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Stability of locally CMFPD homologies under duality

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ABSTRACT

We consider bounded complexes $P_\bullet$ of finitely generated projective A-modules whose homologies have finite projective dimension and are locally Cohen–Macaulay. We give a necessary and sufficient condition so that its dual $P_\bullet^*$ also has the same property.

1. Introduction

Throughout this paper A will denote a Cohen–Macaulay (CM) ring with $\dim A_{\mathfrak{m}} = d \; \forall \; \mathfrak{m} \in Max(A)$. Throughout, “CM” abbreviates “*Cohen–Macaulay*” and “FPD” abbreviates “*finite projective dimension*”, which clarifies the title of the paper.

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To introduce the main results, in this paper, let M denote a finitely generated A-module with $proj\dim(M) = r < \infty$. When A is local, then M is Cohen–Macaulay if and only if $grade(M) = r$. In this case, $Ext^i(M, A) = 0 \quad \forall\, i \neq r$. However, even in the non-local case, grade of M is defined as $grade(M) := \min\{i : Ext^i(M, A) \neq 0\}$ (see [6]). Let $\mathcal{B}$ denote the category of such finitely generated A-modules M, with $proj\dim(M) = grade(M)$. Then, $\forall\, M \in \mathcal{B}$, $Ext^i(M, A) = 0 \quad \forall\, i \neq proj\dim(M)$.

Now suppose $P_\bullet$ is an object in the category $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ of finite complexes of finitely generated projective A-modules, with homologies in $\mathcal{B}$. In this paper, we give a necessary and sufficient condition, for such a complex $P_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A))$, so that its dual $P^*_\bullet$ is also in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$.

To further describe this condition let the homology $H_t(P_\bullet) \neq 0$, at degree t and $\rho_t = proj\dim(H_t(P_\bullet))$. The homomorphism $H_t(P_\bullet) \longrightarrow \frac{P_t}{B_t}$, where $B_t = \partial_{t+1}(P_{t+1})$, induces a homomorphism

$$\iota_t : Ext^{\rho_t}\left(\frac{P_t}{B_t}, A\right) \longrightarrow Ext^{\rho_t}\left(H_t(P_\bullet), A\right)$$

The main Theorem 3.6 states that the dual $P^*_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ if and only if ι_t is an isomorphism, whenever $H_t(P_\bullet) \neq 0$.

The theorem immediately applies to complexes $P^*_\bullet \in Ch^b(\mathcal{P}(A))$ whose homologies are locally CMFPD (see Corollary 3.7). The theorem also applies to a number of interesting subcategories of $\mathcal{B}$, for which ι_t is already an isomorphism, for all such t. Among them are the categories $\mathcal{B}(n) = \{M \in \mathcal{B} : proj\dim(M) = n\}$. Note, $\mathcal{A} := \mathcal{B}(d)$ is the category of modules of finite length and finite projective dimension. The stability of $P_\bullet \in Ch^b_{\mathcal{A}}(\mathcal{P}(A))$, under duality is a theorem in [4].

We underscore that this paper is part of a wider study [4,5] of duality of subcategories of derived categories and their Witt groups. While we have particular interest in the setting of singular varieties and we also provide further insight into nonsingular varieties in these articles. Our interest in Witt theory stems from the introduction of Chow–Witt groups, [1] and developed by Fasel [3], as obstruction groups for projective modules to split off a free direct summand. The readers are referred to [4] for further introductory comments. To be more specific, one of the primary motivations behind this study has been to address the Witt theory, for non-regular (Cohen–Macaulay) schemes X, with $\dim X = d$, while this article addresses the duality aspect of the same. Let $\mathscr{V}(X)$ denote the category of locally free sheaves on X and $D^b(\mathscr{V}(X))$ denote the derived category of finite complexes of locally free sheaves. Let $\mathbb{M}(X, d)$ denote the subcategory of $Coh(X)$ with finite length and finite $\mathscr{V}(X)$-dimension. (*For unexplained notations, readers are referred to* [5,4].) As a consequence of Theorem 3.6, it follows that $D^b_{\mathbb{M}(X,d)}(\mathscr{V}(X))$ is closed under the usual duality induced by $\mathcal{E} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$. This allows us [5,4] to give a definition of shifted Witt groups $W^r(D^b_{\mathbb{M}(X,d)}(X))$ of $D^b_{\mathbb{M}(X,d)}(X)$, while $D^b_{\mathbb{M}(X,d)}(X)$ fails to inherit a triangulated structure. Also note that $\mathbb{M}(X, d)$ has a duality sending $\mathcal{F} \mapsto \mathcal{E}xt^d(\mathcal{F}, \mathcal{O}_X)$ and hence a Witt group $W(\mathbb{M}(X, d))$ is defined. It was established that $W(\mathbb{M}(X,d)) \xrightarrow{\sim} W^{\dim X}(D^b_{\mathbb{M}(X,d)}(X))$ is an isomorphism [5,4]. When X is regular, this would be a result of Balmer and Walter [2], while in the non-regular case, the right side would be meaningful only in the light of Theorem 3.6. Note, when X is regular $\mathbb{M}(X,d)$ is subcategory of schemes of finite length.

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2. Preliminaries

We will borrow some notations from [4].

Notations 2.1. Throughout this article, A **will denote a noetherian commutative ring, such that** $\dim A_{\mathfrak{m}} = d \ \forall \ \mathfrak{m} \in Max(A)$**. All modules will be assumed to be finitely generated, unless stated.** Most often, A will be assumed to be Cohen–Macaulay. We will also use the abbreviation "CM" for "Cohen–Macaulay". We set up the notations:

1. $Mod(A)$ will denote the category of finitely generated A-modules.
2. $MFPD(A)$ will denote the category of finitely generated A-modules with finite projective dimension.
3. For integers $r \geq 0$, denote

$$\mathcal{B}(r) := \mathcal{B}(A)(r) := \{M \in MFPD(A) : proj\dim(M) = grade(M) = r\}$$

and

$$\mathcal{B} := \mathcal{B}(A) := \bigcup_{r \geq 0} \mathcal{B}(r)$$

Note, when A is local and CM, then $\mathcal{B}$ is the category of finitely generated Cohen–Macaulay A-modules with finite projective dimension (see 2.7). In the non-local case, there are modules $M \in MFPD(A)$ which are locally CM, with $M \notin \mathcal{B}$, e.g. $A = k[X,Y], M = \dfrac{k[X,Y]}{(X,Y)} \oplus \dfrac{k[X,Y]}{(X-1)}$.
4. $\mathcal{P}(A)$: category of finitely generated projective A-modules.
5. $Ch^b(\mathcal{P}(A))$ will denote the category of bounded complexes of finitely generated projective A-modules. Also, $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ will denote the category of complexes $P_\bullet$ in $Ch^b(\mathcal{P}(A))$ such that all the homologies $H_i(P_\bullet)$ are in $\mathcal{B}$.
6. We will denote complexes $P_\bullet$ in $Ch^b(\mathcal{P}(A))$ by:

$$\cdots 0 \longrightarrow P_m \xrightarrow{\partial_m} P_{m-1} \longrightarrow \cdots\cdots \longrightarrow P_n \longrightarrow 0 \cdots$$

7. For a complex $P_\bullet$ of projective A-modules $P_\bullet^*$ will denote the usual dual induced by $Hom(\_\_, A)$.

8. Let $B_r = B_r(P_\bullet) := \partial_{r+1}(P_{r+1}) \subseteq P_r$ denote the module of r-boundaries and $Z_r = Z_r(P_\bullet) := \ker(\partial_r) \subseteq P_r$ denote the module of r-cycles (or the r^{th} syzygy).
9. The r^{th}-homology of $P_\bullet$ will be denoted by $H_r = H_r(P_\bullet) := \frac{Z_r}{B_r}$. So, the r^{th}-homology of the dual is $H_r(P_\bullet^*) = \frac{\ker\left(\partial^*_{-(r-1)}\right)}{image(\partial^*_{-r})}$.
10. By width, we mean the number $h - l$ where $P_h \neq 0, P_l \neq 0$ and $P_k = 0$ for all $r \notin [l, h]$.

Definition 2.2. A complex $P_\bullet$ as in 2.1, is called **indecomposable**, if

$$P_\bullet = U_\bullet \oplus V_\bullet \Longrightarrow U_\bullet = 0 \quad \text{or} \quad V_\bullet = 0.$$

We say that $P_\bullet$ is **decomposable**, if $P_\bullet$ is not indecomposable.

Lemma 2.3. *Let $P_\bullet$ be as in 2.1. If B_{r-1} is projective, then $P_\bullet$ is a direct sum of the following two complexes*

$$\cdots \longrightarrow P_{r+1} \longrightarrow Z_r \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow B_{r-1} \longrightarrow P_{r-1} \longrightarrow P_{r-2} \longrightarrow P_{r-3} \longrightarrow \cdots \longrightarrow \cdots$$

Example 2.4. Given an indecomposable projective module P, there is a trivial indecomposable complex

$$0 \longrightarrow P = P \longrightarrow 0$$

which is quasi-isomorphic to 0.

Lemma 2.5. *Let*

$$P_\bullet \qquad \cdots \longrightarrow P_{r+1} \longrightarrow P_r \longrightarrow P_{r-1} \longrightarrow \cdots$$

be an indecomposable complex of projective modules. Assume, for all r, H_r is CM and has finite projective dimension (i.e. $H_r \in \mathcal{B}$).

If $\dim A = 0$, then $width(P_\bullet) = 0$ or $P_\bullet$ is a trivial indecomposable complex (as in 2.4). If $\dim A = 1$, then $width(P_\bullet) = 1$.

Proof. First, assume $d = 0$. Since, $P_\bullet$ is a bounded complex, we can assume $P_r = 0$ for all $r < 0$ and $P_0 \neq 0$. In this case, $H_0 = 0$ or $proj \dim H_0 = 0$. If $H_0 \neq 0$ then $P_0 = B_0 \oplus H_0$. So, $P_\bullet$ has a direct summand

$$H_\bullet : \qquad 0 \longrightarrow H_0 \longrightarrow 0.$$

Since, it is indecomposable, $P_\bullet = H_\bullet$. So, width is zero. If $H_0 = 0$ then $P_1 \xrightarrow{\sim} P_0 \oplus Z_1$. Since $P_\bullet$ is indecomposable, $Z_1 = 0$ and $P_1 \xrightarrow{\sim} P_0$. So, $P_\bullet$ is given by

$$0 \longrightarrow P_1 \xrightarrow{\sim} P_0 \longrightarrow 0.$$

This is isomorphic to

$$0 \longrightarrow P_0 = P_0 \longrightarrow 0.$$

and P_0 is also indecomposable.

Now assume $d = 1$. We assume $P_0 \neq 0$ and $P_r = 0$ for all $r < 0$. If $H_0(P_\bullet) = 0$, by the same argument as above $P_\bullet$ is isomorphic to

$$0 \longrightarrow P_0 = P_0 \longrightarrow 0.$$

and P_0 is also indecomposable. So, assume $H_0 \neq 0$. Then, we have an exact sequence

$$0 \longrightarrow B_0 \longrightarrow P_0 \longrightarrow H_0 \longrightarrow 0.$$

Then, $\mathit{proj}\dim H_0 = 0$ **or** 1. In either case, it follows B_0 is projective. So, $P_1 = Z_1 \oplus B_0$. Note $B_0 \neq P_0$. So, $P_\bullet$ splits into two complexes:

$$Q_\bullet: \qquad \cdots \longrightarrow 0 \longrightarrow B_0 \longrightarrow P_0 \longrightarrow 0$$

$$P'_\bullet: \qquad \cdots \longrightarrow P_2 \longrightarrow Z_1 \longrightarrow 0 \longrightarrow 0$$

and hence $P_\bullet = Q_\bullet$. So the proof is complete. □

2.1. CM modules, grade and finite projective dimension

Readers are referred to [6] for a definition of $grade(M)$ and for a proof of the following lemma.

Lemma 2.6. *Suppose $(A, \mathfrak{m})$ is a Cohen–Macaulay local ring and $\dim A = d$. Suppose M is a Cohen–Macaulay A-module with finite projective dimension.*

1. *Then,*

$$\mathit{proj}\dim M + \dim M = d \quad \textit{and} \quad \mathit{proj}\dim M = grade(M).$$

2. *Let $r = \mathit{proj}\dim M$. Then, $\forall\ i \neq r$, $Ext^i(M, A) = 0$, and $\mathit{proj}\dim Ext^r(M, A) = r$.*

Corollary 2.7. *Suppose* $(A, \mathfrak{m})$ *is a Cohen–Macaulay local ring with* $\dim A = d$. *Suppose* M *is a finitely generated* A*-module with finite projective dimension. Then,* M *is Cohen–Macaulay if and only if*

$$\textit{proj}\dim M = \textit{grade}(M) = \textit{height}(\textit{ann}(M)).$$

Proof. Suppose M is Cohen–Macaulay. Write $I = \textit{ann}(M)$. Then $\textit{depth}M = \dim M = \dim\left(\frac{A}{I}\right) = d - \textit{height}(I)$. Therefore $\textit{proj}\dim(M) = \textit{height}(I)$. Also, $\textit{grade}(M) = \textit{depth}_I(A) = \textit{height}(I)$ (see [6, p. 108]).

Now suppose $\textit{proj}\dim M = \textit{grade}(M) = \textit{height}(I)$. Now, $\textit{height}(I) = d - \dim M$. So, $\textit{proj}\dim M + \dim M = d$. Hence $\dim M = \textit{depth}(M)$. So, M is Cohen–Macaulay. □

The following lemma will be useful for our subsequent discussions.

Lemma 2.8. *Let* A *be a Cohen–Macaulay ring with* $\dim A = d$ (non-local but as in 2.1) *and* $M \neq 0$ *be in* $\mathcal{B}$. *Let* $\wp_0, \wp_1 \in \textit{Supp}(M)$ *and* $\wp_0 \subseteq \wp_1$. *Then,* $\textit{proj}\dim(M_{\wp_0}) = \textit{proj}\dim(M_{\wp_1})$.

We state two lemmas without proof.The proof of the second lemma can be found in ([4, Proposition 2.8]).

Lemma 2.9. *Let* A *be a CM ring and* M, N *be finitely generated* A*-modules such that* M *has finite projective dimension. Let* $r \geq 0$ *be an integer. Then* $\textit{height}(\textit{Ann}(\textit{Ext}^r(M, N))) \geq r$. *In this case,* $\textit{grade}(\textit{Ext}^r(M, N)) \geq r$.

Lemma 2.10. *Let* $P_\bullet$ *be a bounded complex of projective modules such that all the homologies* H_i *have finite projective dimension. Then so do all the boundaries* B_i*, kernels* Z_i *and the cokernels* $\frac{P_i}{B_i}$. *In fact,*

$$\textit{proj}\dim(B_r) \leq \dim A - 1, \quad \textit{proj}\dim(Z_r) \leq \dim A - 2, \quad \forall\ r.$$

Lemma 2.11. *Let* A *be a CM ring and* $P_\bullet$ *be a complex in* $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$. *Let* $t_0, t_1, \ldots$ *be the list of all the degrees so that* $H_{t_i}(P_\bullet) \neq 0$ *and* $\rho_i = \textit{proj}\dim(H_{t_i}(P_\bullet))$. *Then,*

$$\textit{grade}\left(\textit{Ext}^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right)\right) = \rho_k.$$

Proof. To begin with, note that $\frac{P_{t_k}}{B_{t_k}}$ has finite projective dimension by 2.10 and hence, by 2.9,

$$\textit{grade}\left(\textit{Ext}^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right)\right) \geq \rho_k.$$

Now, consider the exact sequence

$$0 \longrightarrow H_{t_k} \longrightarrow \frac{P_{t_k}}{B_{t_k}} \longrightarrow B_{t_k-1} \longrightarrow 0$$

and the corresponding five term exact sequence

$$0 \longrightarrow Ext^{\rho_k}\left(B_{t_k-1}, A\right) \longrightarrow Ext^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right) \longrightarrow Ext^{\rho_k}\left(H_{t_k}, A\right)$$

$$\longrightarrow Ext^{\rho_k+1}\left(B_{t_k-1}, A\right) \longrightarrow Ext^{\rho_k+1}\left(\frac{P_{t_k}}{B_{t_k}}, A\right) \longrightarrow 0$$

Choose $\wp \in Spec(A)$ with $height(\wp) = \rho_k$, such that $Ext^{\rho_k}\left(H_{t_k}, A\right)_{\wp} \neq 0$. Applying 2.9, we get $Ext^{\rho_k+1}\left(B_{t_k-1}, A\right)_{\wp} = 0$. Hence, $Ext^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right)_{\wp} \neq 0$. Hence,

$$grade\left(Ext^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right)\right) = \rho_k.$$

The proof is complete. □

Remark 2.12. Let $s_j := t_j - t_{j-1}$. Note in the previous proof that if $\rho_k \geq 1$, then the 5-term exact sequence can be rewritten as:

$$0 \longrightarrow Ext^{\rho_k+s_k}\left(\frac{P_{t_{k-1}}}{B_{t_{k-1}}}, A\right) \longrightarrow Ext^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right) \longrightarrow Ext^{\rho_k}\left(H_{t_k}, A\right)$$

$$\longrightarrow Ext^{\rho_k+1+s_k}\left(\frac{P_{t_{k-1}}}{B_{t_{k-1}}}, A\right) \longrightarrow Ext^{\rho_k+1}\left(\frac{P_{t_k}}{B_{t_k}}, A\right) \longrightarrow 0$$

and hence it follows that for $\wp \in Spec(A)$ with $height(\wp) < \rho_k + s_k$, we have

$$Ext^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right)_{\wp} \cong Ext^{\rho_k}\left(H_{t_k}, A\right)_{\wp}.$$

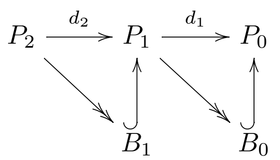
3. Dualizable complexes

In this section, we classify complexes $P_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ whose dual is also in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$. The following lemma will be useful subsequently.

Lemma 3.1. *Suppose* $P_\bullet: \ P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_2} P_0$ *is a complex of projective A-modules. Then, there is an exact sequence*

$$0 \longrightarrow \left(\frac{P_0}{B_0}\right)^* \longrightarrow P_0^* \longrightarrow \left(\frac{P_1}{B_1}\right)^* \longrightarrow H_{-1}(P_\bullet^*) \longrightarrow 0$$

Proof. We have the diagrams:



$$P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{{d_2}^*} P_2^*$$

So,

$$\ker(d_2^*) = \{\lambda \in P_1^* : \lambda_{|B_1} = 0\} = \left(\frac{P_1}{B_1}\right)^*.$$

Also,

$$Image(d_1^*) \cong \{\lambda_{|B_0} : \lambda \in P_0^*\}$$

Let

$$K = \{\lambda \in P_0^* : \lambda_{|B_0} = 0\} = \left(\frac{P_0}{B_0}\right)^*.$$

Then we have an exact sequence

$$0 \longrightarrow K \longrightarrow P_0^* \longrightarrow Image(d_1^*) \longrightarrow 0$$

So,

$$0 \longrightarrow \left(\frac{P_0}{B_0}\right)^* \longrightarrow P_0^* \longrightarrow Image(d_1^*) \longrightarrow 0$$

This completes the proof. □

The following is a key tool to compute the homologies of the dual.

Proposition 3.2. *Suppose*

$$P_\bullet: \qquad P_{s+1} \longrightarrow P_s \longrightarrow P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0$$

is a complex of projective A-modules with $s > 0$ such that $H_i(P_\bullet) = 0$ for $i = 1, \ldots, s-1$ and $H_s(P_\bullet) \in \mathcal{B}$. Then,

$$\forall\ 0 < t < s \quad H_{-t}(P_\bullet^*) = Ext^1\left(\frac{P_{t-1}}{B_{t-1}}, A\right) = Ext^t\left(\frac{P_0}{B_0}, A\right).$$

If $proj\dim(H_s) \geq 1$, then the above statement also holds for $t = s$.

Proof. For $0 < t < s$ the statement follows because the complex is exact between 0 and s. For the last statement, we have an exact sequence

$$0 \longrightarrow H_s \longrightarrow \tfrac{P_s}{B_s} \longrightarrow B_{s-1} \longrightarrow 0.$$

Since $proj\dim(H_s) \geq 1$ and $H_s \in \mathcal{B}$, we conclude that $H_s^* = 0$ and hence, $B_{s-1}^* \cong \left(\frac{P_s}{B_s}\right)^*$. Therefore, in conjunction with 3.1, we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 \to & Ext^0\left(\frac{P_{s-1}}{B_{s-1}}, A\right) & \to & Ext^0(P_{s-1}, A) & \to & Ext^0(B_{s-1}, A) & \to & Ext^1\left(\frac{P_{s-1}}{B_{s-1}}, A\right) & \to 0 \\
& \| & & \| & & \downarrow \wr & & \downarrow \wr & \\
0 \to & Ext^0\left(\frac{P_{s-1}}{B_{s-1}}, A\right) & \to & Ext^0(P_{s-1}, A) & \to & Ext^0\left(\frac{P_s}{B_s}, A\right) & \to & H_{-s}(P_\bullet^*) & \to 0
\end{array}$$

□

Theorem 3.3. *Let $P_\bullet$ be a complex in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$. Let $t_0 < t_1 < t_2 < \ldots$ be the list of all the degrees so that $H_{t_i}(P_\bullet) \neq 0$. Write $\rho_i = proj\dim(H_{t_i}(P_\bullet))$. Assume $\rho_j \geq 1$ for all j. Then, the following are equivalent:*

1. *For all $k \geq 1$ and all j, the module $Ext^k\left(\frac{P_{t_j}}{B_{t_j}}, A\right)$ is in $\mathcal{B}$.*
2. *For all j, the module $Ext^{\rho_j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right)$ is in $\mathcal{B}$.*
3. *The natural homomorphism*

$$\iota_j : Ext^{\rho_j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right) \xrightarrow{\sim} Ext^{\rho_j}\left(H_{t_j}, A\right) \quad \textit{is an isomorphism} \quad \forall j.$$

In this case, for all j, $Ext^0\left(\frac{P_{t_j}}{B_{t_j}}, A\right)$ has finite projective dimension.

Proof. Clearly, (1) $\Longrightarrow$ (2). We will prove (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).
(3) $\Longrightarrow$ (1): Suppose ι_j is an isomorphism for all j. Since the complex is exact before t_0, it splits and hence, B_{t_0-1} is projective. Hence, $P_{t_0} \cong Z_{t_0} \oplus B_{t_0-1}$. So,

$$Ext^k\left(\frac{P_{t_0}}{B_{t_0}}, A\right) \cong Ext^k\left(\frac{Z_{t_0}}{B_{t_0}}, A\right) \oplus Ext^k\left(B_{t_0-1}, A\right) \cong \begin{cases} 0 & k \neq 0, \rho_0 \\ Ext^{\rho_0}\left(H_{t_0}, A\right), & k = \rho_0 \\ B^*_{t_0-1} & k = 0 \end{cases}.$$

Hence, $Ext^k\left(\frac{P_{t_0}}{B_{t_0}}, A\right)$ is in $\mathcal{B}$ for all k. Now we do the induction step. So, we assume the statement holds for all $j \le j_0$ and prove it for $j_1 = j_0+1$. Let $s = t_{j_1} - t_{j_0}$. Corresponding to the short exact sequence:

$$0 \longrightarrow H_{t_{j_1}} \longrightarrow \frac{P_{t_{j_1}}}{B_{t_{j_1}}} \longrightarrow B_{t_{j_1}-1} \longrightarrow 0 \qquad (I),$$

we get a long exact sequence

$$0 \longrightarrow Ext^0(B_{t_{j_1}-1}, A) \longrightarrow Ext^0\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \longrightarrow 0 \cdots\cdots\cdots$$

$$Ext^{\rho_{j_1}-1}(B_{t_{j_1}-1}, A) \to Ext^{\rho_{j_1}-1}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \longrightarrow 0 \longrightarrow Ext^{\rho_{j_1}}(B_{t_{j_1}-1}, A) \longrightarrow$$

$$Ext^{\rho_{j_1}}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \xrightarrow[\sim]{\iota_{j_1}} Ext^{\rho_{j_1}}\left(H_{t_{j_1}}, A\right) \longrightarrow Ext^{\rho_{j_1}+1}(B_{t_{j_1}-1}, A) \to Ext^{\rho_{j_1}+1}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \to 0$$

Since ι_{j_1} is an isomorphism,

$$Ext^{\rho_{j_1}}(B_{t_{j_1}-1}, A) = 0, \quad Ext^{\rho_{j_1}}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \cong Ext^{\rho_{j_1}}\left(H_{t_{j_1}}, A\right), \text{ which is in } \mathcal{B}.$$

$$\text{Also,} \quad Ext^{\rho_{j_1}+1}(B_{t_{j_1}-1}, A) \cong Ext^{\rho_{j_1}+1}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right).$$

Then,

$$\forall k \neq \rho_{j_1}, k \ge 1 \quad Ext^k\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \cong Ext^k(B_{t_{j_1}-1}, A) \cong Ext^{k+s}\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right)$$

which are all in $\mathcal{B}$, by induction hypothesis. Also

$$Ext^0\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \cong Ext^0(B_{t_{j_1}-1}, A) \text{ has finite projective dimension.}$$

(2) $\Longrightarrow$ (3): To prove this we assume that ι_j is not an isomorphism for some j.

So, $\exists \mathfrak{m}$ and j such that $(\iota_j)_{\mathfrak{m}}$ is not an isomorphism. It will be enough to prove that $Ext^{\rho_i}\left(\frac{P_{t_i}}{B_{t_i}}, A\right)_{\mathfrak{m}}$ either has infinite projective dimension or is not CM, for some i. By replacing A by $A_{\mathfrak{m}}$, we assume A is local.

Further, we write $P_\bullet = \bigoplus_{n=1}^r P_\bullet^n$, where $P_\bullet^n$ are indecomposable complexes in $Ch_{\mathcal{B}}(\mathcal{P}(A))$. If $H_{t_j}(P_\bullet^n) \neq 0, \rho_j = grade(H_{t_j}) \le grade(H_{t_j}(P_\bullet^n)) \le proj\dim(H_{t_j}(P_\bullet^n)) \le proj\dim(H_{t_j}) = \rho_j$ and so $proj\dim(H_{t_j}(P_\bullet^n)) = \rho_j$ and also, for some j, n the homomorphism $\iota_j : Ext^{\rho_j}\left(\frac{P^n_{t_j}}{B^n_{t_j}}, A\right) \longrightarrow Ext^{\rho_j}\left(H_{t_j}(P^n_\bullet), A\right)$ is not an isomorphism. We will replace $P_\bullet$ by such a $P^n_\bullet$ and assume that $P_\bullet$ is indecomposable. Let

$$j_1 = \min\left\{ j : Ext^{\rho_j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right) \xrightarrow{\iota_j} Ext^{\rho_j}\left(H_{t_j}, A\right) \text{ is not an isomorphism}\right\}$$

Note that $j_1 > 0$ since $\frac{P_{t_0}}{B_{t_0}} = H_{t_0} \oplus B_{t_0 - 1}$. Let $j_0 = j_1 - 1$ and $s = t_{j_1} - t_{j_0}$. By minimality of j_1, since we already proved ($3 \Longrightarrow 1$), it follows that (1) applies to $\dfrac{P_{t_{j_0}}}{B_{t_{j_0}}}$. So, $Ext^0\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right)$ has finite projective dimension and

$$\forall\ k \geq 1 \quad Ext^k\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right) \quad \text{is in } \mathcal{B}.$$

It follows that $Ext^0\left(B_{t_{j_1}-1}, A\right)$ has finite projective dimension and

$$\forall\ k \geq 1 \quad Ext^k\left(B_{t_{j_1}-1}, A\right) \cong Ext^{k+s}\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right) \quad \text{is in } \mathcal{B}.$$

Corresponding to the short exact sequence (I), we have the five term exact sequence

$$\begin{aligned} 0 \to Ext^{\rho_{j_1}+s}\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right) &\to Ext^{\rho_{j_1}}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \xrightarrow{\iota_{j_1}} Ext^{\rho_{j_1}}\left(H_{t_{j_1}}, A\right) \\ \xrightarrow{\partial} Ext^{\rho_{j_1}+1+s}\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right) &\to Ext^{\rho_{j_1}+1}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right) \to 0. \end{aligned}$$

By 2.11,

$$grade\left(Ext^{\rho_{j_1}}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right)\right) = \rho_{j_1}.$$

For notational convenience, write

$$K = Ext^{\rho_{j_1}+s}\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right) \qquad E = Ext^{\rho_{j_1}}\left(\frac{P_{t_{j_1}}}{B_{t_{j_1}}}, A\right), \qquad C = image(\iota_{j_1}),$$

$$\mathcal{H} = Ext^{\rho_{j_1}}\left(H_1, A\right), \qquad B = image(\partial), \qquad K' = Ext^{\rho_{j_1}+1+s}\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right)$$

If E has infinite projective dimension then E is not in $\mathcal{B}$. So, assume E has finite projective dimension. All the modules listed above have finite projective dimension. Since ι_{j_1} is not an isomorphism, either $K \neq 0$ or $C \neq 0$. Assume $K \neq 0$. We have $g := grade(K) \geq \rho_{j_1} + s$. Let $\wp \in Supp(K)$ such that $height(\wp) = g$. Therefore

$$g = grade(K_\wp) \leq proj\dim(K_\wp) \leq \dim(A_\wp) = g, \quad \text{and so} \quad proj\dim(K_\wp) = g.$$

Since $Tor^{g+1}(\frac{A_\wp}{\wp A_\wp}, C_\wp) = 0$, it follows from the sequence of Tor modules that $proj\dim(E_\wp) = g$. So, $proj\dim(E) \geq proj\dim(E_\wp) = g \geq \rho_{j_1} + s > \rho_{j_1}$. So, E is not CM and hence not in $\mathcal{B}$.

Now assume $K = 0$. Then it follows that $B \neq 0$, otherwise ι_{j_1} would be an isomorphism. Consider the exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{H} \longrightarrow B \longrightarrow 0.$$

Since $B \subseteq K'$, $g := grade(B) \geq grade(K') \geq \rho_{j_1} + s + 1$. Let $\wp \in Supp(B)$ such that $height(\wp) = g$ and $proj\dim(B_\wp) = g$. Since $proj\dim(\mathcal{H}) = \rho_{j_1} \leq g - 2$, it follows $proj\dim(E_\wp) = proj\dim(B_\wp) - 1 = g - 1 > \rho_{j_1}$. So, E is not CM and hence not in $\mathcal{B}$. The proof is complete. □

The following will be needed for our subsequent inductive argument, which follows from the proof of 3.3.

Corollary 3.4. *Let $P_\bullet$ be as in 3.3. Further, assume A is local and $P_\bullet$ is indecomposable. Suppose $\iota_j : Ext^{\rho_j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right) \to Ext^{\rho_j}\left(H_{t_j}, A\right)$ is not an isomorphism for some j.*

Let $j_0 = \min\{j : \iota_j$ is not an isomorphism$\}$. Then,

1. *$E := Ext^{\rho_{j_0}}\left(\frac{P_{t_{j_0}}}{B_{t_{j_0}}}, A\right)$ is not in $\mathcal{B}$.*
2. *For all $k < j_0$, $Ext^{\rho_k}\left(\frac{P_{t_k}}{B_{t_k}}, A\right)$ is in $\mathcal{B}$.*
3. *For all $\wp \in Supp(E)$ with $height(\wp) = \rho_{j_0}$, $E_\wp \in \mathcal{B}$ (for the ring $A_\wp$).*

Proof. The proof follows from the proof of 3.3 and 2.11. □

We proceed to give an equivalence condition for duality. The following proposition will be useful for our inductive argument.

Proposition 3.5. *Let $P_\bullet$ as in 3.3. Further, assume A is local and $P_\bullet$ is indecomposable.*

$$X^{(k)} = \{\wp \in X = Spec(A) : height(\wp) = k\} \quad \text{and} \quad g_j^k = grade\left(Ext^j\left(\frac{P_{t_k}}{B_{t_k}}, A\right)\right).$$

Assume, for some $r_0 \geq 1$ and some t_{i-1}, $Ext^{r_0}\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right)$ is not in $\mathcal{B}$, and

$$\forall \wp \in X^{\left(g_j^{i-1}\right)}, \quad \forall 1 \leq j \leq r_0 \qquad Ext^j\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right)_\wp \quad \text{is in } \mathcal{B}(A_\wp).$$

Then

1. *either* $H_{-r}(P^*_\bullet)$ *is not in* $\mathcal{B}$ *for some* $r > t_{i-1}$.
2. *or there is an* $r_1 \geq 1$, *such that the following hold:*

$$Ext^{r_1}\left(\frac{P_{t_i}}{B_{t_i}}, A\right) \text{ is not in } \mathcal{B} \quad \text{and}$$

$$\forall \wp \in X^{\left(g^i_j\right)}, \quad \forall 1 \leq j \leq r_1 \quad Ext^j\left(\frac{P_{t_i}}{B_{t_i}}, A\right)_\wp \quad \text{is in} \quad \mathcal{B}.$$

In particular, if $t_{i-1} = t_n$ *is the last one, then 1 must hold.*

Proof. Let $s = t_i - t_{i-1}$. If $r_0 \leq s$ then by 3.2,

$$H_{-(r_0+s)}(P^*_\bullet) = Ext^{r_0}\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right) \quad \text{is not in} \quad \mathcal{B}.$$

So, assume $r_0 - s > 0$. The latter part of assertion (2) would follow if $Ext^j\left(\frac{P_{t_i}}{B_{t_i}}, A\right)$ had finite projective dimension, which may not always be the case. The long exact sequence of Ext modules corresponding to the usual exact sequence $0 \longrightarrow H_{t_i} \longrightarrow \frac{P_{t_i}}{B_{t_i}} \longrightarrow B_{t_i-1} \longrightarrow 0$ gives us that:

- $$Ext^l(B_{t_i-1}, A) \xrightarrow{\sim} Ext^l\left(\frac{P_{t_i}}{B_{t_i}}, A\right) \forall l \neq \rho_i, \rho_i + 1.$$

- There is a 5-term sequence

$$0 \longrightarrow Ext^{\rho_i}(B_{t_i-1}, A) \longrightarrow Ext^{\rho_i}\left(\frac{P_{t_i}}{B_1}, A\right) \xrightarrow{\iota_i} Ext^{\rho_i}(H_{t_i}, A) \xrightarrow{\partial}$$

$$Ext^{\rho_i+1}(B_{t_i-1}, A) \longrightarrow Ext^{\rho_i+1}\left(\frac{P_{t_i}}{B_{t_i}}, A\right) \longrightarrow 0$$

We break the proof into various cases.
(Case $r_0 - \rho_i < s$**):** So, $1 \leq r_0 - s < \rho_i$. We have

$$\forall \ \ k \geq 1 \qquad Ext^k(B_{t_i-1}, A) \cong Ext^{k+s}\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right).$$

So, from the long exact sequence above, we extract the following row:

$$0 \longrightarrow Ext^{r_0}\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right) \longrightarrow Ext^{r_0-s}\left(\frac{P_{t_i}}{B_{t_i}}, A\right) \longrightarrow 0$$

With $r_1 = r_0 - s$, our assertion is satisfied.
(Case $\rho_i \leq r_0 - s$): It follows from the long exact sequence of the ext modules

$$\forall 1 \leq j < \rho_i \quad Ext^j\left(\frac{P_{t_i}}{B_{t_i}}, A\right) \cong Ext^{j+s}\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right)$$

For $j < \rho_i$, we have $j + s \leq \rho_i + s < r_0$. Therefore,

$$\forall 1 \leq j < \rho_i,\ \wp \in X^{(g^i_j)} \quad Ext^j\left(\frac{P_{t_i}}{B_{t_i}}, A\right)_\wp \quad \text{has FPD and is CM.} \qquad (I)$$

$$\text{By the grade Lemma 2.11 } grade\left(Ext^{\rho_i}\left(\frac{P_{t_i}}{B_{t_i}}, A\right)\right) = \rho_i \qquad (II)$$

So, $g^i_{\rho_i} = \rho_i$ and we have

$$\forall\ \wp \in X^{(\rho_i)} \quad Ext^{\rho_i}\left(\frac{P_{t_i}}{B_{t_i}}, A\right)_\wp \cong Ext^{\rho_i}(H_{t_i}, A)_\wp \quad \text{is in} \quad \mathcal{B}. \qquad (III).$$

We will show that $E := Ext^{\rho_i}\left(\frac{P_{t_i}}{B_{t_i}}, A\right)$ is not in $\mathcal{B}$. If E has infinite projective dimension, then E is not in $\mathcal{B}$. So, we will assume E has finite projective dimension. For convenience, we use notations

$$\left.\begin{array}{lll} K = Ext^{\rho_i+s}\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right) & E = Ext^{\rho_i}\left(\frac{P_{t_i}}{B_{t_i}}, A\right), & C = image(\iota), \\ \mathcal{H}_1 = Ext^{\rho_i}(H_{t_i}, A), & B = image(\partial), & K' = Ext^{\rho_i+1+s}\left(\frac{P_{t_{i-1}}}{B_{t_{i-1}}}, A\right), \\ E' = Ext^{\rho_i+1}\left(\frac{P_{t_i}}{B_{t_i}}, A\right). & & \end{array}\right\}$$

1. **Now assume $K \neq 0$.** So, we have an exact sequence

$$0 \longrightarrow K \longrightarrow E \longrightarrow C \longrightarrow 0$$

Now $g := grade(K) \geq \rho_i + s$ and hence $C \neq 0$. Let $\wp \in Supp(K)$ and $height(\wp) = g$. Since $\rho_i + s \leq r_0$, $(K)_\wp$ has finite projective dimension. It follows that $proj\dim(K_\wp) = g$. So, $C_\wp$ also has finite projective dimension. From the sequence of Tor modules, it follows that $proj\dim(E_\wp) = g \geq \rho_i + s$. So, $proj\dim(E) \geq \rho_i + s > \rho_i$. Hence E is not in $\mathcal{B}$. With $r_1 = \rho_i$ the assertion of the theorem is valid.

2. **Assume** $K = 0$ **and** $K' \neq 0$**.** Since $K = 0$, we have $r_0 - s \geq \rho_i + 1$. Also, $E \xrightarrow{\sim} C$. The five term exact sequence reduces to

$$0 \longrightarrow E \longrightarrow \mathcal{H}_1 \longrightarrow K' \longrightarrow E' \longrightarrow 0$$

We have $g' := grade(K') \geq \rho_i + 1 + s$.
Assume $B \neq 0$**:** Then, $g := grade(B) \geq grade(K') \geq \rho_i + 1 + s$. Now, we have an exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{H}_1 \longrightarrow B \longrightarrow 0$$

Since $E, \mathcal{H}_1$ have finite projective dimension, so does B. Let $\wp \in Supp(B)$ and $height(\wp) = g$. Then $\dim(A_\wp) = g$ and $proj \dim(B_\wp) = g \geq \rho_i + 1 + s \geq \rho_i + 2$. Hence it follows that $proj \dim(E_\wp) = g - 1 \geq \rho_i + s$. So, $proj \dim(E) \geq \rho_i + s$ and hence E is not in $\mathcal{B}$. In this case also, with $r_1 = \rho_i$ the assertion of the theorem is valid.
Assume $B = 0$**:** Then we get that $E \cong \mathcal{H}_1$, $K' \cong E'$ and choosing $r_1 = r_0 - s$ proves the assertion.
3. **Assume** $K = K' = 0$**.** In this case, $\iota : E \longrightarrow \mathcal{H}_1$ is an isomorphism, $E' = 0$ and hence $r_0 - s \geq \rho_i + 2$. The assertion of the theorem is satisfied with $r_1 = r_0 - s$.

The proof is complete. □

Theorem 3.6. *Let $P_\bullet$ be a complex in $Ch_{\mathcal{B}}(\mathcal{P}(A))$ with notations as in 3.3. Then, $P_\bullet^*$ is in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ if and only if $\iota_j : Ext^{\rho_j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right) \longrightarrow Ext^{\rho_j}\left(H_{t_j}(P_\bullet), A\right)$ is an isomorphism $\forall j$.*

Proof. Assume that ι_j is an isomorphism for all j. Then by Theorem 3.3,

$$\forall k \geq 1, \forall j \quad Ext^k\left(\frac{P_{t_j}}{B_j}, A\right) \quad \text{is in} \quad \mathcal{B}.$$

So, by 3.2,

$$\forall t_j < t \leq t_{j+1} \quad H_t(P_\bullet^*) = Ext^{t-t_j}\left(\frac{P_{t_j}}{B_j}, A\right) \quad \text{is in} \quad \mathcal{B}.$$

So, $P_\bullet^*$ is in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$.

Now suppose ι_j is not an isomorphism for some j. Arguing exactly as in the proof of Theorem 3.3, we assume that A is local and $P_\bullet$ is indecomposable. Now, by Corollary 3.4, there exists $r \geq 1$ and i such that $Ext^r\left(\frac{P_{t_i}}{B_{t_i}}, A\right)$ is not in $\mathcal{B}$, and

$$\forall \wp \in X^{\left(g_j^{i-1}\right)}, \quad \forall 1 \leq j \leq r \qquad Ext^j\left(\frac{P_{t_i}}{B_{t_i}}, A\right)_\wp \quad \text{is in} \quad \mathcal{B}.$$

This is exactly the hypothesis of Proposition 3.5, which guarantees us that either there exists a homology $H_r(P_\bullet^*)$ which is not in $\mathcal{B}$ or that one obtains $r_1 \geq 1$ so that the above hypothesis repeats for $i+1$. Note that for $i = n$, the only possibility is that there exists a homology $H_r(P_\bullet^*)$ which is not in $\mathcal{B}$. Thus, by induction, the above process repeats until we produce a homology $H_r(P_\bullet^*)$ which is not $\mathcal{B}$. So, $P_\bullet^*$ is not in $Ch_{\mathcal{B}}^b(\mathcal{P}(A))$. The proof is complete. □

The following is a version of 3.6 for complexes with CMFPD homologies.

Corollary 3.7. *Let $P_\bullet$ be a complex in $Ch^b(\mathcal{P}(A))$ such that, for all maximal ideals $\mathfrak{m}$ and all degrees t, $H_t(P_\bullet)_\mathfrak{m}$ is a CMFPD module, if nonzero. We denote $\rho_t(\mathfrak{m}) := proj \dim(H_t(P_\bullet)_\mathfrak{m})$. Then the dual $P_\bullet^*$ also has the same property, if and only if, $\iota_t : Ext^{\rho_j(\mathfrak{m})}\left(\frac{P_t}{B_t}, A\right)_\mathfrak{m} \longrightarrow Ext^{\rho_t(\mathfrak{m})}\left(H_t(P_\bullet), A\right)_\mathfrak{m}$ is an isomorphism, whenever $H_t(P_\bullet)_\mathfrak{m} \neq 0$.*

4. Computations on dualizable complexes

In fact, we can say considerably more about the homologies of P^* under the above conditions. We first note that another look at the long exact Ext sequence along with induction, gives us the following theorem.

Theorem 4.1. *Let A be a CM ring. Suppose $P_\bullet$ is a complex in $Ch_{\mathcal{B}}^b(\mathcal{P}(A))$ such that $P_\bullet^* \in Ch_{\mathcal{B}}^b(\mathcal{P}(A))$. Then, with notations as in the equivalence Theorem 3.6,*

1. *$\forall\ t$, we have*

$$\forall\ 1 \leq i, j = 0, 1, \ldots, n \qquad Ext^i\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = 0 \quad or \quad are\ in\ \mathcal{B}.$$

   Also, $Ext^0\left(\frac{P_t}{B_t}, A\right)$ has finite projective dimension.
2. *Further,*

$$Ext^i\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = \begin{cases} Ext^{i+s_j}\left(\frac{P_{t_{j-1}}}{B_{t_{j-1}}}, A\right) & \forall\ 1 \leq i < \rho_j - s_j \\ Ext^{\rho_{j-1}}\left(H_{t_{j-1}}, A\right) & for\ i = \rho_j - s_j \\ 0 & \forall\ \rho_{j-1} - s_j < i < \rho_j \\ Ext^{\rho_j}\left(H_{t_j}, A\right) & for\ i = \rho_j \\ 0 & \forall\ \rho_j < i \end{cases}$$

3. *For all t, the homologies $H_t(P_\bullet^*)$ are in $\mathcal{B}$. In fact, for $t_{j-1} < t \leq t_j$,*

$$H_{-t}(P_\bullet^*) = Ext^{t-t_{j-1}}\left(\frac{P_{t_{j-1}}}{B_{t_{j-1}}}, A\right).$$

Proof. The equation in (3) follows immediately from Proposition 3.2. So it is enough to prove (1, 2).These statements are clearly true for $j = 0$. We assume that (1, 2) are valid upto j and prove them $j + 1$.

By the equivalence Theorem 3.6,

$$\iota_{j+1} : Ext^{\rho_{j+1}}\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}}, A\right) \xrightarrow{\sim} Ext^{\rho_{j+1}}\left(H_{t_{j+1}}, A\right) \quad \text{is an isomorphism.}$$

We have,

$$Ext^{i}\left(\frac{P_{t_j+l}}{B_{t_j+l}}, A\right) \simeq Ext^{i+l}\left(\frac{P_{t_j}}{B_{t_j}}, A\right), \forall\ i \geq 1, l \neq \rho_{j+1}$$

and

$$Ext^{i}\left(B_{t_j+l}, A\right) \simeq Ext^{i+l+1}\left(\frac{P_{t_j}}{B_{t_j}}, A\right), \forall\ i \geq 1,\ 1 \leq l < s_{j+1}$$

which are hence in $\mathcal{B}$ by induction.

As in the previous proofs, looking at the long exact Ext sequence corresponding to $0 \longrightarrow H_{t_{j+1}} \longrightarrow \frac{P_{t_{j+1}}}{B_{t_{j+1}}} \longrightarrow B_{t_{j+1}-1} \longrightarrow 0$ along with induction, gives us all the statements in (1, 2) except the finite projective dimension of $Ext^0\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}}, A\right)$. An inductive argument with the exact sequence

$$0 \longrightarrow Ext^0\left(\frac{P_{t_j+l}}{B_{t_j+l}}, A\right) \longrightarrow Ext^0\left(P_{t_j+l}, A\right) \longrightarrow Ext^0\left(B_{t_j+l}, A\right) \longrightarrow Ext^1\left(\frac{P_{t_j+l}}{B_{t_j+l}}, A\right) \longrightarrow 0$$

for $0 \leq l < s_{j+1}$ and $\frac{P_{t_j+l}}{B_{t_j+l}} \xrightarrow{\sim} B_{t_j+l-1}, 0 < l < s_{j+1}$ gives us that $Ext^0\left(\frac{P_{t_{j+1}-1}}{B_{t_{j+1}-1}}, A\right)$ has finite projective dimension. Then

$$0 \longrightarrow Ext^0\left(B_{t_{j+1}-1}, A\right) \longrightarrow Ext^0\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}}, A\right) \longrightarrow Ext^0\left(H_{t_{j+1}}, A\right) \longrightarrow Ext^1\left(B_{t_{j+1}-1}, A\right) \longrightarrow 0$$

Since all the other terms have finite projective dimension, so does $Ext^0\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}}, A\right)$. The proof is complete. □

We give some explicit formulas for nonzero ext-modules.

Corollary 4.2. *Let A, $P_\bullet$ and all the notations be as in Theorem 4.1. Let $j \geq 0$ be an integer. For $r = 0, \ldots, j$ let*

$$\tau_r^j = \rho_r - (s_j + \cdots + s_{r+1}) = \rho_r - (t_j - t_r).$$

Then

$$Ext^i\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = \begin{cases} Ext^{\rho_r}\left(H_{t_r}(P_\bullet), A\right) & \text{if } i = \tau_r^j \geq 1 \\ 0 & \text{if } 1 \leq i \neq \tau_r^j \ \forall\ r. \end{cases}$$

Proof. Fix j and note that $\tau_r^j > \tau_{r-1}^j$ for all $r = 1, \dots, j$. First, for $i = \tau_j^j = \rho_j$, by Theorem 4.1, we have

$$Ext^i\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = Ext^{\rho_j}\left(H_{t_j}, A\right).$$

For $i = \tau_{j-1}^j = \rho_{j-1} - s_j$ by theorem 4.1, we have

$$Ext^i\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = Ext^{\rho_{j-1}-s_j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = Ext^{\rho_{j-1}}\left(H_{t_{j-1}}, A\right)$$

Also, by Theorem 4.1, $Ext^i\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = 0$ for $\tau_{j-1}^j < i < \tau_j^j$. Now, we use induction. Assume the theorem is established $\forall\ i \geq \tau_{r+1}^j$. We prove it for $\tau_{r+1}^j < i \leq \tau_r^j$. For $\tau_r^j = \rho_r - (s_{r+1} + \cdots + s_j) \leq i$ it follows inductively that

$$Ext^i\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = Ext^{i+s_j}\left(\frac{P_{t_{j-1}}}{B_{t_{j-1}}}, A\right) = Ext^{i+s_{r+1}+\cdots+s_j}\left(\frac{P_{t_r}}{B_{t_r}}, A\right)$$

The proof is complete by application of Theorem 4.1. □

We compute the homologies of the dual for complexes as in Theorem 4.1.

Corollary 4.3. *Let $P_\bullet$ be a complex as in Theorem 4.1. Then,*

$$H_{-t}(P_\bullet^*) = \begin{cases} Ext^{\rho_r}\left(H_{t_r}(P_\bullet), A\right) & \text{if } t = t_r + \rho_r \text{ for some } r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We assume $t_j < t \leq t_{j+1}$. By 3.2,

$$H_{-t}(P_\bullet^*) = Ext^{t-t_j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right)$$

By 4.2, $H_{-t}(P_\bullet^*)$ is non-zero, only if

$$t - t_j = \tau_r^j = \rho_r - (s_j + \cdots + s_{r+1}).$$

That means, if

$$t = \rho_r + [t_j - (s_{r+1} + s_{r+2} + \cdots + s_j)] = t_r + \rho_r.$$

For $t = t_r + \rho_r$, we have

$$H_{-t}(P_\bullet^*) = Ext^{\rho_r-(s_j+\cdots+s_{r+1})}\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = Ext^{\tau_r^j}\left(\frac{P_{t_j}}{B_{t_j}}, A\right) = Ext^{\rho_r}\left(H_{t_r}, A\right).$$

The proof is complete. □

5. Some dualizable categories of complexes

We consider some special categories of complexes and use the equivalence condition in the previous section to show that they are closed under duality. We also show that these conditions are almost the best we can hope for. We begin with some initial lemmas that will be used in the subsequent proofs.

Lemma 5.1. *Let A be a noetherian ring and*

$$P_s \longrightarrow P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0$$

be a complex of projective A-modules. Assume that the complex does not split at degree $i = 1, \ldots, s-1$. Then,

$$s \le proj\dim\left(\frac{P_0}{B_0}\right) - 1.$$

Proof. Write $\rho = proj\dim\left(\frac{P_0}{B_0}\right)$. We have the exact sequence

$$0 \longrightarrow B_{s-1} \longrightarrow P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \frac{P_0}{B_0} \longrightarrow 0$$

If $s \ge \rho$ then B_{s-1} is projective which contradicts the hypothesis that the complex is not split at $i = 1, \ldots, s-1$. Hence, the proof is complete. □

Lemma 5.2. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Let $P_\bullet$ be an indecomposable complex in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$. Let $t_0 < t_1 < t_2 < \cdots$ be the list of all the degrees so that $H_{t_i}(P_\bullet) \neq 0$. Let $\rho_i = proj\dim H_{t_i}$. Assume*

$$\forall\ j \in \mathbb{Z}, \quad \rho_j - \rho_{j+1} \le t_{j+1} - t_j =: s_{j+1}$$

Then,

$$\forall\ j \in \mathbb{Z}, \quad proj\dim H_{t_j} = proj\dim\left(\frac{P_{t_j}}{B_{t_j}}\right).$$

Conversely, for any fixed j*:*

$$proj\dim\left(\frac{P_{t_k}}{B_{t_k}}\right)=\rho_k \quad for \quad k=j,j+1 \quad \Longrightarrow \rho_j-\rho_{j+1}\le s_{j+1}+1.$$

Proof. We assume $t_0=0$. Since $P_\bullet$ is indecomposable, $H_0=\frac{P_0}{B_0}$. So the lemma holds trivially in this case. Now assume that the lemma holds upto t_j. Note that if $\frac{P_{t_j}}{B_{t_j}}$ is projective, then so is B_{t_j}. By our assumption of indecomposability, 2.3 tells us that this is possible only if $P_{t_j+1}=B_{t_j}$ and this is the final term in the complex, in which case we are already done by the induction hypothesis (there is no t_{j+1}). So assume that $proj\dim\left(\frac{P_{t_j}}{B_{t_j}}\right)\ge 1$ and we will prove the equality for t_{j+1}. Then

$$proj\dim(B_{t_j})=proj\dim\left(\frac{P_{t_j}}{B_{t_j}}\right)-1=\rho_j-1.$$

Inductively, we have

$$proj\dim B_{t_{j+1}-1}=(\rho_j-1)-(t_{j+1}-1-t_j)=\rho_j-s_{j+1}\le\rho_{j+1}.$$

Consider the exact sequence

$$0\longrightarrow H_{t_{j+1}}\longrightarrow\frac{P_{t_{j+1}}}{B_{t_{j+1}}}\longrightarrow B_{t_{j+1}-1}\longrightarrow 0$$

We write the long exact sequence:

$$\longrightarrow 0\longrightarrow Tor_{\rho_{j+1}+1}\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}},A/\mathfrak{m}\right)\longrightarrow Tor_{\rho_{j+1}+1}\left(B_{t_{j+1}-1},A/\mathfrak{m}\right)$$

$$\longrightarrow Tor_{\rho_{j+1}}\left(H_{t_{j+1}},A/\mathfrak{m}\right)\longrightarrow Tor_{\rho_{j+1}}\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}},A/\mathfrak{m}\right)\longrightarrow Tor_{\rho_{j+1}}\left(B_{t_{j+1}-1},A/\mathfrak{m}\right)$$

Since the top right term vanishes, and the bottom left term is non-zero, the middle term in the second row cannot vanish. It follows that $proj\dim\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}}\right)=\rho_{j+1}$.

For the converse, $proj\dim\left(\frac{P_{t_j}}{B_{t_j}}\right)=\rho_j$ and hence $proj\dim\left(B_{t_{j+1}-1}\right)=\rho_j-s_{j+1}$. Then the above long exact sequence gives us that

$$\rho_j-s_{j+1}=proj\dim\left(B_{t_{j+1}-1}\right)\le\rho_{j+1}+1.$$

This completes the proof. □

Theorem 5.3. *Let A be a CM ring. Let $P_\bullet$ be a complex in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$. Let $t_0 < t_1 < t_2 < \cdots < t_n$ be the list of the non-vanishing homologies. Let $\rho_i = proj\dim H_{t_i} > 0$. Assume*

$$\rho_j - \rho_{j+1} < t_{j+1} - t_j =: s_{j+1},\ j = 0, 1, 2, \dots, n-1.$$

Then, $P^*_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A))$.

Proof. By equivalence Theorem 3.6, we only need to prove the homomorphisms ι_j are isomorphisms. Note that the condition

$$\rho_j - \rho_{j+1} < t_{j+1} - t_j =: s_{j+1},\ j = 0, 1, 2, \dots, n-1$$

localizes and restricts to indecomposable summands of the complex for the subsequence of $\{0, 1, 2, \dots, n-1\}$ consisting of nonzero homologies, because $\rho_j - \rho_k < t_k - t_j$ for all $j < k$. Hence, we can assume that A is local and $P_\bullet$ is indecomposable. Note that $H_{t_0} = \dfrac{P_{t_0}}{B_{t_0}}$ is an isomorphism and hence,

$$\iota_0 : Ext^{\rho_0}\left(\frac{P_{t_0}}{B_{t_0}}, A\right) \to Ext^{\rho_0}\left(H_{t_0}, A\right)$$

is an isomorphism. By Lemma 5.2, $proj\dim\left(\frac{P_{t_j}}{B_{t_j}}\right) = \rho_j \forall\ j$. Hence by Lemma 5.1, we have $s_{j+1} = t_{j+1} - t_j \le \rho_j - 1$ (which also ensures that $\rho_j \ge 2$) and $proj\dim B_{t_{j+1}-1} = \rho_j - s_{j+1} < \rho_{j+1}$. Thus, the long exact sequence corresponding to the exact sequence: $0 \longrightarrow H_{t_{j+1}} \longrightarrow \frac{P_{t_{j+1}}}{B_{t_{j+1}}} \longrightarrow B_{t_{j+1}-1} \longrightarrow 0$ gives us that

$$\iota_{j+1} : Ext^{\rho_{j+1}}\left(\frac{P_{t_{j+1}}}{B_{t_{j+1}}}, A\right) \xrightarrow{\sim} Ext^{\rho_{j+1}}\left(H_{t_{j+1}}, A\right).$$

Hence, by the equivalence Theorem 3.6, we get that $P^*_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A))$. The proof is complete. □

Corollary 5.4. *Let $P_\bullet$ be a complex as in Theorem 4.1. Suppose u, v be integers and*

$$H_{-u}(P^*_\bullet) \neq 0, \quad H_{-v}(P^*_\bullet) \neq 0, \quad \textit{and} \quad H_{-t}(P^*_\bullet) = 0 \quad \forall\ u < t < v.$$

Then

$$proj\dim H_{-v}(P^*_\bullet) - proj\dim H_{-u}(P^*_\bullet) < v - u.$$

Proof. Note that $(t_r+\rho_r)-(t_{r-1}+\rho_{r-1}) = s_r+\rho_r-\rho_{r-1} > 0$. So, by 4.3, $u = t_{r-1}+\rho_{r-1}$, $v = t_r + \rho_r$ for some r. Also, $v - u = s_r + \rho_r - \rho_{r-1}$. From, 4.3,

$$H_{-v}(P_\bullet^*) = Ext^{\rho_r}\left(H_{t_r}, A\right), \quad H_{-u}(P_\bullet^*) = Ext^{\rho_{r-1}}\left(H_{t_{r-1}}, A\right).$$

So,

$$proj\dim H_{-v}(P_\bullet^*) = \rho_r, \quad \text{and} \quad proj\dim H_{-u}(P_\bullet^*) = \rho_{r-1}$$

Therefore,

$$proj\dim H_{-v}(P_\bullet^*) - proj\dim H_{-u}(P_\bullet^*) = \rho_r - \rho_{r-1} < \rho_r - \rho_{r-1} + s_r = v - u.$$

This completes the proof. □

Putting together all the above theorems, lemmas and corollaries gives us the following theorem:

Theorem 5.5. *Let A be a CM ring. Let $\mathcal{C}$ be the additive subcategory of $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ generated by complexes $P_\bullet$ which are satisfy the condition that:*

$$\rho_j - \rho_{j+1} < t_{j+1} - t_j =: s_{j+1}, \;\; j = 0, 1, 2, \dots, n-1$$

where the non-zero homologies of $P_\bullet$ are in degrees $t_0 < t_1 < t_2 < \dots < t_n$ and $\rho_i = proj\dim H_{t_i} > 0$. Then $\mathcal{C}$ is closed under the duality $Hom_A(\_\_, A)$.

We give some more easy consequences of the above results.

Theorem 5.6. *Let A be a CM ring and let*

$$\mathcal{B}(k) = \{P_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A) : \;\forall\; i \;\, H_i(P_\bullet) = 0 \;\, or \;\, proj\dim(P_\bullet) = k\}.$$

Then the category $Ch^b_{\mathcal{B}(k)}(\mathcal{P}(A))$ is an exact category closed under the duality $Hom_A(\_\_, A)$.

Remark 5.7. When $k = d$, the above category $\mathcal{B}(d)$ is exactly the category of finite length modules with finite projective dimension and in [4], a dévissage theorem for $\mathcal{B}(d)$ was proved.

We end with a lemma that seems to suggest that the conditions in Theorem 5.5 are almost optimal for duality to hold.

Lemma 5.8. *Suppose $P_\bullet : \;\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \cdots$ is a complex of projective A-modules, such that $H_0(P_\bullet), H_1(P_\bullet) \in \mathcal{B}$. Let $\rho_1 = proj\dim H_1$ and $\rho_0 = proj\dim H_0$.*

Assume $\rho_0 = \rho_1 + 1$ *and* $proj\dim B_0 \leq \rho_0 - 1 = \rho_0$. *Then* $Ext^{\rho_1}\left(\frac{P_1}{B_1}, A\right)$ *is not Cohen–Macaulay.*

Proof. Consider the exact sequence $0 \longrightarrow H_1 \longrightarrow \frac{P_1}{B_1} \longrightarrow B_0 \longrightarrow 0$. Since $Ext^{\rho_1}(B_0, A) = Ext^{\rho_1 - 1}(H_1, A) = 0$, from the corresponding long exact sequence of the ext modules, we have an exact sequence

$$0 \longrightarrow Ext^{\rho_1}(B_0, A) \longrightarrow Ext^{\rho_1}\left(\frac{P_1}{B_1}, A\right) \longrightarrow Ext^{\rho_1}(H_1, A) \longrightarrow 0$$

Since $proj\dim(B_0) \leq \rho_0 - 1 = \rho_1$, we have $proj\dim(Z_0) \leq \rho_0 - 2$. So, form the exact sequence $0 \longrightarrow B_0 \longrightarrow Z_0 \longrightarrow H_0 \longrightarrow 0$ we have $Ext^{\rho_1}(B_0, A) \cong Ext^{\rho_0}(H_0, A)$. Now,

$$proj\dim\left(Ext^{\rho_0}(H_0, A)\right) = \rho_0 = \rho_1 + 1, \quad proj\dim\left(Ext^{\rho_1}(H_1, A)\right) = \rho_1$$

It follows

$$proj\dim\left(Ext^{\rho_1}\left(\frac{P_1}{B_1}, A\right)\right) = \rho_0 = \rho_1 + 1.$$

Also, the height of the annihilator of the middle term is the minimum of that of the other two. So,

$$grade\left(Ext^{\rho_1}\left(\frac{P_1}{B_1}, A\right)\right) = \rho_0 = \rho_1.$$

Therefore, by 2.7, $Ext^{\rho_1}\left(\frac{P_1}{B_1}, A\right)$ is not Cohen–Macaulay. The proof is complete. □

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