DE GRUYTER OPEN Spec. Matrices 2014; 2:165–179

Research Article **Open Access**

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Singular M-matrices which may not have a nonnegative generalized inverse

Abstract: A matrix $A \in \mathbb{R}^{n\times n}$ is a GM-matrix if $A = sI - B$, where $0 < \rho(B) \le s$ and $B \in$WPFn i.e., both B and B^t have $\rho(B)$ as their eigenvalues and their corresponding eigenvector is entry wise nonnegative. In this article, we consider a generalization of a subclass of GM-matrices having a nonnegative core nilpotent decomposition and prove a characterization result for such matrices. Also, we study various notions of splitting of matrices from this new class and obtain sufficient conditions for their convergence.

Keywords: Eventually nonnegative, Eventually positive, Perron-Frobenius property, Perron-Frobenius splitting, PFn, WPFn

MSC: 15A09, 15B48

DOI 10.2478/spma-2014-0017
Received March 31, 2014; accepted September 4, 2014

1 Introduction

Let $\mathbb{R}^{n\times n}$ denote the set of all real square matrices of order n. We say that a real matrix A is nonnegative (positive) if it is entry wise nonnegative (positive) and we write $A \ge 0$ ($A > 0$). This notation and nomenclature are used for vectors also. If v is a nonzero and nonnegative column or row vector then we say that v is *semipositive*.

Definition 1.1. *$A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if all the off-diagonal entries of A are nonpositive. If A is a Z-matrix, then A can be expressed in the form $A = sI - B$, where $B \ge 0$ and $s \ge 0$. A Z-matrix A is called an M-matrix if $B \ge 0$ and $0 \le \rho(B) \le s$, where $\rho(B)$ denotes the spectral radius of B.*

The term M-matrix was first introduced by Ostrowski in 1937 with reference to the work of Minkowski who proved that if a Z-matrix A has all its row sums positive, then det $A > 0$. An extensive theory of M-matrices has been developed relative to their role in numerical analysis, in modeling of an economy, optimization and Markov chains [3]. Fifty equivalent conditions for a matrix to be an M-matrix are also given there. The following is a sample of a couple of such equivalent conditions for a matrix to be a nonsingular M-matrix.

Theorem 1.1. *(Theorem 6.2.3, [3]) Let A be a Z-matrix. Then the following statements are equivalent.*
(i) A is a nonsingular M-matrix.
(ii) $A^{-1} \ge 0$.
(iii) A has a convergent regular splitting, that is, A has a representation $A = U - V$, where $U^{-1} \ge 0$, $V \ge 0$, and $U^{-1}V$ is convergent ($\rho(U^{-1}V) < 1$).

In [3], the authors have also proved the following result for a singular M-matrix. For $A \in \mathbb{R}^{n\times n}$, A^D denotes the Drazin inverse of A (see section 2 for a definition).

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Theorem 1.2. *(Lemma 6.4.4, [3]) Let $A \in \mathbb{R}^{n\times n}$ and $A = sI - B$, where $B \geq 0$. Then $\rho(B) \leq s$ if and only if $A^D \geq 0$.*

Several generalizations of M-matrices have been studied in the literature. We recall a few of these, in what follows. In [16], the class of M_v-matrices were introduced and the authors established that M_v-matrices have properties that are analogous to those of M-matrices. A matrix $A \in \mathbb{R}^{n\times n}$ is an M_v-matrix if A can be expressed as $A = sI - B$, where $0 \leq \rho(B) \leq s$ and there exists an integer m_0 such that $B^m \geq 0$ for every integer $m \geq m_0$. This last condition on B is referred to as *eventual nonnegativity*.

In [11], the notion of pseudo M-matrices were introduced. These are matrices of the form $A = sI - B$, where $s > \rho(B) > 0$ and B is *eventually positive*, i.e., there exists a nonnegative integer m such that $B^l > 0$ for all $l \geq m$. The authors show that the inverse of a pseudo M-matrix is eventually positive.

In [12], matrices of the form $A = sI - B$ were considered, where $s > \rho(B)$ with B irreducible and eventually nonnegative. The authors demonstrate that if an eventually nonnegative matrix B is irreducible and the index of the eigenvalue 0 of B is at most 1, then there exists $\beta > \rho(B)$ such that $A = sI - B$ has a positive inverse for all $s \in (\rho(B), \beta)$.

Let us recall that $A \in \mathbb{R}^{n\times n}$ is said to have the *Perron-Frobenius property*, if $\rho(A)$ positive and is an eigenvalue of A such that, there is a nonnegative eigenvector corresponding to this eigenvalue. Let $WPFn$ denote the class of all matrices $B \in \mathbb{R}^{n\times n}$ such that both B and B^t have the Perron-Frobenius property.

In [7], the authors consider yet another extension of M-matrices, namely, GM-matrices. $A = sI - B$ is called a GM-matrix, if $0 < s \leq \rho(B)$ and $B \in WPFn$. The authors prove that A is a nonsingular GM-matrix if and only if $A^{-1} \in WPFn$ and $0 < \lambda_n < Re(\lambda_i)$ for $i = 1, 2, \cdots, n-1$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. This is an analogue to Theorem 1.1, for GM-matrices.

When we attempt at extending the above result to singular matrices, we observe that the group inverse is a better choice. The reason for this is given in the following. If λ is an eigenvalue of a nonsingular matrix A, then we know that λ^{-1} is an eigenvalue of A^{-1}, with the same eigenvector. But not every generalized inverse retains this property. Such a property is referred to as the spectral property [1]. For example, if $0 \neq \lambda$ is an eigenvalue of a singular matrix A, then it is not true always that λ^{-1} is an eigenvalue of $A^{\dagger}$. On the other hand, if λ is an eigenvalue of a singular matrix A, then we know that $\lambda^{\dagger}$ is an eigenvalue of $A^{\#}$, where $\lambda^{\dagger} = \lambda^{-1}$, if $\lambda \neq 0$ and $\lambda^{\dagger} = 0$ if $\lambda = 0$. A precise statement is given in Theorem 2.3. It is for this advantage that we prefer the group inverse to any other generalized inverse, in particular the Moore-Penrose inverse.

In this article, as our first objective, we extend the aforementioned result of [7] to a subclass of singular matrices, which in turn also generalizes Theorem 1.2. This is done in Theorem 3.2. We consider matrices with a nonnegative core nilpotent decomposition, i.e., those matrices A, of index k, which can be written as $A = P\left[\begin{array}{c|c} C & 0 \\ \hline 0 & N \end{array}\right]P^{-1}$, where C and P are nonsingular matrices, N is nilpotent of index k (that is $N^l = 0$ for all $l \geq k$), and O is the zero matrix of appropriate size. P and P^{-1} are nonnegative. Then we consider, among such matrices, only those matrices which have a representation similar to those of GM-matrices and call them as $GM_{\#}$-matrices. Consequently, we prove that A is a $GM_{\#}$-matrix if and only if $A^{\#} \in WPFn$.

In the second part of this article, in Section 4, we consider various splittings of matrices of the type above and obtain sufficient conditions for their convergence. We say that a splitting $A = U - V$ converges if $\rho(U^{-1}V) < 1$ when U is invertible and $\rho(U^{\#}V) < 1$ or $\rho(U^{\dagger}V) < 1$ (as the case may be) when U is a singular matrix (Here $U^{\#}$ denotes the group inverse of U and $U^{\dagger}$ denotes the Moore-Penrose inverse of U. These definitions will be given in the next section).

The paper is organized as follows. In the section that follows the introductory part, we present some preliminary definitions and results. In the third section, we characterize $GM_{\#}$-matrices. In the last section, we give some sufficient conditions for the convergence of splittings of $GM_{\#}$-matrices.

2 Preliminary notions and results

Let $A \in \mathbb{R}^{n\times n}$. The unique matrix $Y \in \mathbb{R}^{n\times n}$ such that $AYA = A$, $YAY = Y$, $(AY)^t = AY$ and $(YA)^t = YA$ is called the Moore-Penrose inverse of A and is denoted by $A^\dagger$. Recall that the smallest positive integer k such that $\mathbb{R}^n = R(A^k) \oplus N(A^k)$, or equivalently, the smallest nonnegative integer k such that rank A^k = rank A^{k+1} is called the index of A and is denoted by Ind(A). It is well known that the index exists for all nonzero matrices. Let Ind$(A) = k$. Then the unique matrix X, which satisfies the equations $XAX = X$, $AX = XA$ and $A^{k+1}X = A^k$ is called the Drazin inverse of A and is denoted by A^D. When $k = 1$, X is known as the group inverse of A and is denoted by $A^\#$. The group inverse of A exists if and only if rank(A) = rank(A^2). The group inverse, if it exists, is unique.

The following Theorem gives a formula to find the Drazin inverse (and hence the group inverse, if it exists) of A from the core nilpotent decomposition of A.

Theorem 2.1. *(Theorem 7.2.1, [5]) If $A \in \mathbb{R}^{n\times n}$ is such that $Ind(A) = k$, then there exists a nonsingular matrix P such that $A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$, where C is nonsingular and N is nilpotent of index k. Further, if P, C and N are any matrices satisfying the above conditions, then $A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$.*

Theorem 2.2. *(Corollary 7.2.2, [5]) For $A \in \mathbb{R}^{n\times n}$, $A^\#$ exists if and only if there exists nonsingular matrices P and C such that $A = P \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$. If $A^\#$ exists then $A^\# = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$.*

The spectral property of the group inverse is given by the following Theorem. Let $\lambda^\dagger$ denote $\frac{1}{\lambda}$ if $\lambda \neq 0$ and 0, if $\lambda = 0$.

Theorem 2.3. *(Theorem 7.4.1, [5]) For $A \in \mathbb{R}^{n\times n}$, with index 1, $\lambda \in \sigma(A)$ if and only if $\lambda^\dagger \in \sigma(A^\#)$. That is, if $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, then $\sigma(A^\#) = \{\lambda_1^\dagger, \lambda_2^\dagger, \dots, \lambda_n^\dagger\}$.*

The reverse order law does not hold for the group inverse in general. However, the commutativity of A and B guarantees that $(AB)^\# = B^\# A^\#$.

Theorem 2.4. *(Theorem 7.8.4, [5]) Let both $A, B \in \mathbb{R}^{n\times n}$ have index 1. If $AB = BA$, then*
(i) $(AB)^\# = B^\# A^\# = A^\# B^\#$
(ii) $A^\# B = BA^\#$, $AB^\# = B^\# A$.

Next, we recall the notion of dominant and strictly dominant eigenvalues of a square matrix A.

Definition 2.1. *For $A \in \mathbb{R}^{n\times n}$, $\sigma(A)$ denotes its spectrum. An eigenvalue $\lambda \in \sigma(A)$ is called* dominant *if $|\lambda| = \rho(A)$ and* strictly dominant *if $\lambda = \rho(A)$, λ is a simple eigenvalue and is strictly larger in modulus than any other eigenvalue, i.e., $|\lambda| > |\mu|$ for all $\mu \in \sigma(A)$, with $\mu \neq \lambda$. The eigenspace of A for the eigenvalue λ is denoted by $E_\lambda(A)$. Thus $E_\lambda(A) = N(A - \lambda I)$, the null space of $A - \lambda I$.*

The definition of a matrix having the Perron-Frobenius property was mentioned in the introduction. We recall a stronger notion, next.

Definition 2.2. *We say that $A \in \mathbb{R}^{n\times n}$ has the strong Perron-Frobenius property if the spectral radius $\rho(A)$ is a strictly dominant eigenvalue and there is a positive eigenvector corresponding to $\rho(A)$. By PFn we mean the collection of matrices A such that both A and A^t have the strong Perron-Frobenius property. As mentioned earlier, WPFn denotes the collection of matrices A such that both A and A^t have the Perron-Frobenius property.*

Recall that a matrix A is said to be *eventually nonnegative (eventually positive)* if $A^k \geq 0$ ($A^k > 0$) for all $k \geq k_0$ for some positive integer k_0.

The inclusions in the following are proper (see [6], Section 5): $PFn \subset$ {nonnilpotent eventually nonnegative matrices} $\subset WPFn$.

For $A \in \mathbb{R}^{n\times n}$, we denote by $G(A)$, the *graph* with vertices $1, 2, \ldots, n$ in which there is an edge (i, j) if and only if $a_{ij} \neq 0$. We say that *vertex i has access to vertex j* if $i = j$ or if there is a sequence of vertices $\{v_1, v_2, \ldots, v_r\}$ such that $v_1 = i$, $v_r = j$ and (v_i, v_{i+1}) is an edge in $G(A)$, for $i = 1, 2, \ldots, r-1$. If i has access to j and j has access to i then we say that i and j *communicate*. Equivalence classes under the communication relation on the set of vertices of $G(A)$ are called *classes* of A. By $A[\alpha]$ we denote the principal sub-matrix of A indexed by $\alpha \subseteq \{1, 2, \ldots, n\}$. The graph $G(A[\alpha])$ is called a *strong component* of $G(A)$ whenever α is a class of A. We say that $G(A)$ is *strongly connected* whenever A has only one class, or equivalently, whenever A is irreducible. We call a class α *basic* if $\rho(A[\alpha]) = \rho(A)$. We call a class α *initial* if no vertex in any other class β has access to any vertex in α and final if no vertex in α has access to a vertex in any other class β.

In the rest of this section we collect results that will be used in the sequel. The next two theorems give a relation between eventually positive (eventually nonnegative) matrices and the matrices with PFn ($WPFn$) property.

Theorem 2.5 (Theorem 2.2, [15])**.** *For any $A \in \mathbb{R}^{n\times n}$, the following properties are equivalent:*
(i) A and A^t possess the strong Perron-Frobenius property.
(ii) A is an eventually positive matrix.
(iii) A^t is an eventually positive matrix.

Theorem 2.6. *(Theorem 2.3, [15]) Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix which is not nilpotent. Then both A and A^t possess the Perron-Frobenius property.*

The following result can be proved using the spectral decomposition. A proof is given in [9]. $G_\lambda(A)$ denotes the generalized eigenspace of A corresponding to the eigenvalue λ.

Theorem 2.7. *(Theorem 2.1, [8]) Let $A \in \mathbb{R}^{n\times n}$ have k distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ where $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k|$. Let P be the projection matrix onto $G_{\lambda_1}(A)$ along $\oplus_{j=2}^{k} G_{\lambda_j}(A)$ (P is called the spectral projector) and let $Q = A - \lambda_1 P$. Then, $PQ = QP$ and $\rho(Q) \leq \rho(A)$. Furthermore, if the index of $A - \lambda_1 I$ is 1 then $PQ = 0$.*

Next we present two results, where the first one gives a necessary and sufficient condition for a matrix to be in PFn, while the second one gives a characterization for a matrix to be in $WPFn$.

Theorem 2.8. *(Theorem 2.2, [8]) For any matrix $A \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*
(i) $A \in PFn$.
(ii) $\rho(A)$ is an eigenvalue of A and in the spectral decomposition $A = \rho(A)P + Q$ we have $P > 0$, rank $P = 1$ and $\rho(Q) < \rho(A)$, where P denotes the spectral projector.

Theorem 2.9. *(Theorem 2.3, [8]) For any matrix $A \in \mathbb{R}^{n\times n}$, the following are equivalent:*
(i) $A \in WPFn$ has a strictly dominant eigenvalue.
(ii) $\rho(A)$ is an eigenvalue of A and in the spectral decomposition $A = \rho(A)P + Q$ we have $P \geq 0$, rank $P = 1$ and $\rho(Q) < \rho(A)$, where P denotes the spectral projector.

The following two results together give another sufficient condition for a matrix to be in $WPFn$.

Theorem 2.10. *(Theorem 3.6, [8]) If the matrix A has a basic and initial class α for which $A[\alpha]$ has a right Perron-Frobenius vector, then A has the Perron-Frobenius property.*

Theorem 2.11. *(Theorem 3.7, [8] If the matrix A has a basic and final class β for which $(A[\beta])^t$ has a right Perron-Frobenius vector, then A^t has the Perron-Frobenius property.*

3 $GM_\#$-matrices

As mentioned earlier, in [7], the authors proposed the notion of a GM-matrix and gave a characterization for a nonsingular GM-matrix. We give the statement of this result for ready reference and later use.

Theorem 3.1. *(Theorem 3.1, [7]) Let $A \in \mathbb{R}^{n\times n}$. Let the eigenvalues of A (when counted with multiplicity) be arranged in the following manner: $|\lambda_1| \geq |\lambda_2| \geq \dots |\lambda_n|$. Then the following are equivalent:*
(i) A is a nonsingular GM-matrix.
(ii) $A^{-1} \in WPFn$ and $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$.

Next, we propose the definition of a nonnegative core-nilpotent decomposition.

Definition 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be of index k. A core-nilpotent decomposition, $A = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1}$ is called a nonnegative core-nilpotent decomposition if $P \geq 0$ and $P^{-1} \geq 0$. Here C is nonsingular, N is nilpotent of index k and O is the zero matrix of the appropriate size.*

We now present the main result of this article, which is an analogue of Theorem 3.1 for singular matrices. First, we consider the class of all matrices for which the group inverses exist.

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be of index 1. Let $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, the non-zero eigenvalues of A, be such that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_m|$, where $1 < m < n$. Further, assume that, $A = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$ is a nonnegative core nilpotent decomposition. Then the following statements are equivalent:*
(i) A can be written as $A = \rho(B)I - B$, with $P^{-1}BP = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, where $B_1 \in WPFm$.
(ii) $A^\# \in WPFn$ and $0 < \lambda_m < Re(\lambda_i)$, for $i = 1, 2, \dots, m-1$.

Proof. We have $A = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$, $P \geq 0$, $P^{-1} \geq 0$ and C is nonsingular. As the index of A is 1, $A^\#$ exists and so $A^\# = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$, by Theorem 2.2. By Theorem 2.3, the nonzero eigenvalues of $A^\#$ are $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_m^{-1}$ (including multiplicities) and 0 is also an eigenvalue of $A^\#$, with $n-m$ as its multiplicity. So, $\rho(A^\#) = |\lambda_m|^{-1} = \rho(C^{-1})$, as the eigenvalues of C^{-1} are same as the nonzero eigenvalues of $A^\#$.

(i) $\Rightarrow$ (ii). Let $A = \rho(B)I - B$, with $P^{-1}BP = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, where $B_1 \in WPFm$. We prove that $\rho(A^\#)$ is an eigenvalue of $A^\#$ with a nonnegative eigenvector corresponding to it. From $A = \rho(B)I - B$, we have
$P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = \rho(B)\begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} - B.$

Thus, $\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} = \rho(B)\begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} - P^{-1}BP = \rho(B)\begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} - \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$.

We thus have $C = \rho(B)I_m - B_1$ and $O = \rho(B)I_{n-m} - B_2$. Thus, B_2 is a diagonal matrix of order $n-m$ with $\rho(B)$ as its diagonal entries. Clearly, $\rho(B) \geq \rho(B_1)$. Since C is nonsingular, we have $\rho(B) > \rho(B_1)$. Also, $B_1 \in WPFm$. So, C is a nonsingular GM-matrix. Therefore, by Theorem 3.1, $C^{-1} \in WPFm$ and $0 < \lambda_m < Re(\lambda_i)$ for $i = 1, 2, \dots, m-1$.

Next we show that $A^\# \in WPFn$. Let $w^0, u^0 \in \mathbb{R}^m_+$ be such that

$$C^{-1}w^0 = \rho(C^{-1})w^0 = |\lambda_m|^{-1}w^0 = \lambda_m^{-1}w^0$$
$$\text{and } (C^{-1})^t u^0 = \rho(C^{-1})u^0 = \lambda_m^{-1}u^0.$$

Set $w := (w^0, 0)^t \in \mathbb{R}^n$. Then $w \geq 0$. Further,

$$A^{\#}(Pw) = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} w = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w^0 \\ 0 \end{bmatrix} = P\begin{bmatrix} C^{-1}w^0 \\ 0 \end{bmatrix} = P\begin{bmatrix} \rho(C^{-1})w^0 \\ 0 \end{bmatrix} = \rho(C^{-1})Pw = \lambda_m^{-1}Pw = \rho(A^{\#})Pw.$$

Thus, $A^{\#}(Pw) = \rho(A^{\#})Pw$, where $Pw \geq 0$ (since $P \geq 0$). Hence, Pw is a right Perron-Frobenius vector for $A^{\#}$. This implies that $A^{\#}$ has the Perron-Frobenius property. In a similar way, we can prove that $(A^{\#})^t$ also has the Perron-Frobenius property. So, $A^{\#} \in WPFn$.

(ii)$\Rightarrow$ (i): Let $A^{\#} \in WPFn$ and $Re(\lambda_i) > \lambda_m > 0$ for $i = 1, 2, \cdots, m-1$. So, there exists $v \geq 0$, $w \geq 0$ in $\mathbb{R}^n$ such that $A^{\#}v = \rho(A^{\#})v = \rho(C^{-1})v$ and $(A^{\#})^t w = \rho(A^{\#})w = \rho(C^{-1})w$. Now $A^{\#} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$. So,

$$\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}v = P^{-1}A^{\#}PP^{-1}v = \rho(A^{\#})P^{-1}v = \rho(C^{-1})P^{-1}v.$$

Let $v^0 \in \mathbb{R}^m$ be defined such that its m coordinates are the first m coordinates of $P^{-1}v$ in that order. Thus $v^0 \geq 0$. We show that $v^0 \neq 0$. Let $v = (v_1, v_2, \ldots, v_n)^t$. As P and P^{-1} are both nonnegative, P and P^{-1} are both monomial matrices, i.e., each row and column has only one nonzero entry. Therefore, $P^{-1}v = (k_{1i}v_i, k_{2j}v_j, \ldots, k_{nl}v_l)^t$, where k_{ji} is the unique positive entry in the j^{th} row of the matrix P^{-1}. If $v^0 = 0$, then $P^{-1}v = (0, k_{m+1s}v_s, \ldots, k_{nl}v_l)^t$ (where 0 denotes a zero vector of appropriate order). From the last equation, we then have

$$(0, 0)^t = \rho(C^{-1})(0, k_{m+1s}v_s, \cdots, k_{nl}v_l)^t,$$

that is, $P^{-1}v = 0$. This implies that $v = 0$, a contradiction. So $v^0 \neq 0$. Hence, $\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v^0 \\ 0 \end{bmatrix} = \rho(C^{-1})\begin{bmatrix} v^0 \\ 0 \end{bmatrix}$. So, $C^{-1}v^0 = \rho(C^{-1})v^0$. This implies that C^{-1} has the Perron-Frobenius property. In a similar way we can prove that $(C^{-1})^t$ also has the Perron-Frobenius property. Thus, $C^{-1} \in WPFm$. Further, the eigenvalues of C are the nonzero eigenvalues of A, which satisfy the condition that $0 < \lambda_m < Re(\lambda_i)$, for $i = 1, 2, \cdots, m-1$. Therefore, by Theorem 3.1, C is a nonsingular GM-matrix. Hence, there exists $B_1 \in \mathbb{R}^{m \times m}$ such that $C = sI - B_1$ with $B_1 \in WPFm$ and $s > \rho(B_1)$. Now, set $B = P\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}P^{-1}$, where $B_2 = sI_{n-m}$. Then $\rho(B) = s$, since $s > \rho(B_1)$. Now, $sI_n - B = s\begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} - P\begin{bmatrix} B_1 & 0 \\ 0 & sI_{n-m} \end{bmatrix}P^{-1} = P\begin{bmatrix} sI_m - B_1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = A$. Thus A has the given property, completing the proof of (ii) $\Rightarrow$ (i). □

Remark 3.1. *Theorem 3.2 holds good when A is of index k where $k > 1$. In this case we must replace $A^{\#}$ by A^D, the Drazin inverse of A.*

We illustrate the above theorem by the following example.

Example 3.1. *Let* $A = \begin{bmatrix} 7 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Then rank A = rank A^2 and so $A^{\#}$ exists. Also,* $\sigma(A) = \{10, 8, 1, 0\}$, $\rho(A) = 10$. *Let* $P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. *Then* $P^{-1}AP = \left[\begin{array}{ccc|c} 7 & -3 & 0 & 0 \\ 1 & 11 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$, *so that* $C = \begin{bmatrix} 7 & -3 & 0 \\ 1 & 11 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $C^{-1} = \frac{1}{80}\begin{bmatrix} 11 & 3 & 0 \\ -1 & 7 & 0 \\ 0 & 0 & 80 \end{bmatrix}$. $\sigma(C) = \{10, 8, 1\}$. *Now, let* $B = \begin{bmatrix} 13 & 0 & 2 & 0 \\ 0 & 19 & 0 & 0 \\ -\frac{3}{2} & 0 & 9 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}$. *For one thing,* $B \ngeq 0$ *and for another,* $B \in WPF_4$. *The latter assertion follows from the fact that the eigenspace corresponding to the eigenvalue 20 is spanned by the vector* $(0, 0, 0, 1)^t$. *Then* $A = \rho(B)I - B$, *where* $\sigma(B) = \{20, 19, 12, 10\}$. *Also,*

$P^{-1}BP = \left[\begin{array}{ccc|c} 13 & 3 & 0 & 0 \\ -1 & 9 & 0 & 0 \\ 0 & 0 & 19 & 0 \\ \hline 0 & 0 & 0 & 20 \end{array}\right]$, *with* $B_1 = \begin{bmatrix} 13 & 3 & 0 \\ -1 & 9 & 0 \\ 0 & 0 & 19 \end{bmatrix}$. $B_1 \in WPF_3$, *since* $\sigma(B_1) = \{19, 12, 10\}$ *and* $\rho(B_1) = 19$ *is an eigenvalue of* B_1 *with an eigenvector* $(0, 0, 4)^t$. *Thus condition (i) of Theorem 3.2 holds. Now* $A^\# = P \left[\begin{array}{c|c} C^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] P^{-1} = \frac{1}{160}\begin{bmatrix} 22 & 0 & 4 & 0 \\ 0 & 160 & 0 & 0 \\ -3 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Since* $\sigma(A^\#) = \{0, 1, \frac{1}{10}, \frac{1}{8}\}$ *and* $\rho(A^\#) = 1$ *is an eigenvalue of* $A^\#$ *with an eigenvector* $(0, 4, 0, 0)^t$, *we have* $A^\# \in WPF_4$ *i.e., condition (ii) of Theorem 3.2 is satisfied.*

The nonnegativity of P and P^{-1}, cannot be dispensed with, in Theorem 3.2. We illustrate this by the following example.

Example 3.2. *Let* $A = \begin{bmatrix} 7 & 0 & \frac{8}{3} \\ 0 & 0 & 0 \\ 12 & 0 & 11 \end{bmatrix}$ *and* $P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -3 & 0 \end{bmatrix}$. *Then* $P^{-1}AP = \left[\begin{array}{cc|c} 7 & -4 & 0 \\ -8 & 11 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$. *Hence,* $C = \begin{bmatrix} 7 & -4 \\ -8 & 11 \end{bmatrix}$. *Let* $B = \begin{bmatrix} 9 & 0 & -\frac{8}{3} \\ 0 & 16 & 0 \\ -12 & 0 & 5 \end{bmatrix}$. *Then* $A = 16I - B$, *where* $\sigma(B) = \{16, 13, 1\}$, $\rho(B) = 16$. $P^{-1}BP = \left[\begin{array}{cc|c} 9 & 4 & 0 \\ 8 & 5 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$, *so that* $B_1 = \begin{bmatrix} 9 & 4 \\ 8 & 5 \end{bmatrix}$. *Since* $B_1 \geq 0$, *we have* $B_1 \in WPF_2$. *On the other hand,* $A^\# = \frac{1}{45}\begin{bmatrix} 11 & 0 & -\frac{8}{3} \\ 0 & 0 & 0 \\ -12 & 0 & 7 \end{bmatrix}$, $\sigma(A^\#) = \{\frac{1}{15}, \frac{1}{3}, 0\}$ *and* $\rho(A^\#) = \frac{1}{3}$. *But the eigenvector corresponding to* $\frac{1}{3}$ *is of the form* $(2\alpha, 0, -3\alpha)^t$, *where* α *is any real number.*

Next we extend the definition of a GM-matrix to any square matrix of index 1.

Definition 3.2. *Let A be a square matrix of index 1, having a nonnegative core nilpotent representation. We say that A is a $GM_\#$-matrix if it satisfies property (i) of Theorem 3.2. A is said to be an inverse $GM_\#$-matrix if $A^\#$ has that property.*

In view of Theorem 3.2, we have the following:

Corollary 3.1. *A matrix $C \in \mathbb{R}^{n\times n}$ is an inverse $GM_\#$-matrix if and only if $C \in WPFn$ and $Re(\lambda^{-1}) > (\rho(C))^{-1}$ for all $\lambda \in \sigma(C)$, $\lambda \neq \rho(C)$. Every nonzero real eigenvalue of an inverse $GM_\#$-matrix is positive.*

4 Splittings of $GM_\#$-matrices

In [10] and [14], the authors studied various splittings of rectangular matrices. All those splittings involve Moore-Penrose inverse. As mentioned in the introduction, only the group inverse has spectral properties similar to those of inverse of a nonsingular matrix. So, we study those splittings of matrices that uses group inverse of matrix.

In this section, we define various splittings of a $GM_\#$-matrix and give sufficient conditions for their convergence. We begin by recalling some definitions.

Definition 4.1. *A splitting $A = U - V$ is called*
(1) a weak (nonnegative) splitting if $U^{-1}V \geq 0$.
(2) a weak-regular splitting if $U^{-1}V \geq 0$ and $U^{-1} \geq 0$.
(3) a regular splitting if $U^{-1} \geq 0$ and $V \geq 0$.

The notion of proper splitting of matrices plays a crucial role in characterizing various generalizations of monotone matrices. Let us recall its definition [4].

Definition 4.2. *Let $A \in \mathbb{R}^{n\times n}$. Then $A = U - V$ is said to be a proper splitting if $R(A) = R(U)$ and $N(A) = N(U)$.*

The following theorem gives some of the properties of a proper splitting in the context of the group inverse. For a proof we refer to [13].

Theorem 4.1. *Let $A = U - V$ be a proper splitting of A. Suppose that $A^{\#}$ exists. Then $U^{\#}$ exists and*
(a) $AA^{\#} = UU^{\#}$; $A^{\#}A = U^{\#}U$; $VU^{\#}U = V$; $UU^{\#}V = V$.
(b) $A = U(I - U^{\#}V) = (I - VU^{\#})U$.
(c) Both $I - U^{\#}V$ and $I - VU^{\#}$ are invertible.
(d) $A^{\#} = (I - U^{\#}V)^{-1}U^{\#} = U^{\#}(I - VU^{\#})^{-1}$.

The next result presents necessary and sufficient conditions for the convergence of proper splitting of a matrix of index 1. This is an extension of Lemma 4.5 of [7] for the case of singular matrices.

Theorem 4.2. *Let $A = U - V$ be a proper splitting of a matrix A of index 1. Then the following are equivalent:*
(i) The splitting is convergent. i.e., $\rho(U^{\#}V) < 1$.
(ii) $\min\{Re(\lambda) : \lambda \in \sigma(A^{\#}V)\} > -\frac{1}{2}$.
(iii) $\min\{Re(\lambda) : \lambda \in \sigma(VA^{\#})\} > -\frac{1}{2}$.

Proof. (i) $\Leftrightarrow$ (ii): Let $A = U - V$ be a proper splitting. Then by Theorem 4.1, $A = U(I - U^{\#}V) = (I - VU^{\#})U$ and $A^{\#} = (I - U^{\#}V)^{-1}U^{\#} = U^{\#}(I - VU^{\#})^{-1}$. So, $A^{\#}V = (I - U^{\#}V)^{-1}U^{\#}V$. Hence, if λ is an eigenvalue of $U^{\#}V$ with the eigenvector v, then $A^{\#}Vv = (I - U^{\#}V)^{-1}U^{\#}Vv = \frac{\lambda}{1-\lambda}v$. Note that $\lambda \neq 1$ (since $I - U^{\#}V$ is invertible). This implies that $\frac{\lambda}{1-\lambda} \in \sigma(A^{\#}V)$. Again $U = A + V = A - (-V)$. This is a proper splitting of U. So $U^{\#}(-V) = (I - A^{\#}(-V))^{-1}A^{\#}(-V)$ i.e., $U^{\#}V = (I + A^{\#}V)^{-1}A^{\#}V$. As above, we can see that if μ is an eigenvalue of $A^{\#}V$ with an eigenvector w, then $\frac{\mu}{1+\mu}$ is an eigenvalue of $U^{\#}V$. Thus $\mu \in \sigma(U^{\#}V)$ if and only if there exists a unique $\lambda \in \sigma(A^{\#}V)$ such that $\mu = \frac{\lambda}{1+\lambda}$. The inequality $\rho(U^{\#}V) < 1$ holds if and only if $|\mu| < 1$ for all $\mu \in \sigma(U^{\#}V)$, which in turn holds if and only if $|\frac{\lambda}{1+\lambda}| < 1$ for all $\lambda \in \sigma(A^{\#}V)$. This is true if and only if $\frac{(Re(\lambda))^2+(Img(\lambda))^2}{(1+Re(\lambda))^2+(Img(\lambda))^2} < 1$ for all $\lambda \in \sigma(A^{\#}V)$, which in turn holds if and only if $Re(\lambda) > -\frac{1}{2}$ for all $\lambda \in \sigma(A^{\#}V)$. Finally, this happens if and only if $\min\{Re(\lambda) : \lambda \in \sigma(A^{\#}V)\} > -\frac{1}{2}$. This proves (i) $\Leftrightarrow$ (ii).
The equivalence of (i) and (iii) follows by observing that the nonzero eigenvalues of $A^{\#}V$ and $VA^{\#}$ are the same or by using the relation $VA^{\#} = VU^{\#}(I - VU^{\#})^{-1}$. □

Corollary 4.1. *Let $A = U - V$ be a proper spitting of a matrix A of index 1. If $A^{\#}V$ or $VA^{\#}$ is an inverse $GM_{\#}$-matrix, then the splitting is convergent.*

Proof. Let $P = A^{\#}V$. If P is an inverse $GM_{\#}$-matrix, then by corollary 3.1, $P \in WPFn$ and $Re((\lambda)^{-1}) > (\rho(P))^{-1} > 0$ for all nonzero $\lambda \in \sigma(P)$, $\lambda \neq \rho(P)$. Thus condition (ii) of Theorem 4.2 is satisfied. Therefore the splitting is convergent. If $P = VA^{\#}$, by a similar argument, it again follows that the splitting is convergent. □

Before we proceed to define splittings of $GM_{\#}$-matrices we give some results that will be used to prove the convergence of such splittings. The following lemma is part of Theorem 2.1 in [2].

Lemma 4.1. *Let $A = U - V$ be a proper splitting of A such that $U^\dagger V \in WPFn$. Then the following are equivalent:*
(i) $\rho(U^\dagger V) < 1$.
(ii) $A^\dagger V$ has the Perron-Frobenius property.
(iii) $\rho(U^\dagger V) = \frac{\rho(A^\dagger V)}{1+\rho(A^\dagger V)}$.

The above result holds good even if we replace the Moore-Penrose inverse by the group inverse, when it exists.

Lemma 4.2. *Let $A = U - V$ be a proper splitting of A such that $A^\#$ exists. If $U^\# V$ has the Perron-Frobenius property, then the following are equivalent:*
(i) $\rho(U^\# V) < 1$.
(ii) $A^\# V$ has the Perron-Frobenius property.
(iii) $\rho(U^\# V) = \frac{\rho(A^\# V)}{1+\rho(A^\# V)}$.

We may make even weaker assumptions in Lemma 4.2, as we show below.

Lemma 4.3. *Let $A = U - V$ be a proper splitting of a matrix A of index 1, such that $V^\#$ exists and $UV = VU$. Suppose that $V^\# U$ is a $GM_\#$-matrix. Then the following are equivalent:*
(i) $\rho(U^\# V) < 1$.
(ii) $A^\# V$ has the Perron-Frobenius property.
(iii) $\rho(U^\# V) = \frac{\rho(A^\# V)}{1+\rho(A^\# V)}$.

Proof. Since $V^\# U$ is a $GM_\#$-matrix, by Theorem 2.4 and Theorem 3.2, $U^\# V = (V^\# U)^\# \in WPFn$. This implies that $U^\# V$ has the Perron-Frobenius property. The equivalence of the statements now follows from Lemma 4.2. □

Theorem 4.3. *Let $A \in \mathbb{R}^{n\times n}$ be with index 1 and $A = U - V$ be a proper splitting of A, such that $U^\# V$ has the Perron-Frobenius property and $U^\# V$ is not nilpotent. Then any one of the following conditions is sufficient for the convergence of the splitting:*

(A1) *$A^\# V$ is eventually positive.*
(A2) *$A^\# V$ is eventually nonnegative.*
(A3) *$A^\# V \in WPFn$.*
(A4) *$A^\# V$ has a simple, positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*
(A5) *$A^\# V$ has a basic and an initial class α such that $(A^\# V)[\alpha]$ has a right Perron-Frobenius eigenvector.*

Proof. We first prove that (**A2**) $\Rightarrow$ (**A3**) $\Rightarrow$ convergence of the splitting. Suppose that $A^\# V$ is eventually nonnegative. We have $A^\# = (I - U^\# V)^{-1} U^\#$ so that $A^\# V = (I - U^\# V)^{-1} U^\# V$. Since $U^\# V$ is not nilpotent, it has at least one nonzero eigenvalue, say λ ($\neq 1$, since $I - U^\# V$ is invertible). Then $\frac{\lambda}{1-\lambda}$ is an eigenvalue of $A^\# V$ showing that $A^\# V$ is not nilpotent. By Theorem 2.6, $A^\# V$ has the Perron-Frobenius property. By Lemma 4.2, it follows that the splitting is convergent. Thus we have the following implications:

$$\begin{array}{ccccccc} (\mathbf{A1}) & \Rightarrow & (\mathbf{A2}) & \Rightarrow & (\mathbf{A3}) & \Rightarrow & \text{convergence of the splitting.} \\ \Updownarrow & & & & \Uparrow & & \\ (\mathbf{A4}) & & & & (\mathbf{A5}) & & \end{array}$$

In the above scheme, (**A1**) holds if and only if $A^\# V \in PFn$ (by Theorem 2.5), which in turn is equivalent to (**A4**) (by Theorem 2.8). Then (**A5**) implies that $A^\# V$ has the Perron-Frobenius property (by Theorem 2.10), which implies the convergence of the splitting (by Lemma 4.2). The other implications are obvious. □

Remark 4.1. *Recall that a regular splitting $A = U - V$ of a monotone (inverse positive) matrix A converges. The result above is a generalization of this situation since we do not require that A be even square.*

The following is an example illustrating the splitting given in Theorem 4.3.

Example 4.1. *Let* $A = \begin{bmatrix} 7 & 0 & -\frac{8}{3} \\ 0 & 0 & 0 \\ -12 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 11 \end{bmatrix} - \begin{bmatrix} 5 & 0 & \frac{8}{3} \\ 0 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix} = U - V$. *Then this is a proper splitting of* A. *Since* $U^{\#}V = \begin{bmatrix} \frac{5}{12} & 0 & \frac{2}{9} \\ 0 & 0 & 0 \\ \frac{12}{11} & 0 & 0 \end{bmatrix} \geq 0$, $U^{\#}V$ *has the Perron-Frobenius property. Also, it is not-nilpotent, since* $\sigma(U^{\#}V) = \{0, \frac{\frac{5}{12} \pm \sqrt{\frac{25}{144} + \frac{32}{33}}}{2}\} = \{0, \frac{\frac{5}{12} \pm \sqrt{\frac{5433}{4752}}}{2}\} = \{0, 0.743, -0.326\}$. *We have* $A^{\#} = \frac{1}{135}\begin{bmatrix} 33 & 0 & 8 \\ 0 & 0 & 0 \\ 36 & 0 & 21 \end{bmatrix}$ *and so,* $A^{\#}V \geq 0$. *In particular,* $A^{\#}V$ *is eventually nonnegative. Hence the splitting is convergent, by condition* **(A2)** *of Theorem 4.3. We can also deduce this directly by noting that* $\rho(U^{\#}V) \approx 0.74 < 1$.

The splitting given in Theorem 4.3 is clearly different from the one studied in [10]. In [10], the authors study splittings of the type $A = U - V$, where $R(A) = R(U)$, $N(A) = N(U)$ and $U^{\dagger}V \geq 0$. Presently, we do not have an example of a matrix which has a splitting of the type considered in Theorem 4.3, and which does not have a splitting of the type above. However, we are able to present an example of a particular splitting corresponding to Theorem 4.3 which is not a splitting of the type above.

The following is an example of a pseudo overlapping splitting.

Example 4.2. *Let* A *be the* $GM_{\#}$*-matrix given in Example 3.1. Then* $A = 20I - B$, *where* $B = \begin{bmatrix} 13 & 0 & 2 & 0 \\ 0 & 19 & 0 & 0 \\ -\frac{3}{2} & 0 & 9 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}$, $P^{-1}BP = \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right]$ *and* $B_1 = \begin{bmatrix} 13 & 3 & 0 \\ -1 & 9 & 0 \\ 0 & 0 & 19 \end{bmatrix}$. *Here* $P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

We have $\sigma(A) = \{10, 8, 1, 0\}$, $\sigma(B) = \{20, 19, 12, 10\}$ *and* $\sigma(B_1) = \{19, 12, 10\}$ *so that* $\rho(B) = 20$, $\rho(B_1) = 19$ *and* $\rho(B) - \rho(B_1) = 20 - 19 = 1 = \lambda_2$. Let $w = (0, 0, 1)^t$ be an eigenvector for the eigenvalue 19 *of* B_1. *Set* $w^0 := (0, 0, 1, 0)^t$ *and* $v = Pw^0$. *Then* $v = (0, 4, 0, 0)^t$. We note that $Bv = (0, 76, 0, 0)^t = 19v$, *i.e.,* $v \in E_{\rho(B)-\lambda_2}(B) = E_{19}(B)$.

Now consider the splitting, $A = \begin{bmatrix} 7 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U - V$. *Then* $U^{\#} = \frac{1}{154}\begin{bmatrix} 22 & 0 & 0 & 0 \\ 0 & 77 & 0 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $U^{\#}V = \frac{1}{154}\begin{bmatrix} 0 & 0 & 44 & 0 \\ 0 & 77 & 0 & 0 \\ -21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Thus* $\sigma(U^{\#}V) = \{0, \frac{1}{2}, \pm 0.1974i\}$, *so that* $\lambda = \rho(U^{\#}V) = \frac{1}{2}$, *is the dominant eigenvalue of* $U^{\#}V$. *We have* $(U^{\#}V)(0, 4, 0, 0)^t == \rho(U^{\#}V)(0, 4, 0, 0)^t$. *Hence* $v \in E_{\lambda}(U^{\#}V) \cap E_{\rho(B)-\lambda_2}(B)$. *That is, the above splitting is a pseudo overlapping splitting of* A. *Further,* $\eta = \frac{\rho(B)-\rho(B_1)}{1-\lambda} = \frac{20-19}{1-\frac{1}{2}} = 2$ *is an eigenvalue of* $U = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $Re(\eta) = 2 > \frac{\rho(B)-\rho(B_1)}{2} = \frac{1}{2}$. *So, the given splitting is convergent.*

Remark 4.2. *In above example, the splitting* $A = \begin{bmatrix} 7 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $= U - V$ *is a pseudo overlapping splitting but not a proper nonnegative splitting. The splitting is proper, but* $U^{\dagger}V = \frac{1}{154}\begin{bmatrix} 0 & 0 & 44 & 0 \\ 0 & 77 & 0 & 0 \\ -21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ngeq 0$

In [7], Elhashah and Szyld proposed the notion of an overlapping splitting of a nonsingular GM-matrix, as given below.

Definition 4.3. *(Definition 4.3, [7]) A splitting $A = U - V$ of a nonsingular GM-matrix $A = sI - B$, where $B \geq 0$ and $B \in WPFn$, is called an overlapping splitting, if the eigenspace $E_\lambda(U^{-1}V)$, corresponding to a dominant eigenvalue λ of $U^{-1}V$, contains a right Perron-Frobenius vector of B, i.e., the vector space $E_\lambda(U^{-1}V) \cap E_{\rho(B)}(B)$ contains a right Perron-Frobenius vector of B.*

Motivated by this definition, we introduce a pseudo-overlapping splitting of a $GM_\#$-matrix.

Definition 4.4. *Let $A \in \mathbb{R}^{n\times n}$ be a $GM_\#$-matrix with $A = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$, a nonnegative core nilpotent decomposition of A, where $C \in \mathbb{R}^{m\times m}$. Here, $A = \rho(B)I - B$ for some matrix $B \in \mathbb{R}^{n\times n}$ with $P^{-1}BP = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, where $B_1 \in WPFm$ and $B_2 = \rho(B)I_{n-m}$. A splitting $A = U - V$ of A is called a pseudo overlapping splitting, if for a dominant eigenvalue λ of $U^\# V$, the vector space $E_\lambda(U^\# V) \cap E_{\rho(B)-\lambda_m}(B)$ contains a nonzero nonnegative vector. Here λ_m is the nonzero eigenvalue of A having the least absolute value.*

Now, we present a necessary and sufficient condition for the convergence of a pseudo overlapping splitting of a $GM_\#$-matrix.

Theorem 4.4. *Let $A \in \mathbb{R}^{n\times n}$ be a $GM_\#$-matrix with $A = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$, a nonnegative core nilpotent decomposition of A. Here, $C \in \mathbb{R}^{m\times m}$ is nonsingular and $A = \rho(B)I - B$ for some matrix $B \in \mathbb{R}^{n\times n}$ with $P^{-1}BP = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ where $B_1 \in WPFm$ and $B_2 = \rho(B)I_{n-m}$. Let $A = U - V$ be a pseudo overlapping splitting with $R(A) \subseteq R(U)$. If λ is a dominant eigenvalue of $U^\# V$, then $\eta = \frac{\rho(B)-\rho(B_1)}{1-\lambda} \in \sigma(U)$ and the splitting $A = U - V$ converges if and only if $Re(\eta) > \frac{\rho(B)-\rho(B_1)}{2}$.*

Proof. A is a $GM_\#$-matrix with nonnegative core nilpotent decomposition, $A = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$, with $C \in \mathbb{R}^{m\times m}$ nonsingular. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the nonzero eigenvalues of A, with $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m|$. Since A is a $GM_\#$-matrix, by Theorem 3.2, $\rho(C^{-1}) = |\lambda_m^{-1}| = \lambda_m^{-1}$ and $A = \rho(B)I - B$, with $B = P\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}P^{-1}$ where $B_1 \in WPFm$ and $B_2 = \rho(B)I_{n-m}$.

First, we prove that $\rho(B_1) = \rho(B) - \lambda_m$. From $A = \rho(B)I - B$, we get $C = \rho(B)I_m - B_1$. Therefore, $\sigma(B_1) = \{\rho(B) - \lambda_1,\ \rho(B) - \lambda_2, \ldots, \rho(B) - \lambda_m\}$. So,

$$\rho(B) - \lambda_m \leq \rho(B_1). \tag{4.1}$$

On the other hand, since $B_1 \in WPFm$, there exists $w^0 \geq 0$ in $\mathbb{R}^m$ such that $B_1 w^0 = \rho(B_1)w^0$. So, $Cw^0 = (\rho(B) - \rho(B_1))w^0$, so that $(\rho(B) - \rho(B_1))^{-1} \leq \rho(C^{-1}) = \lambda_m^{-1}$, i.e., $\rho(B) - \rho(B_1) \geq \lambda_m$ or

$$\rho(B) - \lambda_m \geq \rho(B_1). \tag{4.2}$$

From (4.1) and (4.2), we get,

$$\rho(B) - \lambda_m = \rho(B_1). \tag{4.3}$$

Since $A = U - V$ is a pseudo overlapping splitting, there exists a nonnegative nonzero vector $v \in \mathbb{R}^n$ such that

$$Bv = (\rho(B) - \lambda_m)v \tag{4.4}$$

and

$$U^{\#}Vv = \lambda v, \tag{4.5}$$

where λ is a dominant eigenvalue of $U^{\#}V$. Now, from (4.4), we have $(\rho(B)I - B)v = Av = (U - V)v$, so that $\rho(B)v - (\rho(B) - \lambda_m)v = U(I - U^{\#}V)v$, since $V = UU^{\#}V$ as $R(A) \subseteq R(U)$. Using (4.5), we then have

$$\lambda_m v = U(I - U^{\#}V)v = U(1 - \lambda)v. \tag{4.6}$$

Thus, $Uv = \frac{\lambda_m}{1-\lambda}v$. If $\lambda = 1$, (4.6) becomes $\lambda_m v = 0$, a contraction to $\lambda_m \neq 0$ and $v \neq 0$. So, $\lambda \neq 1$. Hence $\frac{\lambda_m}{1-\lambda} \in \sigma(U)$, i.e., $\eta = \frac{\rho(B)-\rho(B_1)}{1-\lambda} \in \sigma(U)$.

Since $\eta = \frac{\rho(B)-\rho(B_1)}{1-\lambda}$, we get $\lambda = \frac{\eta-(\rho(B)-\rho(B_1))}{\eta} \in \sigma(U^{\#}V)$. Thus the pseudo overlapping splitting is convergent if and only if $\rho(U^{\#}V) = |\lambda| < 1$. This holds if and only if $|\eta - (\rho(B) - \rho(B_1))| < |\eta|$ for some $\eta \in \sigma(U)$, which in turn holds if and only if $Re(\eta) > \frac{\rho(B)-\rho(B_1)}{2}$. Equivalently, η lies in the right half plane determined by the bisector of the segment on the real axis whose end points are 0 and $\rho(B) - \rho(B_1)$. ☐

Corollary 4.2. *Let $A = U - V$ be a pseudo overlapping splitting of a $GM_{\#}$-matrix A. Suppose further that $U^{\#}V \in WPFn$. Then $\rho(U^{\#}V) < 1$, i.e., the splitting is convergent.*

Proof. We have $U^{\#}V \in WPFn$. So, $\rho(U^{\#}V)$ is a dominant eigenvalue of $U^{\#}V$. Therefore, as in Theorem 4.4, we can prove that $\frac{\rho(B)-\rho(B_1)}{1-\rho(U^{\#}V)} \in \sigma(U)$. As, $1 - \rho(U^{\#}V) < 2$, the inequality $\frac{1}{1-\rho(U^{\#}V)} > \frac{1}{2}$ holds. As, $\rho(B) \geq \rho(B_1)$, this implies that $\frac{\rho(B)-\rho(B_1)}{1-\rho(U^{\#}V)} > \frac{\rho(B)-\rho(B_1)}{2}$. Now $\eta = \frac{\rho(B)-\rho(B_1)}{1-\rho(U^{\#}V)}$. Then $Re(\eta) = \eta > \frac{\rho(B)-\rho(B_1)}{2}$. Therefore, by Theorem 4.4, the splitting $A = U - V$ converges. ☐

In Theorem 4.4, if A is a nonsingular GM-matrix then the pseudo overlapping splitting $A = U - V$ is nothing but an overlapping splitting. We have the following result for the convergence of such an overlapping splitting of a nonsingular GM-matrix.

Corollary 4.3. *(Proposition 4.13, [7]) If $A = sI - B$ is a GM-matrix and the splitting $A = U - V$ is an overlapping splitting for which $E_\lambda(U^{-1}V) \cap E_{\rho(B)}(B)$ contains a right Perron-Frobenius eigenvector of B and $|\lambda| = \rho(U^{-1}V)$, then such a splitting is convergent if and only if there is an $\eta = \frac{s-\rho(B)}{1-\lambda} \in \sigma(U)$ such that $Re(\eta) > \frac{s-\rho(B)}{2}$.*

The following is an example of a pseudo overlapping splitting.

Example 4.3. *Let A be the $GM_{\#}$-matrix given in Example 3.1. Then $A = 20I - B$, where $B = \begin{bmatrix} 13 & 0 & 2 & 0 \\ 0 & 19 & 0 & 0 \\ -\frac{3}{2} & 0 & 9 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}$,*

$P^{-1}BP = \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right]$ and $B_1 = \begin{bmatrix} 13 & 3 & 0 \\ -1 & 9 & 0 \\ 0 & 0 & 19 \end{bmatrix}$. Here $P = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. We have $\sigma(A) = \{10, 8, 1, 0\}$, $\sigma(B) = \{20, 19, 12, 10\}$ and $\sigma(B_1) = \{19, 12, 10\}$ so that $\rho(B) = 20$, $\rho(B_1) = 19$ and $\rho(B) - \rho(B_1) = 20 - 19 =$

$1 = \lambda_2$. Let $w = (0,0,1)^t$ be an eigenvector for the eigenvalue 19 *of* B_1. *Set* $w^0 := (0,0,1,0)^t$ *and* $v = Pw^0$. *Then* $v = (0,4,0,0)^t$. We note that $Bv = (0,76,0,0)^t = 19v$, *i.e.,* $v \in E_{\rho(B)-\lambda_2}(B) = E_{19}(B)$.

Now consider the splitting, $A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{2} & 0 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U - V$. *Then* $U^\# = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

and $U^\# V = \frac{1}{8}\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Thus* $\sigma(U^\# V) = \{0, 0, \frac{1}{8}, \frac{1}{2}\}$, *so that* $\lambda = \rho(U^\# V) = \frac{1}{2}$, *is the dominant eigenvalue of* $U^\# V$. *We have* $(U^\# V)(0,4,0,0)^t == \rho(U^\# V)(0,4,0,0)^t$. *Hence* $v \in E_\lambda(U^\# V) \cap E_{\rho(B)-\lambda_2}(B)$. *That is, the above splitting is a pseudo overlapping splitting of* A. *Further,* $\eta = \frac{\rho(B)-\rho(B_1)}{1-\lambda} = \frac{20-19}{1-\frac{1}{2}} = 2$ *is an eigenvalue of* $U = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $Re(\eta) = 2 > \frac{\rho(B)-\rho(B_1)}{2} = \frac{1}{2}$. *So, the given splitting is convergent.*

In [7], Elhashah and Szyld proposed a generalization of a regular splitting, *viz,* a G-regular splitting and obtained sufficient conditions for the convergence of the G-regular splitting of a nonsingular GM-matrix. Let us recall that result.

Definition 4.5. *A splitting* $A = U - V$ *of a nonsingular GM-matrix* $A = sI - B$ *is called a G-regular splitting, if* U^{-1} *and* V *are in WPFn and* V *is not nilpotent.*

Analogous to the above, we introduce a splitting of a $GM_\#$-matrix.

Definition 4.6. *Let* A *be a* $GM_\#$*- matrix. A proper splitting* $A = U - V$ *of* A *is called a* pseudo $G_\#$-regular *splitting if* $U^\#$ *and* V *are in WPFn and* V *is not nilpotent.*

In order to prove the concluding result on pseudo $G_\#$-regular splittings, let us consider three types of conditions given below: Let $A = U - V$ be a splitting of A.

Type I conditions:

(D1) $U^\#$ and V are eventually positive.
(D2) $U^\#$ and V are eventually nonnegative with V non-nilpotent.
(D3) Each of $U^\#$ and V has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.
(D4) Let $X = U^\#$ or V. Then, X satisfies the following: X has two classes α and β, not necessarily distinct, such that

(a) α is a basic and initial class such that $X[\alpha]$ is non-nilpotent and has the Perron-Frobenius property.
(b) β is a basic and final class such that $(X[\beta])^t$ is non-nilpotent and has the Perron-Frobenius property.

Type II conditions:

(E1) $U^\# V$ is eventually positive.
(E2) $U^\# V$ is eventually nonnegative and non-nilpotent.
(E3) $U^\# V$ is non-nilpotent and is in $WPFn$.
(E4) $U^\# V$ has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.
(E5) $U^\# V$ has a basic and initial class α such that $(U^\# V)[\alpha]$ is non-nilpotent and has the Perron-Frobenius property.

Type III conditions

(F1) $A^{\#}V$ is eventually positive.
(F2) $A^{\#}V$ is eventually nonnegative and non-nilpotent.
(F3) $A^{\#}V$ is non-nilpotent and is in $WPFn$.
(F4) $A^{\#}V$ has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.
(F5) $A^{\#}V$ has a basic and initial class α such that $(U^{\#}V)[\alpha]$ is non-nilpotent and has the Perron-Frobenius property.

Theorem 4.5. *Let A be a $GM_{\#}$-matrix.*
(a) Let $A = U - V$ be a proper splitting of A. Then any one of the Type **I** *conditions is sufficient for the splitting to be a pseudo $G_{\#}$-regular splitting.*
(b) Let $A = U - V$ be a pseudo $G_{\#}$-regular splitting. If the splitting satisfies any one of the Type **II** *conditions, then any one of the Type* **III** *conditions is sufficient for the convergence of the splitting.*

Proof. (a) Suppose that $A = U - V$ satisfies **(D1)**. Then, by Theorem 2.5, $U^{\#}$ and V both belong to $PFn \subseteq WPFn$. Since V is eventually positive, it is not nilpotent. So $A = U - V$ is a pseudo $G_{\#}$-regular splitting. For the remaining conditions we use the following implication diagram to outline the proof.

$$\begin{array}{ccccccc}
(\mathbf{D1}) & \Rightarrow & (\mathbf{D2}) & \Rightarrow & \begin{array}{c} U^{\#}, V \in WPFn \\ V \text{ is nonnilpotent} \end{array} & \Leftrightarrow & \begin{array}{c} A = U - V \text{ is a} \\ G_{\#} - \text{regular splitting} \end{array} \\
\Updownarrow & & & & \Uparrow & & \\
(\mathbf{D3}) & & & & (\mathbf{D4}) & &
\end{array}$$

From the definitions, it follows trivially that $(\mathbf{D1}) \Rightarrow (\mathbf{D2})$. Since A is a $GM_{\#}$-matrix, $A^{\#}$ exists and it belongs to $WPFn$. Since, $A = U - V$ is a proper splitting, $U^{\#}$ exists. Further, rank U^k = rank U^{k+1} for $k = 1, 2, \cdots$, so that U is not nilpotent, which in turn implies that $U^{\#}$ is not nilpotent (since $U^{\#}$ can be expressed as a polynomial in U). This, together with $(\mathbf{D2})$, imply that $U^{\#}$ and V are in $WPFn$ and V is not nilpotent, i.e., the splitting $A = U - V$ is pseudo $G_{\#}$-regular.

The equivalence $(\mathbf{D1}) \Leftrightarrow (\mathbf{D3})$ follows from Theorem 2.5 and Theorem 2.8. Finally, $(\mathbf{D4})(a)$ implies that $U^{\#}[\alpha]$ and $V[\alpha]$ have a right Perron-Frobenius vector. By Theorem 2.10, each of $U^{\#}$ and V has the Perron-Frobenius property. Again $(\mathbf{D4})(b)$ implies that $U^{\#}[\beta]^t$ and $V[\beta]^t$ have a right Perron-Frobenius vector. By Theorem 2.11, each of $U^{\#^t}$ and V^t has the Perron-Frobenius property and so $U^{\#}$ and V are in $WPFn$. Further, V is not nilpotent, since its basic class $V[\alpha]$ is non-nilpotent. Thus, **D4** implies that $A = U - V$ is a $G_{\#}$-regular splitting.

(b) We first assume that $(\mathbf{E1})$ and $(\mathbf{F1})$ are true. Since $U^{\#}V$ is eventually positive, $U^{\#}V$ has the strong Perron-Frobenius property. Therefore, the splitting is the one given by Lemma 4.2. Further, $A^{\#}V$ is eventually positive so that $A^{\#}V$ has the strong Perron-Frobenius property. Therefore, by Lemma 4.2, the splitting is convergent.

With regard to the remaining conditions, we use the following implication diagram to outline the proof. In the diagram below by PF property we mean the Perron-Frobenius property.

$$\begin{array}{ccccccc}
 & & & & & & (\mathbf{E5}) \\
 & & & & & & \Downarrow \\
(\mathbf{E1}) & \Rightarrow & (\mathbf{E2}) & \Rightarrow & (\mathbf{E3}) & \Rightarrow & U^{\#}V \text{ has PF property} \\
\Updownarrow & & & & & & \Downarrow \\
(\mathbf{E4}) & & & & & & \text{Splitting as given in Lemma } 4.2
\end{array}$$

$$\begin{array}{ccccccc}
 & & & & & & (\mathbf{F5}) \\
 & & & & & & \Downarrow \\
(\mathbf{F1}) & \Rightarrow & (\mathbf{F2}) & \Rightarrow & (\mathbf{F3}) & \Rightarrow & A^{\#}V \text{ has PF property} \\
\Updownarrow & & & & & & \Downarrow \\
(\mathbf{F4}) & & & & & & \text{Splitting is convergent}
\end{array}$$

The implications (**E1**) $\Rightarrow$ (**E2**) and (**F1**) $\Rightarrow$ (**F2**) follow from the definition of eventual positivity. The implications (**E2**) $\Rightarrow$ (**E3**) and (**F2**) $\Rightarrow$ (**F3**) follow from Theorem 2.6. The equivalences (**E1**) $\Leftrightarrow$ (**E4**) and (**F1**) $\Leftrightarrow$ (**F4**) follow from Theorem 2.5 and Theorem 2.8.

Finally, from Theorem 2.10, the implications (**E5**) implies that $U^{\#}V$ has the Perron-Frobenius property and (**F5**) implies that $A^{\#}V$ has the Perron-Frobenius property. The other implications in the above diagrams are obvious. □

Acknowledgement: The authors thank the referees for their suggestions and comments. This has resulted in a clearer presentation of the material.

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