-Linear Mech. 38 (2003) 1269-1283.
[13] SO. Rice, The mathematical analysis of random noise I and II, Bell System Technical journal. 23 (1994) 282-332; 24 (1945) 46-156.
[14] S. Narayanan, K. Jayaraman, Chaotic Vibration in a Non-Linear Oscillator with Coulomb Damping, Journal of Sound and Vibration. 146(1) (1991) 17-31.
[15] I. Rychlik, Note on cycle counts in irregular loads, Fatigue Fract. Engng. Mater. Struct. 16 (1993) 377-390.