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# Stability Analysis Of A Cantilevered Plate In Randomly Fluctuating Flow

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**Abstract**

Elastic structures like thin plates oscillating in fluid flow are susceptible to instabilities like flutter. The fluid-structure interaction between plate and the flow causes self-sustaining oscillations when the axial flow velocity exceeds a critical value. Such oscillations can be of very high amplitude and can be detrimental to structural safety. On the other hand, recent studies have revealed the possibility of energy being harvested from such oscillations. In either case, knowledge of the stability boundary of the system is crucial for appropriately designing the system. Most existing approaches in this regard consider the flow to be steady and hence do not accurately reflect the real life scenario where fluid flow is accompanied by random fluctuations. This study focuses on characterizing the response when the flow is accompanied by random fluctuations, which would serve as a first step in obtaining the stability boundaries. The fluctuating flow is modelled using Karhunen Loeve expansion (KLE) and the response is constructed using Polynomial Chaos expansion (PCE). Stochastic collocation approach has been employed for obtaining the PCE coefficients. The results are validated using Monte Carlo simulations.


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*Keywords:* Fluid-Structure Interaction; Plate Flutter; Polynomial Chaos expansion; Sparse Grid Collocation; Uncertainty Quantification.

## 1. Introduction

The use of the traditional horizontal axes wind turbines for energy harvesting requires large investments in terms of infrastructure and costs. This has led to interests on developing alternative energy harvesting devices, such as, vertical axis wind turbines, oscillating wing windmills [1], energy harvesting eels [2] etc. One of the prominent approaches that take advantage of the dynamic instability of a vibrating system is the so called flutter mill [3]. The flutter-mill typically consists of a cantilevered plate like structure in an axial flow. As the velocity of the flow increases beyond a critical velocity, the initially undisturbed plate exhibits self-sustaining oscillations. Energy can be harvested from these oscillations using smart technologies. For quantifying the energy that can be harvested from these self-sustaining oscillations, a thorough understanding of the dynamical behaviour of the system is essential. Since the structure is a flexible member in an axial flow, the system needs to be studied as a problem in fluid-structure interactions. Analytical/numerical studies carried out by [4] have revealed that the self-sustaining oscillations exhibited by the

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plate beyond a critical velocity are due to dynamical instability resulting in flutter. For low flow speeds, an initially undisturbed plate remains in equilibrium, and for small initial perturbations, the oscillations die down. However, as the flow speed increases, the damping effects of the fluid decreases and beyond a critical flow velocity, the system undergoes a Hopf supercritical bifurcation.

The design of energy harvesters requires a thorough understanding of the dynamical behaviour and stability regimes of the dynamical system at different flow conditions. Studies carried out in the literature have considered only steady flow conditions. However, in real life situations, the flows are usually accompanied by fluctuations which exhibit both long term and short term variations. The random fluctuations that occur at small time scales affect the behaviour of the dynamical system and alter the bifurcation and stability characteristics. It has been shown in the literature that sudden gustiness in the loadings can lead to a dynamical system exhibiting flutter instability at a wind velocity below the linear critical wind velocity as well [5]. In nonlinear systems, such behavior may also cause a premature transition to chaos [6]. Moreover, it has been recently established that aero-elastic limit cycle oscillations are very sensitive to the presence of uncertainty [7, 8]. Therefore, a thorough understanding of these phenomena is essential for developing reliable energy harvesting systems. The focus of this study is to gain an understanding of the behaviour of a cantilevered plate in a randomly fluctuating wind flow, and how the random fluctuations in the flow affect the stability regimes.

The aero-elastic governing equations of motion for fluttering plates represent a set of coupled nonlinear partial differential equations, with the coupling between the structure and fluid degrees of freedom being explicitly modeled through analytical semi-empirical models [4]. In this study, we focus on incorporating into the model the additional complexities arising due to the fluid flow being modeled as random processes in time. The corresponding uncertainties associated with the fluid flow propagate through the nonlinear dynamical system and manifests itself in terms of the structure response. Since the deterministic computational models available for plate flutter systems use complicated governing equations whose solutions are time consuming, the use of Monte Carlo simulations (MCS) as a tool for uncertainty quantification can be computationally prohibitively expensive. Instead, in this study, we use polynomial chaos expansion (PCE) based methods for uncertainty quantification. PCE has been used for uncertainty quantification in a host of fluid structure interaction problems [9–14]. The underlying principle in this approach lies in transforming the problem into a space spanned by a set of random orthogonal basis functions and obtaining expressions for the response quantities in terms of a series expansion, which serves as a meta-model for the complicated nonlinear dynamical system. The crux in this approach lies in estimating the corresponding projections for the response along the random basis functions. Rather than using the intrusive stochastic Galerkins approach, which leads to a new set of coupled equations whose solution itself may require development of new computational algorithms, we use the stochastic collocation based approach. This approach requires the solution of the forward problem corresponding to the collocation points, which can be carried out using the algorithms available for the solution of the deterministic system. However, as the stochastic dimension associated with the problem increases, the number of collocation points at which the forward problem needs to be solved increases as a tensor product and can soon become as computationally intensive as MCS. To mitigate this so called "curse of dimensionality we use a sparse grid based collocation approach [15, 16].

The paper is structured as follows: The formulation for the deterministic plate flutter problem as per [4] is briefly explained in Section 2.1. Section 2.2 discusses Karhunen- Loeve expansion of the random flow and brief descriptions of the Polynomial Chaos Expansion and the stochastic collocation are presented in Section 2.3. The results and inferences are discussed in section 3.

## 2. Deterministic modelling and uncertainty quantification

### *2.1. Modelling flow-plate interaction*

A flat flexible thin plate fixed at the leading edge and free at the trailing edge is assumed to have been placed in an axial fluid flow. The plate is considered to be of finite length and of infinite width so that the investigations can be restricted to a two-dimensional analysis. To account for large oscillations in the structure, a nonlinear structural model has been used. Since the plate is thin and the strains generated are low, the plate is modelled as an inextensible Euler-Bernoulli beam. The fluid loads on the plate are calculated through an unsteady lumped vortex model, considering

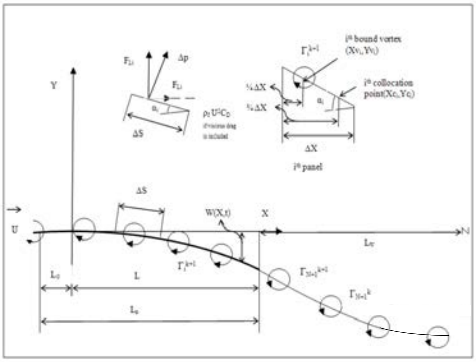


Fig. 1. (a) Schematic of lumped vortex model

flow to be inviscid and incompressible with discrete lumped vortices (Fig.1). The damping behavior is modeled using the Kelvin-Voigt model theory and the viscous effects of the flow are incorporated in the model through a non-dimensional drag coefficient $C_D$. The bound vortices on the panels, instantaneously formed wake vortices and $C_D$ together contribute to the lift and drag forces. The non-dimensional equations of motion are given by

$$\ddot{w} + \left(1 + \alpha\frac{\partial}{\partial\tau}\right)[w''''(1 + w'^2) + 4w'w''w''' + w'^3] + w'\int_0^s(\dot{w}'^2 + w'\ddot{w}')ds$$
$$-w''\int_s^1\left[\int_0^s(\dot{w}'^2 + w'\ddot{w}')ds\right]ds = \mu U_R^2\left(f_L - w'f_D + w''\int_s^1 f_D ds\right). \quad (1)$$

Here, $w$ is the non-dimensional transverse displacement, $v$ is the non-dimensional longitudinal displacement, $s$ is non-dimensional coordinate along the plate centerline, $\tau$ is the non-dimensional time, $\mu$ is the mass ratio, $\alpha$ is the non-dimensional material damping coefficient, and $f_L$ and $f_D$ are respectively the lift and drag forces [4]. To solve for $w(x,\tau)$, the transverse deformation is expanded in terms of generalized coordinates $q_m(\tau)$ and Galerkin eigenfunctions $\phi_m(x)$ as

$$w(x,\tau) = \sum_{m=1}^{M} q_m(\tau)\phi_m(x), \quad (2)$$

where, $M$ is the number of dynamical modes considered. Substituting for in Eq.(1) and applying Galerkins method, a set of coupled ordinary differential equations are obtained as

$$\ddot{q} + A_i q_i + \alpha A_i\dot{q}_i + B_{imnl}q_m q_n q_l + \alpha B_{imnl}(\dot{q}_m\dot{q}_n\dot{q}_l + q_m\dot{q}_n q_l + q_m q_n\dot{q}_l) + C_{imnl}q_m(\dot{q}_n\dot{q}_l + q_n\ddot{q}_l) = f_i, \quad (3)$$

where, $A_i$, $B_{imnl}$, $C_{imnl}$ are constant coefficients derived from the modes and their derivatives. $f_i$ are the generalized loads calculated from the lift force and drag force across the plate centerline.

The Houbolt method [4, 17] is used for the numerical solution of Eq. (3). The underlying principle here lies in representing the first and second derivatives of the state variables based on two backward approximations at time step k+1 and are expressed as

$$\ddot{q}^{k+1} = \frac{\sum_{j=1}^{4} a_j q^{k-j+2}}{\Delta\tau^2}, \quad \dot{q}^{k+1} = \frac{\sum_{j=1}^{4} b_j q^{k-j+2}}{\Delta\tau}. \quad (4)$$

Here, $a_j$,$b_j$ are constants specific to the fourth order scheme given above. Substituting these approximations into Eq.(3) yields

$$D_i q_i + \lambda_{3i} + E_{imnl}q_m q_n q_l + F_{imnl}\lambda_{2m}q_n q_l + C_{imnl}q_m(\lambda_{2n}\lambda_{2l} + q_n\lambda_{1l}) = \Delta^2 f_i, \quad (5)$$

where, $\lambda_{1i}$, $\lambda_{2i}$, $\lambda_{3i}$ are time dependent coefficients, $D_i$, $E_{imnl}$ and $F_{imnl}$ are constant coefficients dependent on time step size $\Delta\tau$, $A_i$, $B_{imnl}$ and $C_{imnl}$. Solving these equations require appropriate selection of the following parameters: number of modes $M$, number of panels considered on the plate $N$, time step size $\Delta\tau$ and nondimensional length of truncated wake street $lW$. Eq. (5) was numerically solved in MATLAB for 200 panels, 6 modes, using a time step size of 0.001 and a non-dimensional truncated wake street length of 40. The other parameters considered were mass ratio of 0.2, reduced velocity of 13, material damping coefficient of 0.004 and viscous drag coefficient was assumed to be zero.

### 2.2. Modelling random flow using Karhunen-Loeve Expansion

Uncertainty quantification in any system is a three step process involving parametrization of uncertainty, simulation of system response and probabilistic analysis of the response [16]. Karhunen-Loeve expansion (KLE) involves a bi-orthogonal decomposition of the correlation function of the process. A stochastic process $U$ is expanded using KLE as,

$$U(x,\theta) = \sum_{i\geq 1} \sqrt{\lambda_i u_i(x) \eta_i(\theta)}, \tag{6}$$

where, $x$ may be a scalar or vector, $\lambda_i$ and $u_i(x)$ are the eigenvalues and eigenvectors of the correlation function of the process $U(x,\theta)$ and $\eta_i(\theta)$ are zero mean, unit variance and mutually uncorrelated random variables. In this study, the fluctuating fluid flow is assumed to be a Gaussian process with an auto-correlation function,

$$R_{xx}(\tau) = \sigma^2 \exp(-c\tau^2), \tag{7}$$

where, $\sigma^2$ is the variance of the process, $c$ is a measure of the correlation time and $\tau$ is the time lag for which correlation is calculated. The values for $\sigma$, and $c$ have been taken to be 0.1 and 1. The time step used in the simulation is 0.001 non-dimensional units and the correlation time is about 2.5. The eigenvectors of the correlation matrix thus obtained and Gaussian random variables generated through pseudo-random number generator together provide the bi-orthogonal bases using which the KLE of the inflow fluctuations is constructed.

### 2.3. Modelling the response through Polynomial Chaos Expansion (PCE)

Polynomial chaos expansions (PCE) are typically used to analyze the propagation of uncertainties from the input to the system response. In PCE, a stochastic process is expanded in terms of mutually orthonormal polynomial functionals of random variables. Typically, this involves writing the process as

$$U(x,\theta) = \sum_{k=0}^{N} u_k(\theta)\Psi_k(\theta), \tag{8}$$

where, $N$ is the number of terms in the expansion, $\Psi_k(\theta)$ is the $k$th probabilists' Hermite polynomial of the random variable(s) $\theta$ which constitute the random basis functions and $u_k(\theta)$ is the coefficient of the *kthterm*. The choice of the polynomials is based on the Askey scheme [16]. The coefficients of the polynomials can be found using stochastic Galerkin method which involves solving a set of coupled equations. For most problems, this may require high computational effort and special solvers and hence non-intrusive methods like stochastic collocation are generally preferable for computing the coefficients. The coefficients $u_k(\theta)$ are found using the relation,

$$u_k(x) = \frac{\langle U_i(x_i), \Psi_k \rangle}{\langle \Psi_k^2 \rangle}, \tag{9}$$

where

$$\langle U_i(x_i), \Psi_k \rangle = \int_{\Theta} U(x,\theta)\Psi_k(\theta) dP(\theta). \tag{10}$$

The dimensions of these integrals depend on the number of random variables considered in PCE. Closed form approximations of these integrals are not possible and one has to resort to numerical techniques. Most commonly

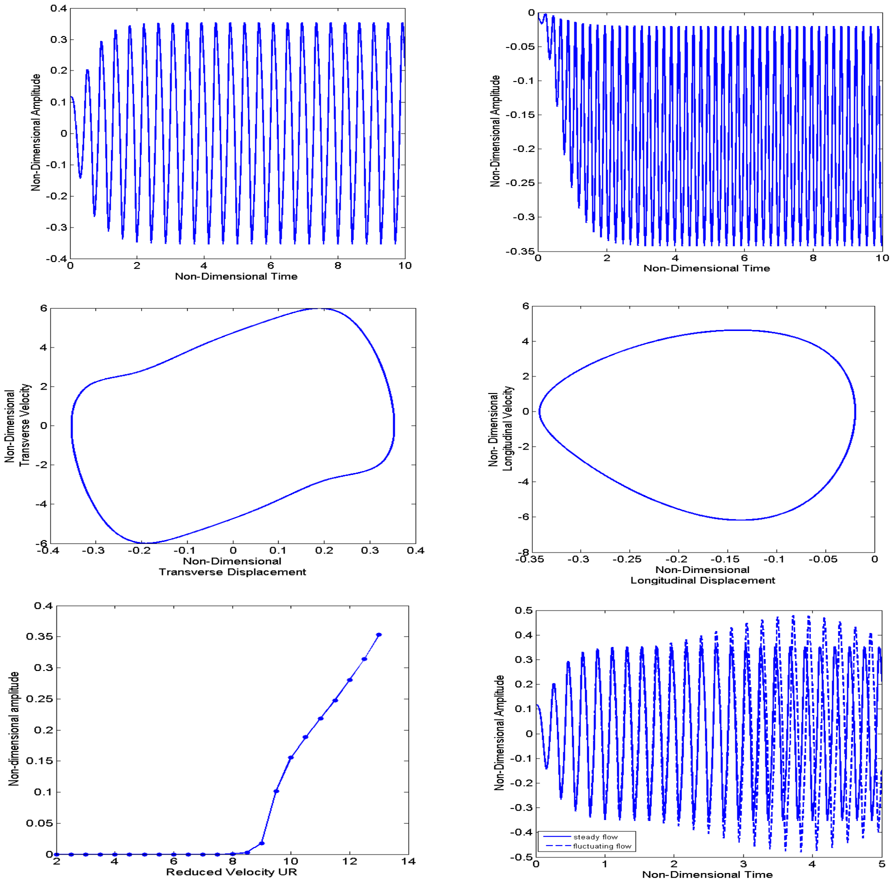

Fig. 2. (a) to (d) Time histories and phase plots of transverse and longitudinal displacements respectively for deterministic case with reduced velocity 13;. (e) shows bifurcation diagram for the deterministic case; (f) compares transverse displacements for a steady inflow and randomly fluctuating inflow.

used numerical quadrature schemes are Gauss-Hermite, Gauss- Lagrange and Gauss-Laguerre rules. For performing multi-dimensional integrals, a tensor product of the 1-dimensional rule is generally used. However, this leads to an exponential increase in the number of quadrature points for high-dimensional systems. To alleviate this problem Smolyak construction based sparse grid techniques can be used. While in conventional collocation, tensor products of one-dimensional quadrature form the $n$-dimensional quadrature scheme, Smolyak construction is based upon tensor product of difference formulas of quadratures and results in fewer number of quadrature points for any given order [15]. In Smolyaks construction, $n$-dimensional quadrature is constructed as the summation of tensor products of all possible set of 1-d difference quadrature rules, whose orders $l_i$ satisfy the following rule,

$$\sum_{i}^{N} l_i \leq l + N - 1. \tag{11}$$

Here, $l$ is the required order of $n$-d quadrature and $N$ is the number of random variables (stochastic dimension) considered [16]. Since it is convenient to construct sparse grids over nested quadrature rules, Gauss-Patterson, Clenshaw-Curtis and Fejer rules are commonly used in place of Gauss-Hermite rule. In this work, both Gauss-Hermite based tensor grid and Gauss Patterson based sparse grid approaches have been employed.

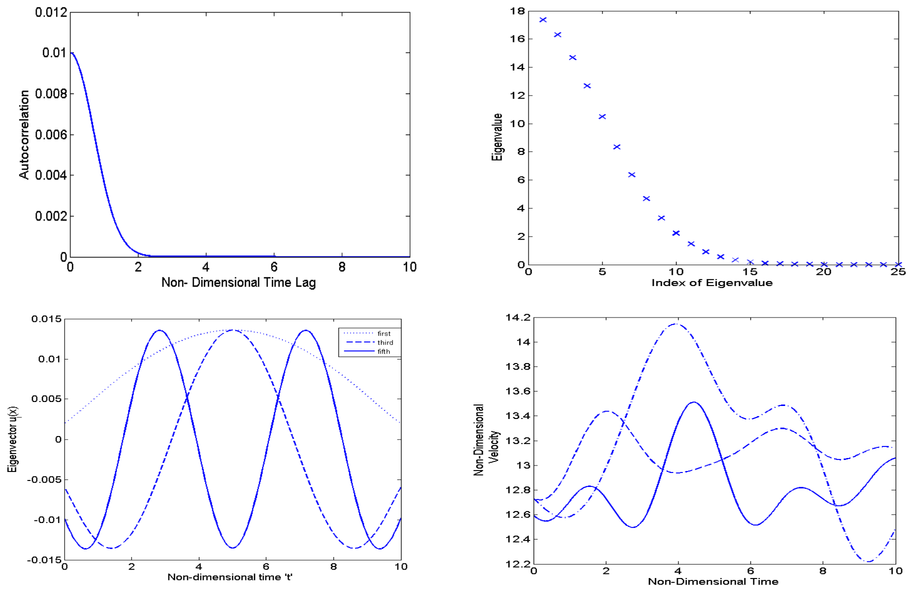


Fig. 3. (a) Autocorrelation assumed for the random inflow; (b)-(c) Eigenvalues and sample eigenvectors of the correlation matrix; (d) Sample realizations of random inflow simulated.

## 3. Results and discussion

The deterministic formulation was simulated for the parameter values given in section 2.1 for flow velocities varying from non-dimensional values of 2 to13. For the given parameters, the onset of flutter is observed at non-dimensional velocity of 9.5 (Fig.2 (e)). Fig. 2(a) shows the time history of transverse displacement at a post-critical velocity of 13. Fig.2(b) shows the longitudinal displacement at the same velocity and Fig.2(c)-(d) show the phase plots for transverse and longitudinal displacements respectively. These figures are an indication the system undergoes limit cycle oscillations post the critical velocity. The fluid forces compensate the structural damping and the system thus has self-sustained oscillations which grow in amplitude with increasing velocity. As discussed in section 1, such oscillations are reported to be sensitive to uncertainties in fluid flow. Fig.2 (f) indicates that the random fluctuations in flow cause the response flutter amplitude also to fluctuate. It seems that the deviation of the fluctuating limit cycle from the steady one increases with time, indicating the effect of uncertainty on the system dynamics accumulates over time.

Figure 3(a) shows the assumed correlation function for the flow fluctuations. The correlation time is approximately 2.5 non-dimensional units. Fig.3 (b) shows the values of the largest 25 eigenvalues and Fig.3(c) shows the first, third and fifth eigenvectors of the correlation matrix. The number of eigenvectors to be considered for simulating the inflow is taken such that the following relationship is satisfied:

$$\sum_{i}^{n} \lambda_i \geq 0.9 \sum_{i}^{N} \lambda_i. \tag{12}$$

Here, $n$ refers to the number of eigenvectors considered and $N$ refers to the total number of eigenvectors. Thus, 8 eigenvectors were used in KLE. Some sample simulated inflow realizations have been shown in Fig.3 (d) for the chosen values of $\sigma$ and $c$ in Eq.(7). It should be noted that the number of eigenvectors considered depends on the values of $c$ and a higher $c$ corresponds to more eigenvectors for a given condition similar to Eq.(11). The PCE of the response is constructed over two random variables and the tensorial grid for the orders 3,4,5 of 1-dimensional Gauss-Hermite Quadrature are shown in Fig.4(a). The sparse grids for orders 3,4 of Gauss-Patterson quadrature are shown in Fig.4 (b). The Gauss-Patterson quadratures of order 3 and 4 contain 7 and 15 quadrature points respectively, and are nested. Fig. 4(c) and Fig.4(d) show the probability density functions for PCE and MCS for tensored and sparse grid respectively. The tensor grid is built upon 5th order Gauss-Hermite rule and the sparse grid is built upon a 4th order Gauss Patterson rule. The figures show that PCE matches the MCS reasonably well in both cases. It must be noted that the number of eigenvectors (and hence random variables) considered in KLE is 8 whereas the number of random variables in PCE is taken to be 2. This indicates that a lower order PCE is sufficient to approximate the effect of higher order *i.e.*, input random variable-random fluctuating flow. This optimal stochastic dimension of PCE and the optimal collocation grid order are possible subjects for further exploration.

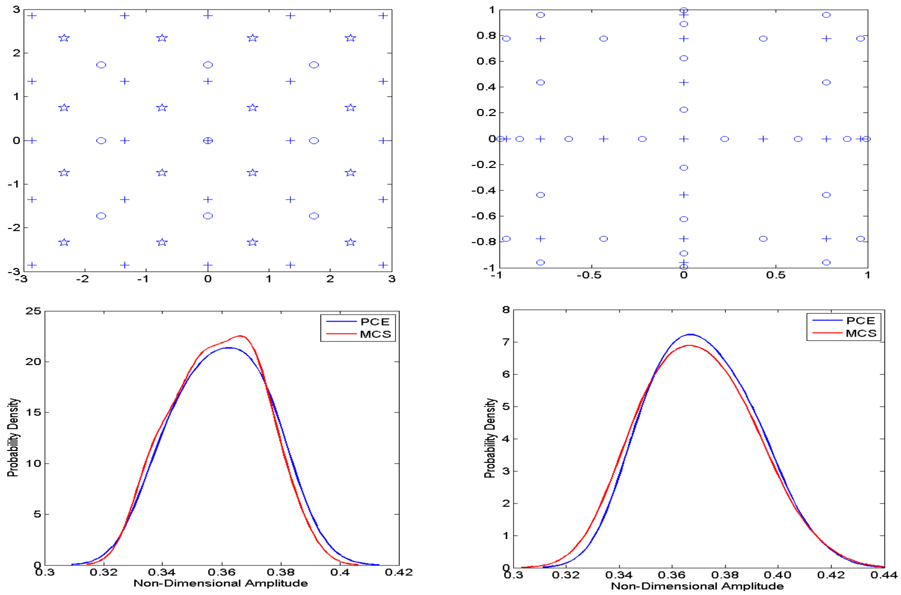


Fig. 4. (a) Tensor grid of 1-d Gauss-Hermite quadrature rules of 3,4,5; (b) sparse grid of order 3,4 Gauss-Patterson quadrature rules; (c) (d) Probability density functions of the extreme values for MCS and PCE for tensor grid collocation of 5th order and sparse grid collocation of 4th order respectively.

## 4. Conclusion

Effect of uncertainty on the flutter of a cantilevered plate in randomly fluctuating axial flow has been discussed. The fluctuating inflow has been simulated using Karhunen Loeve expansion and a polynomial chaos based method has been employed for uncertainty quantification of the system. Stochastic collocation, both tensor-grid based and sparse-grid based, has been employed and the estimates of the PDF of the extreme values of response obtained using both these approaches match reasonably well with the estimates obtained from Monte Carlo simulations. The development of sparse grid collocation for the problem is significant since it enables us to perform uncertainty quantification for inflows with shorter correlation length, thus requiring more KLE eigenvectors. This, in turn, means more random variables in PCE leading to prohibitively large number of collocation points in the case of tensored grid. Hence the proposed sparse grid collocation based PCE for the plate flutter problem promises to be a computationally efficient alternative to MCS for uncertainty quantification.

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