Shock formation in the presence of entropy gradients in fluids exhibiting mixed nonlinearity

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The nonlinear steepening of a finite amplitude disturbance in a quiescent gas with very high specific heat in its near-critical regime is analyzed. The atypical phenomenon of rarefaction shocks are found to occur in the region where the nonlinearity parameter is negative. The undisturbed medium is assumed to be at rest with entropy gradients (temperature gradients). The steepening of the wave front in such a nonhomentropic medium, where a variation in the nonlinearity parameter is present, is investigated using the technique of wave front expansion. A calorically imperfect gas governed by an arbitrary equation of state is considered. An exact closed form solution is obtained for the evolution of the slope of the disturbance. In particular, results have been discussed for a van der Waal's fluid in its gaseous phase in the near-critical region. The distortion of both compression and rarefaction wave forms are examined and the corresponding shock formation distances are calculated. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1795272]

I. INTRODUCTION

The nonlinear distortion of a wave in a single phase gas was found to depend critically on the nonlinearity parameter Γ,[1] where

$$\Gamma = \frac{1}{a}\left[\frac{\partial(\rho a)}{\partial \rho}\right]_s \tag{1}$$

and

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s . \tag{2}$$

In the derivation of shock inequalities in gas dynamics, it is implicitly assumed that Γ is positive. It can be shown from thermodynamic arguments that only compression shocks can occur in such fluids.[2] In fact, for a polytropic gas, Γ can easily be reduced to $(\gamma+1)/2$, where $\gamma=c_p/c_v$. The sign of Γ being positive, forbids the existence of rarefaction shocks in such gases. However, Bethe,[3] and Zeldovich and Raizer[4] proposed the existence of gases for which Γ indeed, attains negative values. They found that for a van der Waal's gas with high specific heats ($C_v/R > 17$), such a transition happens in the near critical region. This results in an inverse behavior leading to the reversal of most of the shock inequalities. Contrary to the behavior when Γ is positive, it turns out that a compression shock is unstable in a gas in the region of negative nonlinearity ($\Gamma < 0$) and splits up immediately into a centered compression fan. In such cases, the only stable solutions for jump discontinuities of the flow parameters is that of the rarefaction shock. Thompson[5] was the first to recognize the importance of Γ and demonstrated the requirement of an antithroat to accelerate negative nonlinearity fluids to supersonic speeds. Thomson and Lambarkis[6] were the first to specify examples of hydrocarbons and fluorocarbons for which Γ attains negative values in the near critical region. Further, Cramer[7] listed out seven such hydrocarbons which are used for commercial purposes. The fluids for which $\Gamma < 0$ are also known as Bethe–Zeldovich–Thompson (BZT) fluids. BZT fluids are in general characterized by high molecular weights and specific heats.

A number of interesting phenomena which are not possible when Γ is strictly positive or strictly negative are found to occur in the vicinity of the region where Γ changes sign. A detailed study of these cases including their dissipative natures have been discussed by Cramer and co-workers in a sequel of papers.[8–11] The usage of BZT fluids in the region around $\Gamma=0$ significantly reduces the effects due to nonlinearity, thus delaying the formation of shock. As indicated by Cramer,[12] this reduction in nonlinearity can benefit processes such as acoustically enhanced solubility, diffusion, heat transfer, and agglomeration. Elimination of harmonic generation, nonlinear resonance, and chaos due to the reduction in nonlinearity may again find applications in gasdynamic springs and shock absorbers.[12] The suppression of shock induced separation of the laminar boundary layer on an adiabatic flat plate has been demonstrated by Cramer.[13] Further, Monaco *et al.*[14] have demonstrated the possibility of shock-free supersonic cascade flows. The disintegration of compression shocks in the negative nonlinearity region was found to decrease the adverse pressure gradients, which are otherwise responsible for boundary layer separation and shock irreversibilty. Brown and Argrow[15] have discussed the advantages of using BZT fluids in realistic turbine cascades

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of organic Rankine cycle engines where they are found to increase the overall efficiency. The above mentioned advantages are much ascribed to the natural dynamics of BZT fluids and do not demand excessive redesign of the turbine cascades exemplifying the potential advantages of the usage of BZT fluids.

Experimental confirmation of the existence of rarefaction shock in the single phase gas regime does not exist to the best of the authors' knowledge. However, Fergason *et al.*[16] have described a procedure which provides a starting point for the experimental verification of the nonclassical phenomenon. They have also given examples of cases where Γ is positive both upstream and downstream of a rarefaction shock. However, the above possibility is subject to the condition that the adiabat curve connecting the states must pass through the negative nonlinearity region. Further, they have simulated the evolution of a rarefaction shock wave by integrating the two-dimensional Euler equations for FC-70 (perfluro-tripentlyamine) using van der Waal's equation of state confirming their conceptual model. While Fergason *et al.*[16] begin with a rarefaction shock wave (RSW) and describe its dynamics, the present study concerns the formation of a RSW from a finite amplitude nonshock wave.

The propagation of a wave in a uniform homentropic medium have been investigated by several authors.[6,9,11] Kluwick and Cox[17] have analyzed the evolution of small amplitude waves in a medium having temperature stratification. They analyzed the local evolution of a wave in the medium using perturbation techniques. In the present study, however, the interest is on the global evolution of the wave, and moreover, there is no restriction on the amplitude of the wave. An exact, closed form solution is obtained for the steepening of the wave front of a finite amplitude wave traveling in a quiescent medium with density or temperature gradients. Such a case may often be encountered in a duct having BZT fluids with an axial temperature variation. The steepening of a disturbance in the medium is followed using the technique of wave front expansion (Whitham).[18] Lin and Szeri,[19] and Tyagi and Sujith[20] have investigated the effect of entropy gradients and area variation on the nonlinear distortion of a wave. They consider a finite amplitude wave having a first order discontinuity at the wave front and follow its motion. Their analysis, however, is valid only for the case of a polytropic gas. Since the assumption of ideal gas breaks down in the near-critical region, a more general analysis is required. Hence, in the present paper, the analysis is further generalized to describe the case of a gas with an arbitrary equation of state with varying specific heats, i.e., specific heat is a function of both temperature and pressure. The study of shock formation in an inhomogeneous medium, i.e., entropy or temperature gradients, for a gas exhibiting a region of negative nonlinearity in its near critical regime is the primary goal of this work.

The rest of the paper is organized as follows: In Sec. II, the method of wave front expansion is used to determine the time evolution of a wave front. A brief discussion of the results for a homentropic case is presented in Sec. III A. In Sec. III B, the steepening of the wave in a nonhomentropic environment is discussed. The effect of variation in Γ on the distortion of the wave and the corresponding shock formation distances are presented for some interesting cases.

II. THEORY

A. Governing equations

The governing equations describing the flow of an inviscid, nonconducting, isentropic gas are the following: Continuity,

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0; \tag{3}$$

momentum,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0; \tag{4}$$

energy,

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0. \tag{5}$$

Equations (3)–(5) form a hyperbolic system and with the equation of state, they completely describe the flow. The undisturbed medium is assumed to be at rest. To study the propagation and distortion of a wave in the medium, Eqs. (3)–(5) are manipulated and written along their characteristics in the (t,x) plane:[21]

$$\rho\frac{d^+u}{dt} + a\frac{d^+\rho}{dt} = \left(\frac{\partial p}{\partial s}\right)_\rho s_x \quad \text{on} \quad C_+: \frac{d^+x}{dt} = u + a, \tag{6}$$

$$\rho\frac{d^-u}{dt} - a\frac{d^-\rho}{dt} = \left(\frac{\partial p}{\partial s}\right)_\rho s_x \quad \text{on} \quad C_-: \frac{d^-x}{dt} = u - a, \tag{7}$$

$$\frac{dp}{dt} - a^2\frac{d\rho}{dt} = 0 \quad \text{on} \quad C_0: \frac{dx}{dt} = u. \tag{8}$$

In the above set of equations, d^+/dt, d^-/dt, and d/dt are derivatives taken along the C_+, C_-, and C_0 characteristics, respectively.

In the present problem, a finite amplitude wave with a compact support having a discontinuity in its first derivative at the wave front is considered. It can then be shown that the leading edge of the wave propagates along the characteristics C_+ and C_- with velocities a and $-a$, respectively.[18] Since the undisturbed medium is at rest, C_0 characteristics are absent at the wave front. The rate of steepening of the leading edges is followed using the technique of wave front expansion. A shock forms when the slope at the leading edge becomes infinity. The disadvantage of the present method is that it neglects the possibility of formation of shock in the middle of the wave. It, however, illustrates the effect of variation in Γ on the nonlinear distortion of the wave. Besides, the method yields a closed form solution which is of valued significance.

B. Wave front expansion

In the neighborhood of the wave front, a different coordinate system, $\xi = x - X(t)$, is defined, where $X(t)$ is the position of the wave front. Physically, ξ represents the distance measured from the wave front. Hence, (i) $\xi=0 \Rightarrow x=X(t)$, describes the motion of the wave front; (ii) $\xi>0$ is the region of the undisturbed quiescent medium into which the wave propagates; and (iii) $\xi<0$ is the region behind the wave front where the flow is unsteady. As mentioned in the preceding section, the motion of the leading edge of the right running wave is governed by the following equation:

$$\dot{X}(t) = a_0[X(t)], \tag{9}$$

where "·" indicates time derivative and the subscript "0" indicates the value of the flow variable in the undisturbed medium. Henceforth, the analysis is performed for the right running wave. The left running wave can be analyzed in a similar fashion. As the wave propagates, a flow variable λ for $\xi<0$ can be expanded in terms of its derivatives at the wave front as shown below.[18]

$$\lambda(\xi,t) = \lambda_0[X(t)] + \xi\lambda_1(t) + \frac{\xi^2}{2}\lambda_2(t) + \cdots, \tag{10}$$

where λ indicates p, ρ, or u, and λ_1, λ_2, ... denote the corresponding spatial derivatives behind the wave front. Since $\xi>0$ is an undisturbed region, $u_0[X(t)]=0$. Further, to clarify, since a discontinuity in the first and the higher derivatives are present at the wave front,

$$\lambda_1(t) \neq \lambda_0'[X(t)], \quad \lambda_2(t) \neq \lambda_0''[X(t)], \ \ldots, \tag{11}$$

where "′" indicates spatial derivatives. Here the left-hand side of the inequalities in Eq. (11) are derivatives behind the wave front (disturbed region) and right-hand side are derivatives in front of the wave front (undisturbed region). It is also known from thermodynamics that $a=a(p,\rho)$. Hence,

$$a(p,\rho) = a_0[X(t)] + \xi p_1\left(\frac{\partial a_0}{\partial p}\right)_\rho + \xi\rho_1\left(\frac{\partial a_0}{\partial \rho}\right)_p + \cdots. \tag{12}$$

The derivatives with respect to t can be obtained using

$$\left[\frac{\partial}{\partial t}\right]_x = \left[\frac{\partial}{\partial t}\right]_\xi + \left[\frac{\partial \xi}{\partial t}\right]_x\left[\frac{\partial}{\partial \xi}\right] = \frac{\partial}{\partial t} - a_0[X(t)]\frac{\partial}{\partial \xi}. \tag{13}$$

The flow terms in Eqs. (3)–(5) are expanded using Eqs. (10), (12), and (13). Comparing the coefficients of ξ^0 and ξ^1, one finds

ξ^0 terms,

$$(\rho_0' - \rho_1)\dot{X}(t) + \rho_0 u_1 = 0; \tag{14}$$

$$p_1 = \rho_0 u_1 \dot{X}(t); \tag{15}$$

$$(p_0' - p_1)\dot{X}(t) - (\rho_0' - \rho_1)\dot{X}(t)a_0^2 = 0; \tag{16}$$

ξ^1 terms,

$$[\dot{\rho}_1 - \rho_2\dot{X}(t)] + \rho_0 u_2 + 2\rho_1 u_1 = 0; \tag{17}$$

$$\rho_0 u_2 \dot{X}(t) + \rho_1 u_1 \dot{X}(t) - \rho_0 \dot{u}_1 - \rho_0 u_1^2 - p_2 = 0; \tag{18}$$

$$[\dot{p}_1 - p_2\dot{X}(t)] + p_1 u_1 = 2a_0\left(p_1\frac{\partial a}{\partial p} + \rho_1\frac{\partial a}{\partial \rho}\right)(\rho_0' - \rho_1)\dot{X}(t) + a_0^2[\dot{\rho}_1 - \rho_2\dot{X}(t) + \rho_1 u_1]. \tag{19}$$

The matrix formed by the coefficients of first derivatives in Eqs. (14)–(16) is singular. Hence, the first set of equations reduces to $p_0'=0$, which is as expected in a stationary undisturbed flow. Similarly, coefficients of the second derivatives in Eqs. (17)–(19) form a singular matrix. Hence, on elimination of the second derivatives, Eqs. (17)–(19) yields

$$\frac{du_1}{dt} + u_1^2\left[1 + \frac{\rho_0}{a_0}\left(\frac{\partial a}{\partial \rho}\right)_p + \rho_0 a_0\left(\frac{\partial a}{\partial p}\right)_\rho\right] + u_1\left[\frac{\ddot{X}(t)}{2\dot{X}(t)} + \frac{\rho_0'\dot{X}(t)}{2\rho_0} + \rho_0'\left(\frac{\partial a}{\partial \rho}\right)_p\right] = 0. \tag{20}$$

This is a nonlinear Riccati equation. A change of the thermodynamic state description variables from (ρ,p) to (ρ,s) is performed using

$$\left[\frac{\partial}{\partial \rho}\right]_p = \left[\frac{\partial}{\partial \rho}\right]_s - a^2\left[\frac{\partial}{\partial p}\right]_\rho. \tag{21}$$

The coefficient of u_1^2 in Eq. (20), then precisely reduces to Γ in its classical form [Eq. (1)]. The last term in the coefficient of u_1 reduces to $\partial a/\partial x$ as the pressure is a constant in the undisturbed medium. To trace the evolution of the wave front as function of its propagation distance, a change of variable, t to $y=X(t)$ is performed.[20] As a result, Eq. (20) reduces to

$$\frac{du_1}{dy} + \frac{\Gamma_0(y)}{a_0(y)}u_1^2 + u_1\left[\frac{3a_0'(y)}{2a_0(y)} + \frac{\rho_0'(y)}{2\rho_0(y)}\right] = 0, \tag{22}$$

which can be further reduced to its linear form:

$$\frac{d}{dy}\left(\frac{1}{u_1}\right) - \left[\frac{3a_0'(y)}{2a_0(y)} + \frac{\rho_0'(y)}{2\rho_0(y)}\right]\frac{1}{u_1} = \frac{\Gamma_0(y)}{a_0(y)}. \tag{23}$$

The solution to this equation with an initial slope of $u_1(0)$ at the leading edge can be written as

$$\frac{1}{u_1(y)} = \frac{IF(0)}{IF(y)u_1(0)} + \frac{1}{IF(y)}\int_0^y \frac{IF(\hat{y})\Gamma_0(\hat{y})}{a_0(\hat{y})}d\hat{y}, \tag{24}$$

where

$$IF(y) = a_0(y)^{-3/2}\rho_0(y)^{-1/2}.$$

The above expression describes the evolution of the leading edge of a traveling wave into a stationary gas governed by an arbitrary equation of state in the presence of entropy gradients. A shock forms when $u_1(y) \to \infty$. If y_s indicates the shock formation distance, then the shock formation time can be obtained from

$$t_s = \int_0^{y_s} \frac{dy}{a_0(y)}.$$

Thus, both the time and the location of shock formation are obtained. The above derivation is applicable for the case of constant area duct. An area variation term for a nonuniform duct can be easily included as shown in Appendix A.

III. EXAMPLES

A. Homentropic environment

An infinite homentropic medium is considered first to illustrate the effect of Γ on the distortion of a wave. For a uniform medium ($\Gamma = \Gamma_0$=constant), Eq. (24) reduces to

$$u_1(t) = \frac{u_1(0)}{1 + \Gamma_0 u_1(0) t}. \quad (25)$$

For the formation of shock, u_1 must become singular in finite time. If $\Gamma_0 > 0$, a compression wave steepens into a shock and rarefaction wave relaxes. On the other hand, if $\Gamma_0 < 0$, compression wave relaxes and rarefaction wave steepens into a shock. Hence, the sign of Γ is found to dictate the nature of evolution of a wave. Further, the rate of steepening of the leading edge is obtained by differentiating Eq. (25):

$$\dot{u}_1(t) = -\Gamma_0 u_1^2(t). \quad (26)$$

It is seen from the above expression that $|\Gamma|$ decides the extent of nonlinear distortion. In particular, it is to be noted that when the quiescent medium is in vicinity of $\Gamma = 0$, there is very little steepening at the wave front. However, in these cases, it is interesting to note that the local values of Γ within the waveform varies between positive and negative values leading to highly nonclassical behavior. The order of complicacy involved in solving the problem for shock formation further increases. Although the steepening of the leading edge of the wave is still exactly given by Eq. (25), the assumption of shock formation, first at the leading edge, may not hold true anymore. The local evolution of the wave cannot be neglected in these cases.

If a shock forms, the time of shock formation and the distance traveled by the wave before it turns into a shock are given by

$$\bar{t} = -\frac{1}{u_1(0)\Gamma_0}, \quad \bar{x} = -\frac{a_0}{u_1(0)\Gamma_0}. \quad (27)$$

It is clear from the above expression that the present problem does not have any natural length or time scales. However, before proceeding to the discussion of nonhomentropic environment, Eqs. (3)–(5) are nondimensionalized with a reference length L_0, velocity a_0, and density ρ_0. Hence,

$$a^* = a/a_0, \quad u_1^* = u_1 L_0/a_0, \quad x^* = x/L_0,$$

$$t^* = t a_0/L_0, \quad \rho^* = \rho/\rho_0, \quad p^* = p/(\rho_0 a_0^2).$$

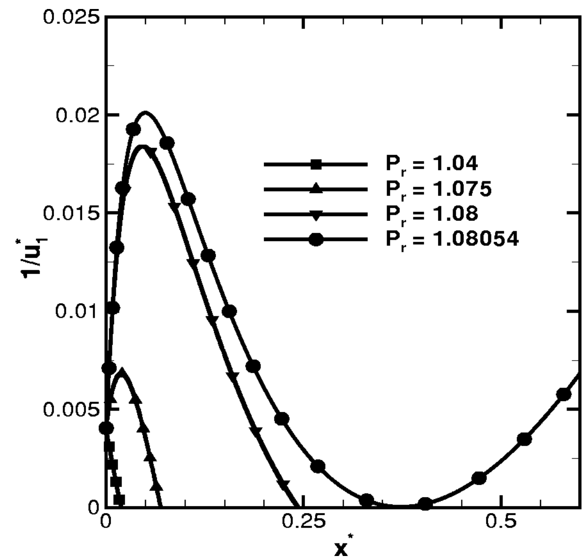

FIG. 1. [van der Waals gas (FC-70).] Evolution of the leading edge of rarefaction waves in a medium having a linearly increasing temperature variation of $1.02T_c$ to $1.1T_c$ [i.e., $T(x^*=0) = 1.02T_c$ and $T(x^*=1) = 1.1T_c$] at different pressures are plotted. A shock forms when $1/u_1^* \to 0$.

B. Nonhomentropic environment

In the present section, the formation of shock in a nonhomentropic environment is investigated. Since the pressure is a constant in the quiescent medium, variation of any one of the properties, say temperature or density, completely describes the quiescent medium. The derivation in Sec. II B, though valid for a general equation of state, the gas model in the present paper is taken to be of that defined by van der Waal's equation of state. Further, a power law variation of specific heat[6] is assumed. The details of the thermodynamic model and the symbols used are given in Appendix B. The results shown henceforth are for FC-70. The critical values for FC-70 are $p_c = 10.2$ atm, $T_c = 608.2$ K, $C_{v\infty}/R = 118.7$, and $n = 0.4930$ obtained from Cramer.[7] Further, Eqs. (23) and (24) are nondimensionalized with $a_0 = 10$ m/s, $\rho_0 = \rho_c$, and $L_0 = 10$ m.

It is seen from Eq. (24) that steepening of a wave depends explicitly on the variation of sound speed, density, and Γ of the undisturbed medium. Discussion of a few examples will show that the effect of variation in Γ dominates over the effects due to variation of other thermodynamic properties on the distortion of the wave.

1. Rarefaction wave

In general, a rarefaction wave steepens in a region where $\Gamma < 0$ and relaxes in a region where $\Gamma > 0$. The variation of the properties is chosen such that both behaviors can be well illustrated. For the present illustration, a linearly increasing temperature variation of $1.02T_c$ to $1.1T_c$ is chosen across a length of 10 m.

A right running rarefaction wave ($u_1^* > 0$) originating at $x^* = 0$ will evolve in an increasing temperature field. Figure 1 illustrates the evolution of leading edge of rarefaction waves at different pressures for the above mentioned variation in temperature. At $P_r = 1.04P_c$, the wave steepens throughout its

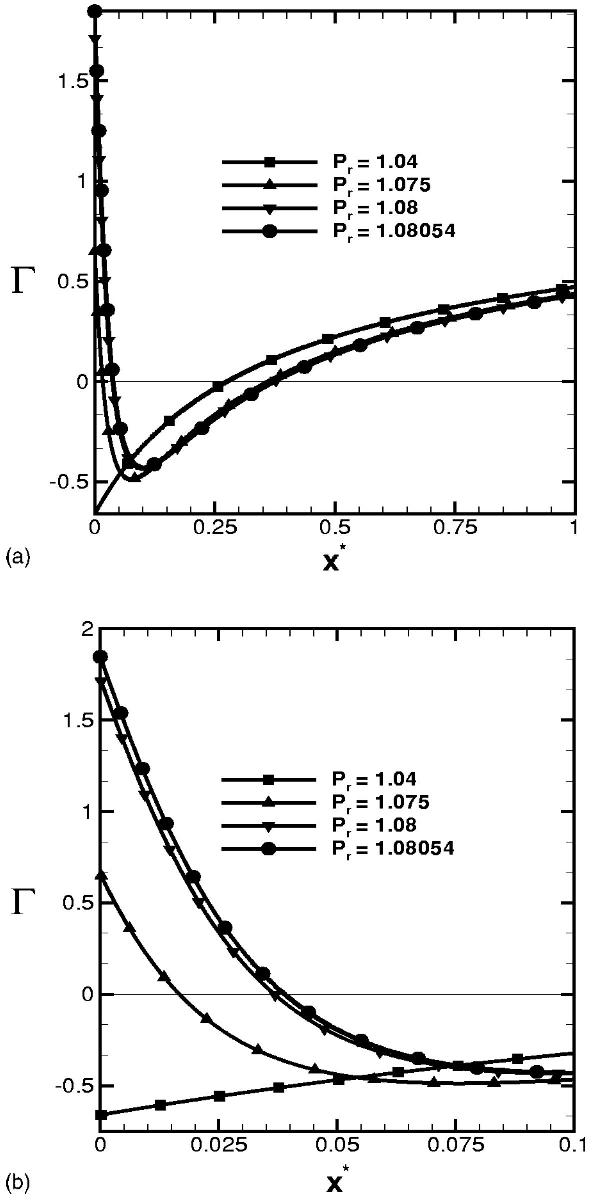

FIG. 2. (a) [van der Waals gas (FC-70).] Γ vs X^* is plotted for the same variation of temperature and pressures as in Fig. 1. At higher pressures and lower temperatures, $\Gamma>0$. At high pressures, the region of negative Γ is sandwiched between the regions of positive Γ. Also, drastic changes in the value of Γ is observed for lower temperatures and higher pressures. (b) The values of Γ from $x^*=0$ to $x^*=0.1$ is scaled up and plotted again for clarity.

period of evolution before turning into a shock. As the pressure increases, one finds a region where the wave relaxes before the steepening begins and then eventually turns into a shock. The relaxation of the wave can be correlated with the fact that Γ is positive [see Fig. 2(b)] in the region. As the pressure is increased furthermore, the wave form just misses to form a shock and is seen as a tangent to the x axis. At the location of tangency, both $1/u_1^*$ and $d(1/u_1^*)$ vanish. It is seen from Eq. (24) that this is possible only when $\Gamma=0$ at the location of tangency. This also shows that Γ must be negative at the location of formation of a rarefaction shock for the wave form of the nature considered having a discontinuity in its first derivative at the wave front, to steepen into a shock. However, the location of minimum u_1^*, is slightly offset from the location where Γ turns negative. In such a narrow region, interestingly, a rarefaction wave continues to relax even when Γ is negative. This is due to the effect of variation in sound speed and density, affecting the first term of Eq. (24) explicitly. Except in this narrow region, the behavior of the wave, i.e., whether it steepens or relaxes, is much decided by the sign of Γ and other properties indeed have a minor influence.

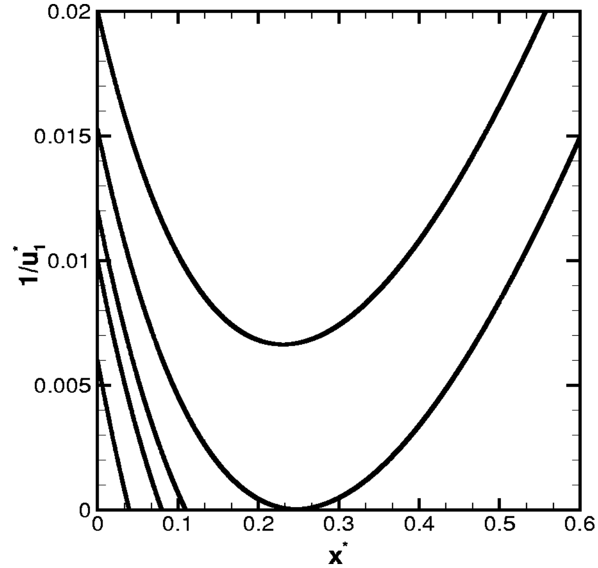

FIG. 3. [van der Waals gas (FC-70).] Steepening of the leading edges of rarefaction waves having different initial strength $[1/u_1^*(0)]$ is plotted for the linear variation of temperature as in Fig. 1. The pressure of the quiescent medium is 1.03 P_c. As u_1^* increases, the shock formation distance decreases.

Figure 3 illustrates the evolution of wave with different $u_1^*(0)$. As the strength of the wave decreases, the shock formation distance increases. A wave with a critical strength just misses to form a shock and is seen as tangent to the x axis. Thus the initial strength of the wave must be above a certain value for it to develop into a shock within the region of interest.

Figure 4 depicts the variation in shock formation distances with changes in the value of the initial slope $[u_1^*(0)]$ at various pressures. The effect of pressure change is seen more clearly with an initial decrease and then a sudden increase in the shock formation distances at higher pressures. This is due to the rapid variation of Γ in the corresponding region [see Fig. 2(b)] due to small changes in pressure. The very high sensitivity of Γ to the changes in pressure for the region discussed is clearly depicted in Fig. 5. It is seen that drastic changes in the shock formation distances can be much attributed to the initial region of evolution, where the temperature is around $1.02T_c$, which is characterized by very high values of $(\partial\Gamma/\partial p)_T$. Since the value of Γ has a direct bearing on the shock formation distances, the region discussed here is highly sensitive to variation in pressures.

The calculations performed for the steepening of left running waves did not show any qualitative differences from that of right running waves. The sign and the magnitude of Γ was again found to be the dominating parameter in deciding the nature and the extent of nonlinear distortion.

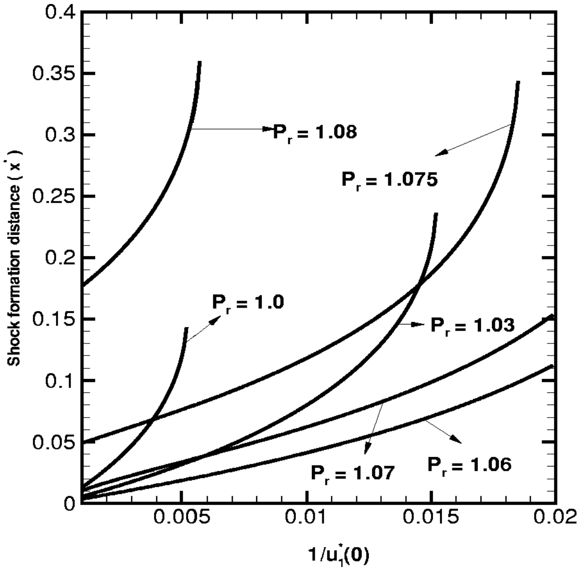

FIG. 4. [van der Waals gas (FC-70).] Shock formation distance of right running rarefaction waves at different pressures for the same linear temperature variation as in Fig. 1 is plotted against the inverse of the initial strength of the wave $[1/u_1^*(0)]$. The disturbances originate at $x^*=0$ and travel into the medium. A sudden increase in the shock formation distances at higher pressures is observed.

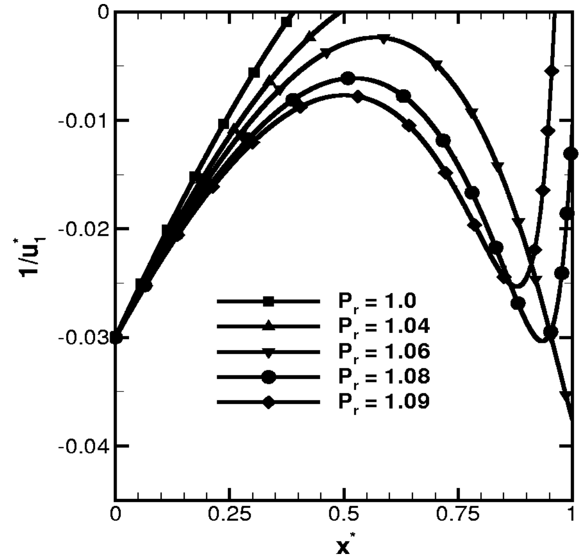

FIG. 6. [van der Waals gas (FC-70).] Steepening of the leading edges of right running compression waves is plotted for different pressures. The initial strength of the waves are −0.03. The quiescent medium has a linearly decreasing temperature field of $1.08T_c$ to $1.02T_c$.

2. Compression wave

The behavior of the compression waves is based on the same principles as that of the rarefaction wave except for a change in sign. Hence, only two cases have been investigated below. A right running compression wave ($u_1^*<0$) is chosen to evolve in a linearly decreasing temperature field with temperature variation of $1.08T_c$ to $1.02T_c$ within the length of 10 m.

The evolution of the leading edge of the wave at different pressures is shown in Fig. 6. As the pressure is increased, the shock formation distance increases and above a critical pressure the waves start relaxing in the region of lower temperatures, consequently not developing into a shock. This happens because of the increase in the spread of the region in which Γ is negative with the increase in the pressure. However, as the pressure is increased further more, since a region of positive Γ develops at lower temperatures, the relaxing waves steepen again to form a shock. This is seen as a sudden increase in the shock formation distances at higher pressures as depicted in Fig. 7.

The above examples examine the results in a quiescent flow with a linear variation in temperature for some interesting cases. A medium having any smooth variation of temperature or density can be analyzed in a similar fashion, sub-

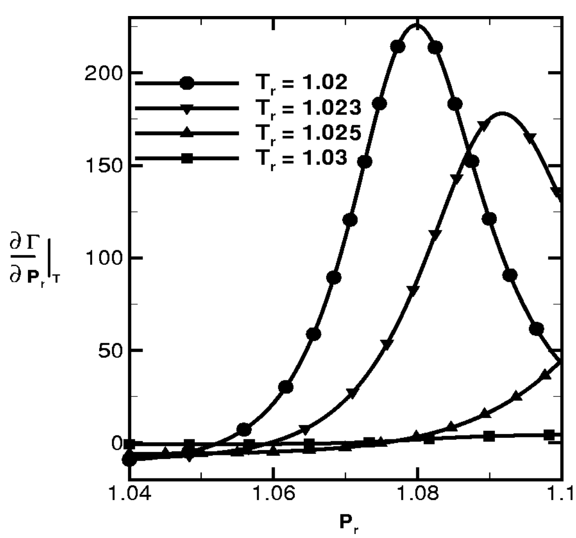

FIG. 5. Variation of $(\partial\Gamma/\partial pr)_T$ is plotted for the same variation of temperature and pressures as in Fig. 1. The sensitivity of Γ for changes in pressure at low temperatures and high pressures is depicted.

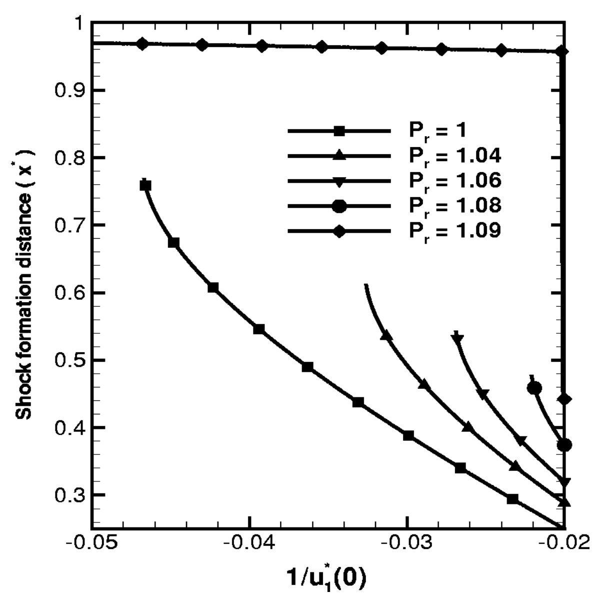

FIG. 7. [van der Waals gas (FC-70).] Shock formation distances of right running compression waves are plotted against the inverse of its initial strength $[1/u_1^*(0)]$ for different pressures. The temperature variation is same as that in Fig. 6. The waves originate at $x^*=0$.

ject to the only restriction that the range of temperatures in the medium should not enter the liquid or the two phase regime.

IV. SUMMARY

The effect of variation of Γ on the nonlinear steepening of the wave front of a finite amplitude wave in a quiescent medium in the presence of entropy gradients has been analyzed. A closed form solution is obtained for the steepening of the leading edge of wave using the method of wave front expansion. The analysis has been carried out for a single phase fluid whose behavior is defined by a general equation of state with varying specific heats. In particular, the results have been discussed for a van der Waal's gas with a power-law variation in specific heats in the near-critical region. The effect of variation in pressure and temperature, resulting in the variation of Γ, on the evolution of both compression and rarefaction wave forms have been investigated. Unlike in the classical case of positive Γ, rarefaction shocks emerge out as a natural consequence of the phenomenon of negative nonlinearity ($\Gamma<0$). Except in a very narrow region, the sign of Γ was found to be the dominating factor in deciding whether a wave steepens or relaxes. In this narrow region, a rarefaction wave continues to relax even if $\Gamma<0$. This is due to the effect of variation in sound speed and density on the distortion of the wave. Further, certain regions were found to be highly sensitive to changes in pressure. Such regions are in turn characterized by rapid variation of Γ, resulting in a sudden change in the shock formation distances. It was also found that for a medium having a given variation in temperature, and a given pressure, the strength of a wave (both compression, and rarefaction wave) must be above a critical value for it to develop into a shock within the region of interest.

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APPENDIX A: VARIABLE AREA DUCTS

The above analysis can easily be extended to a nonuniform duct.[20] The continuity equation has an added term $[(\rho u/A)(dA/dx)]$, where $A=A(x)$, is the variation in area. In writing the governing equations for such a system, the flow is still assumed one dimensional, with a planar wave front. Such an assumption will be approximately true as long as the variation in the area is gradual. Then, following similar steps of derivation as done before in Sec. II B, one finds that the only change required to be made is in Eq. (24),

$$IF(y)=a_0(y)^{-3/2}\rho_0(y)^{-1/2}A_0(y)^{-1/2}.$$

APPENDIX B: THERMODYNAMIC MODEL

1. van der Waal's equation of state

$$P_r=\frac{8T_r}{3\nu_r-1}-\frac{3}{\nu_r^2}. \tag{B1}$$

Z_c, the critical compressibility, is assumed to be a universal constant with a value $3/8$ for a van der Waal's gas. In Eq. (B1), $P_r=P/P_c, T_r=T/T_c$, and $\nu_r=\nu/\nu_c$ are the reduced pressure, temperature, and specific volume, respectively. In the near critical region, variation in specific heat is found to approximately obey power law.[6]

2. Power law of specific heats

$$c_{\nu\infty}(T_r)=c_{\nu\infty}(1)T_r^n, \tag{B2}$$

where $c_{\nu\infty}(T_r)$ is the zero pressure specific heat at a given temperature, and n is the power law exponent. The specific heat of van der Waal's gas is independent of the pressure and is a function of temperature only.[6] Further, to compute Γ, it is found useful to write Γ in the following form:

$$\Gamma=-\frac{\nu_r}{2}\frac{\left(\frac{\partial^2 p_r}{\partial \nu_r^2}\right)_s}{\left(\frac{\partial p_r}{\partial \nu_r}\right)_s}. \tag{B3}$$

With the knowledge of the state equation in the from $p=p(\nu_r,T_r)$, Γ and a can be computed easily using

$$\left(\frac{\partial^2 P_r}{\partial \nu_r^2}\right)_s=\left(\frac{\partial^2 P_r}{\partial \nu_r^2}\right)_T-3Z_c\epsilon T_r\left(\frac{\partial P_r}{\partial T_r}\right)_\nu\frac{\partial^2 P_r}{\partial \nu_r\,\partial T_r}+3Z_c^2\epsilon^2T_r^2\left(\frac{\partial P_r}{\partial T_r}\right)_\nu^2\left(\frac{\partial^2 P_r}{\partial T_r^2}\right)_\nu+Z_c^2\epsilon^2T_r\left(\frac{\partial P_r}{\partial T_r}\right)_\nu^3\left[1-\frac{T_r}{c_\nu}\left(\frac{\partial c_\nu}{\partial T_r}\right)_\nu\right] \tag{B4}$$

and

$$\left(\frac{\partial P_r}{\partial \nu_r}\right)_s=\left(\frac{\partial P_r}{\partial \nu_r}\right)_T-Z_c\epsilon T_r\left(\frac{\partial P_r}{\partial T_r}\right)_\nu^2, \tag{B5}$$

where $\epsilon=R/c_{\nu\infty}(T_c)$.

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