Sample-Based Algorithm to Determine Minimum Robust Cost Path with Correlated Link Travel Times

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Travel time reliability is an important and desirable property in route and departure time choice, especially for a risk-averse traveler. Thus, optimizing for reliability has seen growing interest in the recent past in transportation and also in the fields of computer science, stochastic optimization, and so forth. The present study addressed reliability optimization under uncertainty, in which travel time distributions were represented with a sample. The weighted mean–standard deviation measure (robust cost) was adopted as a metric of reliability. The minimum robust cost path problem with link travel times following a general correlation structure was addressed. A sampling-based approach, which had been relatively unused, was adopted from the literature to capture and represent spatial correlations. A novel network transformation and pruning procedure was proposed to determine an exact solution to the problem while circumventing the high dimensionality of the formulations in the literature. Computation experiments demonstrated the efficacy of the algorithm on real-world networks. The impact of the sample approximation on finding the true optimal solution of the population was quantified and found to be acceptable.

Uncertainty in transportation networks can be attributed to supply- and demand-side characteristics. Supply-side factors include capacity variability, signal controls, incidents, accidents, and weather; demand-side factors include traveler behavior, spatial and temporal demand variation, and special events. The stochastic interactions lead to highly variable system performance and high travel times, especially in peak periods. To mitigate such impacts, there is growing interest in optimizing performance in an uncertain environment. In that regard, this paper addresses the minimum robust cost path (MRCP) problem with stochastic link travel times having arbitrary distributions and general correlation structures.

In this study, the robust cost of a path, a measure of reliability, is defined as a weighted average of the mean and standard deviation of travel times on the path. This measure is chosen because the travel times are not restricted to prespecified distributions (e.g., normal and lognormal). Further, the relative weight for standard deviation allows a straightforward interpretation of risk aversion to unreliable travel times. Another advantage of this measure is that it circumvents the need to determine path travel time distributions with high dimensional multivariate integrals.

The motivation behind this study is threefold. First, it is widely accepted that users of transportation networks place a high value on reliability when making their travel decisions (*1, 2*). Second, the studies that optimize reliability measures under uncertainty make restrictive assumptions of link travel time independence or specific correlation structures. Third, the MRCP problem is difficult to solve for reasons listed below and is intellectually challenging. Empirical evidence suggests that travel times are significantly correlated across links and should not be disregarded (*3*). Unfortunately, the quantification of travel time distribution and the estimation of general correlation matrices with empirical data are nontrivial because of the difficulty in estimating valid correlation structures with empirical data and the intractability of estimating path travel time distributions analytically. In addition, the presence of correlations makes the MRCP problem difficult to solve as several desirable properties, such as linearity, link-separability, subpath optimality, and subpath nondominance (in conventional shortest path problems), do not hold.

As a result of these data issues and algorithmic challenges, practical applications for optimizing reliability-based objectives are not yet sufficiently well developed for realistic network sizes. In particular, the MRCP problem has not received adequate attention in the literature. Important exceptions include Seshadri and Srinivasan (*4*), who propose an exact solution for special correlation cases, and Xing and Zhou (*5*), who address the general correlation case but with a heuristic solution procedure. This study attempts to build on the advantages of both studies and proposes a new exact solution procedure for the general correlation case.

Specifically, this study addresses three objectives: (*a*) propose a new subpath elimination criterion for the MRCP problem, (*b*) develop a new and efficient algorithm to solve the MRCP problem based on the subpath elimination criterion, and (*c*) investigate the computation performance of the proposed algorithm on real-world networks.

To handle correlations, a sampling-based approach with implicit correlation representation based on the literature is adopted as it circumvents correlation quantification issues and leads to increased computational tractability (as explained in a later section) (*5, 6*). However, the solution approach in this study differs from and contributes to the literature in the following respects: A new norm-based subpath elimination criterion is developed for the robust cost objective to identify and eliminate subpaths that cannot be part of the optimal. On the basis of that criterion, a pruning algorithm based on label-correcting procedures is developed. The proposed approach leads to significant enhancements in algorithm performance by applying network transformation techniques (Dial's efficiency, Johnson's

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transformation, and localization) and integrating them with the sample-based quantification technique. Consequently, the new solution procedure yields a substantial dimensionality reduction from an m objective (m being the number of links) to a two-objective problem, and an exact solution is guaranteed. In addition, the quality of the solution from the sample approximation formulation is investigated in contrast to previous studies.

The rest of the paper is organized as follows. A brief review of the related literature is presented, and gaps in relation to the existing study are identified. Problem definition and formulation are then discussed, followed by the solution procedure, which is presented along with implementation details. Computation experiments and results are discussed next, and conclusions end the paper.

LITERATURE REVIEW

Travel time reliability has received considerable emphasis in the recent literature. Quantification of travel time variability at the network level involves addressing various challenges including limited sample size, spare spatial data availability, and separation of sources of variability. [See Boyles for a more detailed discussion (*7*).]

In the literature, stochastic networks have been adopted to appropriately model uncertainty for network optimization. Various objective functions were chosen for optimization including the least expected travel time (LET), mean–variance measures, and probability measures of reliability. The LET path problem under en route guidance was addressed by Polychronopoulos and Tsitsiklis (*8*), Cheung (*9*), and Waller and Ziliaskopoulos (*10*); dynamic programming–based algorithms with exponential worst-case complexity were implemented. The LET path problem in stochastic and time dependent networks was addressed by Hall (*11*), Miller-Hooks and Mahmassani (*12*), and Pretolani (*13*) with the use of a stochastic nondominance criterion along with label-correcting procedures.

Although the LET path problem models decision making under uncertainty, the LET path problem fails to take into consideration the users' risk-averse behavior. Evidence from multiple studies points to the fact that users value reliability considerably (*1, 2, 14*). To model this behavior, Sivakumar and Batta introduced a variance constraint into the standard LET problem (*15*). The resulting integer programming problem assuming link independence was solved with a branch and bound procedure. Sen et al. adopted a similar formulation and proposed a solution procedure applicable for the correlation structures with a cycle covariance property (variance of a path decreases after removal of a cycle), but the performance on large networks was not tested (*16*). Other studies have adopted multiattribute utility functions as objective functions including early and late schedule delays and variability as attributes (*17, 18*). Nikolova et al. showed that this class of problems is NP-hard even for simple quadratic penalty and disutility functions without link correlations (*17*).

Probabilistic measures (on-time arrival reliability) have also been considered as an objective function in other studies (*19–23*). The need to quantify path travel time distributions and correlations restricts these approaches to a few common distributions. See Seshadri and Srinivasan for a more detailed review of studies that optimize probabilistic reliability metrics (*21*).

To avoid some of the quantification issues noted above, a linear function of the mean and a measure of variability have also been used as a metric of reliability (as in the present study) in the literature. The MRCP problem without link correlations can be solved by determining the nondominated path set in the objectives mean and variance by exploiting the subpath nondominance property, which was adopted in Chen et al. (*24, 25*). But the significance of correlations and their effect on optimality have been demonstrated by Seshadri and Srinivasan (*4, 21*). Seshadri and Srinivasan proposed an algorithm and a heuristic for the mean–variance trade-off problem for the case in which Cholesky coefficients of the link covariance matrix are positive, which is too restrictive and seldom holds in transportation networks (*4*). The problem was reformulated as a multiple-objective shortest path problem in m dimensions (where m is the number of links) and solved by determining the nondominated set. Xing and Zhou proposed two approximation methods to solve the MRCP problem with and without link correlations (*5*). A sample approximation approach was used to model correlations, and a Lagrangian substitution-based lower bound heuristic was proposed for optimization. An average error of more than 5.4% and suboptimality in 25.5% of cases were reported in the study.

In summary, compared with the deterministic path optimization models, fewer studies aim to optimize reliability-related objectives. Of the few studies that focus on such stochastic optimization, link travel time is often assumed to be independent because of data and optimization difficulties, which is restrictive and unrealistic (*24*). The issues of distribution quantification, correlation estimation, and reliability quantification with empirical data impose statistical and computation challenges. Unfortunately, the restrictive assumption of link independence or limited correlations can lead to suboptimal solutions (*21*). As a result of these difficulties, practical applications (for reliability-based optimization) on real networks based on empirical data remain elusive. This study tries to address some of those gaps.

PROBLEM DEFINITION AND FORMULATION

Context and Scope

The transportation network is represented as a directed graph $G(N, A)$, where $N = \{1, 2, 3, \ldots, n\}$ represents the set of nodes and A ($|A| = m$) represents the set of m directed arcs and links. The link travel times are assumed to be random variables that follow a multivariate distribution, $\mathbf{t} \sim f(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ is the mean travel time vector and $\boldsymbol{\Sigma}$ is the link travel time covariance matrix.

The present study addresses a static (time invariant) context, and the minimum robust cost path is considered to be an a priori path whereby recourse decisions are not permitted.

Minimum Robust Cost Path

The mean and variance of the travel time of path P are represented by μ_P and σ_P and are given by

$$\mu_P = \sum_{i \in P} \mu_i \tag{1a}$$

$$\sigma_P^2 = \sum_{i \in P} \sigma_i^2 + \sum_{\forall j \neq i \in P} \rho_{ij} \sigma_i \sigma_j \tag{1b}$$

where μ_i and σ_i are the mean and standard deviation, respectively, of the travel time on link i, $\forall i \in A$, and ρ_{ij} represents the correlation coefficient between the travel times of links $i, j \in A$. The robust cost of path P, RC_P, given a weight δ, is defined as a linear combination of the mean and standard deviation of the path travel time.

$$RC_P = \delta \mu_P + (1 - \delta) \sigma_P \tag{2}$$

The parameter δ, which is between 0 and 1, represents the degree of risk aversion of the user. A low value of δ represents a highly risk averse user, while a high value represents a less risk averse user. The objective of the minimum robust cost path problem is to determine the path P^* such that its robust cost RC_{P^*} is lower than the robust cost RC_P of any other path P from origin r to destination s. The mathematical formulation is given by the following:

MRCP:

$$\min RC_p(\boldsymbol{x}) = \delta\left(\sum_{(u,v)\in A} \mu_{uv} x_{uv}\right) + (1-\delta)\left(\sum_{(u,v)\in A} \sigma_{uv}^2 + \sum_{\forall (u,v)\neq(u',v')\in A} \rho_{uv-u'v'}\sigma_{uv}\sigma_{u'v'}x_{uv}x_{u'v'}\right)^{0.5} \qquad (3a)$$

$$\sum_{v:(u,v)\in A} x_{uv} - \sum_{v:(u,v)\in A} x_{vu} \begin{cases} 1 & \text{if } u = r \\ -1 & \text{if } u = s \\ 0 & \text{if } u \in N - \{r, s\} \end{cases} \qquad (3b)$$

where the decision variable $x_{uv} = 1$ if arc (u, v) belongs to optimal path P^* and 0 otherwise. The flow conservation constraints (Equation 3b) ensure that the series of links carrying nonzero flows on the optimal solution constitutes a directed path from the source to the sink.

As noted earlier, subpath optimality and subpath nondominance properties do not hold for the MRCP problem. As an illustration, consider the network shown in Figure 1, in which the mean and variances of the link travel times are given on the arcs. Let the correlations between the arcs (1, 3) and (3, 4) be −0.5 and between arcs (2, 3) and (3, 4) be 0.5. Let the weight (δ) be equal to 0.2. For the path 1-3-4 the robust cost is not additive. Also the path 1-3-4 is the optimal path to Node 4 from 1 even though its subpath (1-3) is suboptimal, violating the subpath optimality condition. Also, the subpath 1-3 is dominated on both objectives, violating the subpath nondominance principle.

Sample Approximation–Based Reformulation

The MRCP problem as formulated in Equations 3*a* and 3*b* requires the mean, variance, and correlations of link travel times. However, empirical estimation of the correlation structure is data intensive and nontrivial for reasons noted previously.

To circumvent these difficulties, a sampling-based approach is adopted and the objective is accordingly reformulated below.

Let t_{di} represent the travel time realization on day d, on link i, measured over D days. The sample mean is given as $\hat{\mu}_i = D^{-1}\sum_{i=1}^{D} t_{di}$. Here, the link travel time distributions are assumed to be stationary, which means that all measurements t_{di} come from the same multivariate distribution. The mean and variance of travel time on path P are estimated from the sample by

$$\hat{\mu}_P = \sum_{i\in P} \hat{\mu}_i \qquad (4a)$$

$$\hat{\sigma}_P^2 = \frac{1}{D-1}\sum_{d=1}^{D}\left(t_{dP} - \hat{\mu}_P\right)^2 = \frac{1}{D-1}\sum_{d=1}^{D}\left(\sum_{i\in P} t_{di} - \sum_{i\in P}\hat{\mu}_i\right)^2 \qquad (4b)$$

Path travel time realization on day d, t_{dP}, can be expressed as the sum of link travel times on that day $t_{dP} = \sum_{i\in P} t_{di}$.

With Equations 4*a* and 4*b*, the robust cost objective in Equation 3*a* can be reformulated by using the sample mean and variance instead of the population values as follows:

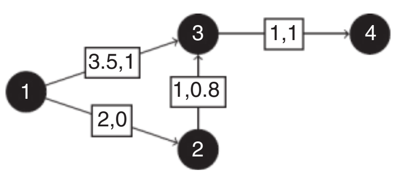
$$\min RC(\mathbf{x}) = \delta\left(\sum_{i\in A}\hat{\mu}_i x_i\right) + (1-\delta)\left(\frac{1}{D-1}\sum_{d=1}^{D}\left(\sum_{i\in P} t_{di}x_i - \sum_{i\in P}\hat{\mu}_i x_i\right)^2\right)^{0.5} \qquad (5)$$

where $\mathbf{x}$ represents the vector of m binary variables $x_i \in \{0,1\}$, which satisfy the flow constraints of unit flow from source r to sink s. The above expression can be written in a concise form as follows:

$$\min RC(\mathbf{x}) = \delta\left(\hat{\boldsymbol{\mu}}^T\mathbf{x}\right) + (1-\delta)\left(\frac{1}{D-1}\mathbf{x}^T\mathbf{C}^T\mathbf{C}\mathbf{x}\right)^{0.5} \qquad (6)$$

where $\hat{\boldsymbol{\mu}}$ represents the vector of sample means of all links and $\mathbf{C}$ is a $D\times m$ matrix of deviations from the mean whose elements are given as $c_{di} = t_{di} - \hat{\mu}_i$. The variance of a path in Equations 5 and 6 is represented as the sum of the squares of D separate link objectives c_{dP}. Thus the problem has been reformulated as a multiobjective ($D + 1$, including the mean) problem.

Typically multiday estimates of travel time data are collected by many traffic agencies by using probe vehicles, bluetooth sources, and historical archives from other intelligent transportation system sensors. These data can be readily used to optimize robust cost according to the reformulated objective in Equation 5 without quantifying distributions or correlations explicitly.



(a)

Path	Mean	Var.	Robust Cost
1-3	3.5	1	1.5
1-2-3	3	0.8	1.31
1-3-4	4.5	1	1.7
1-2-3-4	4	2.7	2.11

(b)

FIGURE 1 Failure of subpath optimality and subpath nondominance: (*a*) example of network and (*b*) path and subpath costs (var. = variance).

PROPOSED ALGORITHM FOR DETERMINING MRCP

In this section an overview of the algorithm is provided and the subpath elimination criterion and proof of correctness are discussed followed by the implementation details. The pseudocode of the algorithm is depicted in Figure 2.

Overview of Algorithm

The input to the algorithm is the origin–destination (O-D) pair $(r; s)$ for which the minimum robust cost path and the travel time realizations over D days are sought.

The algorithm involves two main procedures, namely, network transformation to reduce the dimensionality of the problem and network pruning to find the optimal path.

Network transformation involves two sequential operations: (*a*) identification of Dial efficient links with link costs as sample mean travel times, a step that leads to reduced network dimensionality (~50%) and an acyclic subnetwork, and (*b*) Johnson's transformation applied only to Dial efficient links with cost vector $\mathbf{c}_i$ (defined as the difference between travel time on day d from the sample mean) to obtain reduced cost vectors, $\mathbf{w}_i$ [defined in Carrion and Levinson (*14*)]. The advantage of this transformation is twofold: first the transformed costs, which are nonnegative for Dial efficient links unlike the cost vector $\mathbf{c}_i$ and, second, a super sink node s^* and a new link (s, s^*) are added to the transformed network and its costs $\mathbf{w}_{(s, s^*)}$ are set so that all negative cost elements associated with $\mathbf{c}_i$ are localized on this link.

By exploiting this localization of negative costs onto a single arc, a new subpath elimination criterion is proposed for this transformed network with reduced costs (discussed in a later subsection) and the correctness is established. This criterion eliminates subpaths that satisfy either of the following conditions:

1. The lower bound on the variance of any path containing the current subpath exceeds a norm-based threshold, and the lower bound on the robust cost of the path is larger than the corresponding upper bound on the optimal path (as per Equation 7*a*).
2. The lower bound on the weighted mean component of the robust cost of any path containing the current subpath exceeds the upper bound on the robust cost of the optimal path (as per Equation 7*b*).

Finally, based on this subpath elimination criterion, a new pruning procedure is implemented on the transformed network. The pruning procedure eliminates suboptimal subpaths by using a modified label-correcting approach (as described in the subsection on network pruning).

Subpath Elimination Criterion and Proof of Correctness

Description of Subpath Elimination Criterion

In this subsection the norm-based subpath elimination criterion is explained. The elimination criterion is dependent on only two link separable objectives and thus reduces the dimension from $D + 1$ objectives in Equation 5.

Consider an intermediate node u and a subpath P_{r-u} from source r to node u with two attributes, $d^m(u)$ and $d^v(u)$. Here, $d^m(u)$ represents total expected travel time of the subpath and $d^v(u)$ represents the sum of norm squares ($\Sigma_i \|\mathbf{w}_i\|^2$, $i \in P_{r-u}$, obtained from network transformation) of the transformed cost vectors of the subpath. Let $\mathrm{SP}^m(u)$ denote the shortest path label (value) from u to sink s with

1. **Inputs**:
   a. Network $G(N, A)$, origin r and destination s.
   b. Travel time realizations, t_{di} where $d \in D$ and $i \in A$.
2. **Network Transformation**: To transform cost-vectors to component wise positive
   a. Dial Efficient Links: Perform single sink shortest path algorithm with mean travel times $(\hat{\mu}_i)$ as link costs. Determine set of links which are Dial efficient, A'.
   $i \equiv (u,v) \in A'$ if $d(u) - d(v) > 0$ where $d(u)$ represent shortest path distance from u to sink s.
   b. Johnson's Transformation: Perform Johnson's transformation for each of day $d \in D$.
   For each $d \in D$,
      i. Compute the shortest path labels, $l_d(v)$, $v \in N$ with $c_{di} = t_{di} - \hat{\mu}_i, \forall i \in A'$ as costs.
      ii. Compute the transformed costs $w_{di} = c_{di} + l_d(u) - l_d(v), \forall i \equiv (u,v) \in A'$ and $\beta_d = -w_{d(s,s^*)} = l_d(r) - l_d(s)$
   This results in a transformed network $G_2(V,E)$ where $V = N \cup s^*$ and $E = A' \cup (s,s^*)$.
3. **Network Pruning to eliminate subpaths which are suboptimal:** Perform procedure outlined in Figure \ref{fig:pruning_pro}.
4. **Finding optimal Robust Cost Path**: Identify the minimum robust cost path from the path (label) set at sink s by calculating the robust cost of all the paths in it.

FIGURE 2 MRCP algorithm—correlated links.

link costs as $\hat{\mu}_i$, and let $SP^v(u)$ denote the shortest path distance from u to sink s with link costs as $\|\mathbf{w}_i\|^2$. Let RC_{UB} be an upper bound on the optimal robust cost.

The elimination criterion at node u is given by

Condition 1

$$\left[d^v(u) + SP^v(u)\right] \geq ||\boldsymbol{\beta}||^2 \qquad \text{AND}$$

Condition 2

$$\delta\left[d^m(u) + SP^m(u)\right] + \frac{(1-\delta)}{(D-1)^{0.5}}\left[\left(d^v(u) + SP^v(u)\right)^{0.5} - ||\boldsymbol{\beta}||\right] \geq RC_{UB} \tag{7a}$$

Condition 3

$$\delta\left[d^m(u) + SP^m(u)\right] \geq RC_{UB} \tag{7b}$$

where $-\boldsymbol{\beta}$ represents the localized cost vector on the link (s, s^*) in the transformed network. Condition 1 checks whether the lower bound of the variance of the path P with P_{r-u} as the subpath is greater than the variance on the link (s, s^*). Condition 2 evaluates whether the lower bound of the robust cost of the path P with P_{r-u} as the subpath is greater than the upper bound on the minimum path robust cost. Condition 3 checks whether the lower bound on the weighted mean component of path P with P_{r-u} as the subpath is greater than the upper bound on the minimum path robust cost. The correctness of this elimination criterion is established in the next section.

Proof of Correctness

Proposition 1 establishes that if either Conditions 1 and 2 (Equation 7*a*) or Condition 3 (Equation 7*b*) holds at node u for some subpath P_{r-u}, that subpath cannot be a part of the optimal path from r to s. Proposition 2 establishes that the optimal robust cost from r to s is one of the paths that remain after the subpaths that satisfy the elimination criterion are eliminated.

Proposition 1. Any path from source r to an intermediate node u, given by P_{r-u}, that satisfies Condition 7 cannot be a subpath of the minimum robust cost path from r to sink s^* on the transformed network.

Proof. The proposition is proved by demonstrating that the left-hand side (LHS) of the second inequality in Equation 7*a* and the LHS in the inequality in Equation 7*b* are lower bounds on the robust cost of any path P from source r to sink s^*, which includes P_{r-u} as a subpath where $u \neq s^*$. Consequently, if the lower bound on the robust cost of any path P with P_{r-u} as a subpath is greater than the upper bound on the optimal path robust cost, then the subpath P_{r-u} cannot be contained in the optimal path. ■

First, the validity of Equation 7*a* is demonstrated.

Let the first inequality in Equation 7*a* be true. The variance of a path $P - s^*$ on the transformed network from source r to sink s^* with P_{r-u} as the subpath from r to node u is given as

$$(D-1)\,\text{var}_{P-s^*} = \left\|\sum_{i\in P}\mathbf{w}_i - \boldsymbol{\beta}\right\|^2 \geq \left\|\sum_{i\in P}\mathbf{w}_i\right\|^2 + ||\boldsymbol{\beta}||^2 - 2\left\|\sum_{i\in P}\mathbf{w}_i\right\|||\boldsymbol{\beta}||$$
$$= \left(\left\|\sum_{i\in P}\mathbf{w}_i\right\| - ||\boldsymbol{\beta}||\right)^2 \tag{8}$$

In Equation 8, the inequality follows from the norm identity and Cauchy–Schwarz inequality. All elements of the transformed cost vector are nonnegative as a result of transformation $\mathbf{w}_i \succcurlyeq 0$ [see Carrion and Levinson (*14*)].

By assuming that Condition 1 holds and using the fact that P_{r-u} is the subpath of $P - s^*$ and because the first inequality in Equation 7*a* has been assumed to be true, one can write

$$\left\|\sum_{i\in P}\mathbf{w}_i\right\|^2 \geq \sum_{i\in P}||\mathbf{w}_i||^2 \geq \sum_{i\in P_{r-u}}||\mathbf{w}_i||^2 + SP^v(u) \geq ||\boldsymbol{\beta}||^2 \tag{9}$$

where $SP^v(u)$ denotes the shortest path distance from u to s with link costs as $\|\mathbf{w}_i\|^2$. By applying the inequalities (Equation 9) to the last expression in Equation 8, one can write the following:

$$\left(\left\|\sum_{i\in P}\mathbf{w}_i\right\| - ||\boldsymbol{\beta}||\right)^2 \geq \left[\left(\sum_{i\in P}||\mathbf{w}_i||^2\right)^{0.5} - ||\boldsymbol{\beta}||\right]^2$$
$$\geq \left[\left(\sum_{i\in P_{r-u}}||\mathbf{w}_i||^2 + SP^v(u)\right)^{0.5} - ||\boldsymbol{\beta}||\right]^2 \tag{10}$$

From Equations 8 and 10 and given Equation 9, the following can be written:

$$(D-1)\,\text{var}_{P-s^*} \geq \left[\left(\sum_{i\in P_{r-u}}||\mathbf{w}_i||^2 + SP^v(u)\right)^{0.5} - ||\boldsymbol{\beta}||\right]^2 \tag{11}$$

Thus a lower bound on travel time variance of any path $P - s^*$ with P_{r-u} as its subpath has been obtained. Similarly the lower bound on the mean travel time of $P - s^*$ can be obtained as follows:

$$\hat{\mu}_{P-s^*} = \sum_{i\in P_{r-u}}\hat{\mu}_i + SP^m(u) \tag{12}$$

where $SP^m(u)$ denotes the shortest path distance from u to s with link costs as $\hat{\mu}_i$. With Equations 12 and 11 the following can be written:

$$RC_{P-s^*} \geq \delta\left[\sum_{i\in P_{r-u}}\hat{\mu}_i + SP^m(u)\right]$$
$$+ \frac{(1-\delta)}{(D-1)^{0.5}}\left[\left(\sum_{i\in P_{r-u}}||\mathbf{w}_i||^2 + SP^v(u)\right)^{0.5} - ||\boldsymbol{\beta}||\right] \tag{13}$$

Thus, the LHS of the second inequality in Expression 7*a* is a lower bound on the robust cost of any path $P - s^*$ with P_{r-u} as a subpath. The LHS of inequality in Expression 7*b* is a lower bound on mean travel time of any path $P - s^*$ with P_{r-u} as a subpath. As the variance of a random variable is always nonnegative, the LHS of Expression 7*b* is also a lower bound on the robust cost of any path $P - s^*$ with P_{r-u} as a subpath.

Proposition 2. The minimum robust cost path P^* between source r and sink s will be present in the path set at the sink at the termination of the pruning procedure.

Proof. This claim is proved by contradiction. Assume that the minimum robust cost path P^* is not present in the path set at sink s. The implication is that any one of its subpaths, P^*_{r-u}, was discarded because it satisfied the elimination criterion in Equation 7, which contradicts Proposition 1. Thus, the minimum robust cost path P^* will be present in the path set at the sink after the termination of the pruning procedure. ■

Implementation of the Algorithm

The steps of the algorithm, presented in Figure 2, are discussed below.

Network Transformation

Network transformations involve two operations: (*a*) eliminating the Dial inefficient links and (*b*) transforming the original cost vectors into reduced costs by applying Johnson's transformation.

Identifying Dial Efficient Links In this step, links that do not satisfy a criterion similar to Dial's criterion used in the STOCH algorithm are eliminated (*26*). A weaker condition is used in the present study in which the links that take the user farther from the given destination in average time are eliminated. These links are found by solving for the single destination shortest path problem from all sources to sink s with mean travel times from the sample as link costs. A link (u, v) is Dial efficient only if its tail node is closer to the sink than the head node [i.e., optimal distance labels $d(u) - d(v) > 0$]. There are two advantages from this transformation: the network reduces by 40% to 50% (number of arcs) and the resulting subnetwork is acyclic.

Johnson's Transformation to Obtain Reduced Costs First, the Dial efficient subnetwork above is transformed into a network G_1 as follows:

- A new super source node r^* is added.
- Dummy arcs connecting the node r^* to all nodes are created with zero cost.
- Transformed costs $\mathbf{c}_i$ are defined for the Dial efficient links (referred to as real links) as follows: $\mathbf{c}_i = \{c_{1i}, c_{2i}, \ldots, c_{Di}\}$ where $c_{di} = t_{di} - \hat{\mu}_i$ represents the difference between observed link travel time on day d, t_{di}, and sample mean on link i, $\hat{\mu}_i$, over all days. These costs c_{di} will be positive on some days and negative on others. Therefore, the associated path costs and optimal costs on paths with only real links may be either positive or negative for a given day d.

For the cost vector $\mathbf{c}_i$, the path from r^* to any node v in network G_1 consists of real paths (paths containing only Dial efficient links) and the virtual path with dummy arc (r^*, v). For a given day d, if the minimum path cost on real paths is positive, then the virtual path (r^*, v) is optimal as it has a distance label of zero (by construction). However, if the minimum path cost on real paths is negative, then the optimal distance label to node v will be negative. Thus, the optimal distance labels (based on cost vector $\mathbf{c}_i$) to any node will be either zero or negative.

Johnson's transformation is applied on this transformed network G_1 to convert the costs, $\mathbf{c}_i$, into reduced costs, $\mathbf{w}_i$, which are nonnegative. More formally, as per Johnson's transformation, the reduced cost on a given arc $i \equiv (u, v)$ on day d, w_{di}, is given as

$$w_{di} = c_{di} + \pi_d(v) - \pi_d(u) \tag{14}$$

where node potential for node u on day d is given as $\pi_d(u) = -l_d(u)$ where $l_d(u)$ is the shortest path cost from r^* to u with c_{di} as link costs.

The resulting reduced costs on all arcs in G_1 can be shown to be nonnegative for each day d. To see this, one must note that as per the shortest path optimality principle, the optimal distance labels satisfy the following inequality for all arcs $i \equiv (u, v)$ in network G_1.

$$l_d(v) \le l_d(u) + c_{di} \tag{15}$$

Substituting $\pi_d(u) = -l_d(u)$ and $\pi_d(v) = -l_d(v)$ and rearranging Equation 15 yields

$$w_{di} = c_{di} + \pi_d(v) - \pi_d(u) \ge 0$$

Next, a new super sink node s^* is added to the network G_1, and a dummy arc (s, s^*) between the sink s and super sink s^* is created. This augmented network is referred to as $G_2 = G_1 \cup (s, s^*)$.

The reduced cost on this dummy link on day d is set as $w_{d(s,s^*)} = l_d(s) - l_d(r)$ and is denoted as $-\beta_d$ for ease of notation. This term may be either positive or negative. Thus, the reduced cost transformation ensures that all negative cost elements are localized to arc (s, s^*) in network G_2.

Next, it is shown that the path cost on day d from r to s on network $G(N, A)$, based on cost vector $\mathbf{c}_i$, is the same as the path cost from r to s^* on the transformed network G_2 with the reduced costs $\mathbf{w}_i$.

Consider a path P from r to s in the network G and the corresponding path $P' = P \cup (s, s^*)$. The original path cost on P for day d is given as $c_{dP} = \sum_{i \in P} c_{di}$. The reduced path cost on P' for the same day is given as

$$\begin{aligned} w_{P\cup(s,s^*)} &= \sum_{i\in P} w_{di} - \beta_d = \sum_{i\equiv(u,v)\in P} [c_{di} + l_d(u) - l_d(v)] - \beta_d \\ &= \sum_{i\in P} c_{di} + l_d(r) - l_d(s) - \beta_d = \sum_{i\in P} c_{di} \end{aligned} \tag{16}$$

where $-\beta_d$ represents the cost on the dummy link on day d.

As a result of this equivalence between path costs on network G from r to s and reduced costs on network G_2 from r to s^*, the optimization and subpath elimination criteria are developed by using network G_2 and reduced costs $\mathbf{w}_i$ where all negative cost elements are localized onto the link (s, s^*).

Network Pruning

The transformed network $G_2(V, E)$ is pruned by using a label-correcting-based approach and the subpath elimination criterion (see previous subsection with description of this criterion). The first step in pruning is precomputation of the following quantities.

First, obtain $\mathrm{SP}^m(u)$ and $\mathrm{SP}^v(u)$, $\forall u \in V \backslash s^*$. The time $\mathrm{SP}^m(u)$ is the minimum expected travel time from node u to s, and $\mathrm{SP}^v(u)$

is the shortest path cost from u to s where cost on links i are the norm square ($\|\mathbf{w}_i\|^2$) of the transformed cost vectors. These times are obtained by solving the single sink shortest path problem with the label-correcting procedure, with corresponding costs on the links. Next, the upper bound on optimal robust cost, RC_{UB}, is computed as the robust cost of the LET path.

After precomputation, the network G_2 is pruned on the basis of the norm-based subpath elimination criterion given in Equation 7. As shown in Proposition 2, the pruning procedure returns a set of candidate paths at the sink node s that is guaranteed to contain the optimal path. A label-correcting approach is proposed for pruning; it is depicted in Figure 3 and explained in the next section.

A scan-eligible list is maintained consisting of only those nodes whose subpaths are not discarded by the elimination criterion. The nodes in the scan-eligible list are processed sequentially. Specifically, for each node u in the scan-eligible list, each of the outgoing arcs (u, v) is processed to obtain the labels for mean and variance for connected downstream node v. Specifically, $\boldsymbol{d_k(u)} = [d_k^m(u), d_k^v(u)]$ is maintained for each node $u \in V$. For subpath k, $d_k^m(u)$ and $d_k^v(u)$ represent the mean travel time and sum of link transformed cost vector's norm squares ($\sum_{i\in k} \|\mathbf{w}_i\|^2$), respectively.

Node v is added to the scan-eligible list if at least one of the new (not present at v) subpaths obtained by adding link (u, v) to subpaths at u does not satisfy the elimination criteria. The algorithm is terminated when the scan-eligible list is empty.

Unlike in the standard label-correcting approaches, multiple labels and predecessors may be maintained for each node. The optimal path can be shown to be contained in the set of paths that remain at termination (Proposition 2). Therefore, once the label-correcting algorithm terminates, the optimal path is identified by computing the robust cost of all nonpruned paths to the destination.

Computational Complexity

The main steps involve Dial efficient link identification ($O(mn)$), network transformation ($O(mnD + mD)$), upper bound computation ($O(m + n\log n)$), pruning procedure ($O(mn\kappa)$), and robust cost evaluation of path set at termination ($O(\kappa)$). Here κ is the maximum size of the nonpruned path set size from origin to some nonpruned node. The number of arcs, nodes, and days in analysis are denoted by m, n, and D, respectively.

1. **Inputs**:
    a. Transformed Network $G_2(V, E)$, source r and sink s.
    b. Link attributes which include sample mean of travel times $\hat{\mu}_i$ and transformed cost-vectors $\mathbf{w}_i$, $\forall\ i \in E$.
2. **Precomputation**:
    a. Compute the shortest path distances $SP^m(u)$ with $\hat{\mu}_i$ as link costs and $SP^v(u)$ with $\|\mathbf{w}_i\|^2$ as link cost from every node $u \in V$ to sink s.
    b. Let $RC_{UB} = RC_{P_{\text{LET}}}$ where P_{LET} is the least expected travel time path from r to s
3. **Initialization:**
    a. Set first label of source r to zero for both the objectives
    $d_l^m(r) = d_l^v(r) = 0$
    Initialize the set of labels at remaining nodes as null sets $\boldsymbol{D}(u) = \varnothing, \forall u \in N \setminus \{r\}$
    b. Add source r to node list $\boldsymbol{L} = \{r\}$
4. **Node Selection**: IF set $\boldsymbol{L}$ is empty $\boldsymbol{L} = \varnothing$, GOTO Step 5. ELSE, select a node $u \in \boldsymbol{L}$
5. **Node Processing**: For all nodes v such that $(u, v) \equiv i \in E$, do the following:
    a. Compute Set $\hat{\boldsymbol{D}}(v)$
    b. For every label k of node u,
    c. Compute the temporary label $\hat{d}_k(v) = d_k(u) \cup (u,v)$ to node v obtained by adding arc (u,v) to label $d_k(u)$
    d. $\hat{d}_k^m(v) = d_k^m(u) + \hat{\mu}_i$; $\hat{d}_k^v(v) = d_k^v(u) + \|\mathbf{w}_i\|^2$
    e. Update labels of $\boldsymbol{D}(v)$
    f. For each acyclic temporary label, $\hat{d}_k(v) \notin \boldsymbol{D}(v)$ check ***Subpath Elimination Criterion***.
    g. IF NOT $\boldsymbol{D}(v) = \boldsymbol{D}(v) \cup \{\hat{d}_k(v)\}$
6. **Update List**:
    a. IF set of labels of node v, $\boldsymbol{D}(v)$ has been modified and node $v \notin \boldsymbol{L}$, $\boldsymbol{L} = \boldsymbol{L} \cup v$.
    b. Delete node u from list: $\boldsymbol{L} = \boldsymbol{L} \setminus u$.
    c. GO TO Step 2
7. **Return**: The path (label) set $\boldsymbol{D}(s)$ at sink s.

FIGURE 3 Label-correcting–based pruning.

This procedure gives an overall complexity $O(mn(D+1)+mD+(m+n\log n)+mn\kappa+\kappa)$, which is pseudopolynomial in path set size κ.

COMPUTATION EXPERIMENTS

In this section the performance of the proposed algorithm on real transportation networks is quantified in computation experiments followed by an analysis of sample approximation on the optimal solution quality.

The experiments were conducted on a Windows 7 (32-bit) computer with two 2.9-GHz central processing units (Core-2 Duo) and 3 GB of RAM.

Performance on Real-World Networks

The performance of the pruning algorithm is studied on real-world transportation networks obtained from the website hosted by Bar-Gera (http://www.bgu.ac.il/~bargera/tntp/). The link travel time realizations over D days were assumed to follow a multivariate shifted lognormal distribution. The univariate shifted lognormal random variable used for generating these travel times, t_i, is given by

$$t_i = \gamma_i + \exp(\mu_i + \sigma_i z) \tag{17}$$

where $z \sim N(0,1)$ is a standard normal random variable and γ_i represents free-flow travel time. For the current study, the shifted lognormal parameters for each link on the network were generated uniformly in the following ranges (65–125 s/km) for γ, (20–200 s/km) for excess mean ($E[t_i] - \gamma_i$), and (0.05–0.75) for the coefficient of variation. These values were obtained on the basis of data from the 2008 Chennai, India, Comprehensive Traffic Study (CCTS). The correlation matrix was generated uniformly for each network from the space of positive definite matrices. Then, the travel time realizations were generated on all links for $D = 100$ days. On each network, the algorithm was applied to find MRCPs for 25 randomly generated O-D pairs.

The performance metrics for analysis include average computation time and average nonpruned path set size (at intermediate nodes as well as at the destination). The results from these computation experiments are presented in Table 1. The computation (comp) times were found to be less than 5 s for 98% of the O-D pairs on moderate to small networks, suggesting acceptable performance for moderately sized (<1,500 nodes) networks. On the large network in Austin, Texas, the average computation time is about 1.2 min, which is still acceptable for a stand-alone application.

However, the path set sizes may be higher for a few O-D pairs, especially on longer paths. Network density (arc-to-node ratio) significantly affects the algorithm's performance as seen from results on the networks of Chicago, Illinois; Barcelona, Spain; and Winnipeg, Manitoba, Canada.

Further computation experiments suggested a weak influence of network structure on computation times and nonpruned path set size (variables such as assortativity, betweenness centrality, closeness centrality, and path size were found to be influential). Because of space limitations and the relatively weak influence, the experiments are not reported in the paper.

In contrast, other methods in the literature, namely, the PIND procedure (*4*) and Lagrangian relaxation (*5*), do not guarantee the optimal solution. It has been reported previously in the literature that Lagrangian relaxation reported an average error of more than 5.4% and suboptimality in 25.5% of cases.

Analysis of the Effect of Sample Approximation on Solution Quality

The travel time distributions being approximated through a sample raise questions about possible bias and the precision of the solution obtained from the sample relative to the population. The extent of the error is quantified as described below.

A second set of computation experiments was performed on a real network of Chennai with 1,279 nodes and 3,644 links. The empirically observed travel times on 94 links during a period of 100 days (from November 2012 to February 2013) were used in this set. On the other links of the networks, independent travel times were generated from the marginal distributions estimated for each lane type by using the data obtained from the 2008 CCTS, which follow shifted lognormal distribution.

These travel time realizations (during a 100-day period) over the complete network were assumed to constitute the population in the current experiment. Then the sample-based MRCP algorithm was applied by varying the sample size systematically from 5 to 25 days (in steps of five). For each sample size, the MRCP algorithm was applied with 100 draws. This process was done for 15 O-D pairs in groups of five selected on the basis of their shortest path distances of less than 10 km, between 10 and 20 km, and more than 20 km.

Let P_s^* and P_p^* be the optimal paths in the sample and the population, respectively. To quantify the quality of the solution with the

TABLE 1 Performance of the Algorithm on Real-World Networks

			Computation Time			Path Set at Node			Path Size at Destination		
Network Name	n	m	Average	SD	90%	Average	SD	90%	Average	SD	90%
Anaheim, California	416	914	0.01	0.02	0.06	7.8	12.5	35.3	45.6	92.1	246.4
Chicago, Illinois	933	2,950	6.26	30.51	1.41	53.5	161.9	138.5	928.1	3,748.1	1,207.8
Barcelona, Spain	1,020	2,522	0.12	0.26	0.64	22.4	39.9	77.2	146.8	257.1	569.6
Winnipeg, Manitoba, Canada	1,052	2,836	0.67	2.17	2.64	30.6	67.2	114.4	307.3	732.6	1,476.8
Chennai	1,279	3,644	11.63	55.32	5.25	88.4	294.1	180.0	1,011.9	3,766.1	1,804.0
Austin, Texas	7,388	18,961	72.94	152.08	404.64	218.7	403.5	921.8	3,259.4	6,703.7	13,999.2

NOTE: SD = standard deviation.

TABLE 2 Quantification of Sampling Errors Across O-Ds

Path Length	Sample Size	Sample Error (%)		Variance Error (%)		Total Error (%)	
		Mean	SD	Mean	SD	Mean	SD
<10 km	5	1.99	4.28	5.05	23.44	1.40	2.56
	10	1.56	3.58	3.15	18.81	1.17	2.31
	15	1.33	2.97	2.46	15.62	1.34	2.51
	20	1.04	2.24	1.73	13.90	1.38	2.58
	25	0.87	1.93	1.24	12.50	1.64	2.72
10–20 km	5	6.16	6.87	2.11	12.58	4.35	4.10
	10	4.13	5.31	0.83	8.97	3.32	3.62
	15	2.82	4.02	0.64	7.14	2.72	3.35
	20	2.19	3.05	0.79	6.35	2.30	3.01
	25	1.84	2.75	0.59	5.49	1.90	2.75
>20 km	5	8.15	6.67	1.20	10.22	3.91	2.99
	10	4.99	4.66	0.66	7.19	2.94	2.49
	15	3.63	3.66	0.51	5.82	2.38	2.16
	20	3.03	3.22	0.29	4.99	2.15	2.05
	25	2.49	2.69	0.28	4.37	1.85	1.91

NOTE: Errors are expressed as percentages.

sample-based objective, the metrics measured were bias (difference in robust cost between P_s^* in the sample and P_p^* in the sample), variance error (difference in robust cost between P_p^* in the sample and P_p^* in the population), and total error (difference in robust cost between P_s^* in the population and P_p^* in the population).

In Table 2, the percentages of error values averaged over the O-D pairs of similar path length are presented. As expected, as the sample size increases, the mean and the standard deviation of all errors decrease. The path lengths seem to influence the bias and variance components and, hence, the total error. The O-D pairs with shorter path lengths have lower sample errors and total errors as the population optimal is more likely to be found in the sample. The mean total errors across samples for shorter paths lie in the range of 1.17% to 1.64%. In contrast, errors of 1.85% to 4.35% were observed for the longer paths and may be explained by the smaller size of the nonpruned path set. The O-D pairs with longer paths have a lower variance error (0.28% to 1.20%) compared with shorter paths (1.24% to 4.28%), possibly reflecting a greater tendency to regress toward the mean in the former case. More important, the quality of the solution for the sample-based robust cost objective is found to be acceptable (1% to 2%) even with modest sample sizes in most cases.

CONCLUSION

This paper presents a new sample-based algorithm for the minimum robust cost path on a network with link travel time correlations. The problem was formulated as a separable multiobjective problem in which a sample-based approach was adopted to represent the link travel time distributions. The sample-based approach adopted was found to capture the path correlations implicitly and thus obviate the explicit estimation of the link travel time correlation matrix. For this objective, solution procedures based on subpath optimality and nondominance are shown to be inapplicable.

A new subpath elimination criterion was developed to eliminate suboptimal subpaths. This criterion is applied to a suitably transformed network with reduced costs, and a pruning algorithm is developed to discard suboptimal subpaths for the robust cost objective. The transformations together with the elimination criterion enable the conversion of the MRCP problem from a $D + 1$ dimensional problem to a two-objective problem leading to a significant gain in computational efficiency. The proposed algorithm is shown to yield an exact solution, and its computational complexity is shown to be pseudopolynomial in regard to the number of nonpruned paths. The approach could also be extended with repetitive application to multiday and quasi-dynamic scenarios.

The empirical findings from computation experiments performed led to the following insights:

1. The algorithm is guaranteed to find the optimal and performs reasonably well on small to medium (<1,500 nodes) size networks with computation time being less than 5 s for 98% of the O-D pairs. On large networks, the computation times are higher, particularly for O-D pairs with substantial spatial separation. In such cases, the number of nonpruned paths increases considerably.
2. The average error in optimal path resulting from sampling approximation was found to be less than 5% for various sample sizes. The O-D pairs with shorter path lengths have a smaller bias than those pairs with longer path lengths. In contrast, the variance component of such O-D pairs was found to be larger. The total error was found to increase with increasing path lengths and may potentially be improved by applying the algorithm repeatedly on the basis of real-time traffic state updates and reoptimization.

The directions for future research include the transformation of costs vectors into positive components while negative cycles exist. Extending this present algorithm to time-dependent networks, online routing, and nonstationary distributions offers interesting and valuable scope for further investigations.

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REFERENCES

1. Bates, J., J. Polak, P. Jones, and A. Cook. The Valuation of Reliability for Personal Travel, *Transportation Research Part E,* Vol. 37, 2001, pp. 191–229.
2. Recker, W., Y. Chung, J. Park, L. Wang, A. Chen, Z. Ji, H. Liu, M. Horrocks, and J.S. Oh. *Considering Risk-Taking Behavior in Travel Time Reliability.* Technical Report UCB-ITS-PRR-2005-3. California PATH Research Report, University of California, Berkeley, 2005.
3. Nicholson, A. *Estimating Travel Time Reliability: Can We Safely Ignore Correlation?* Delft University, Netherlands, 2012.
4. Seshadri, R., and K. Srinivasan. An Algorithm for the Minimum Robust Cost Path on Networks with Random and Correlated Link Travel Times. *Network Reliability in Practice,* 2012, pp. 171–208.
5. Xing, T., and X. Zhou. Finding the Most Reliable Path With and Without Link Travel Time Correlation: A Lagrangian Substitution Based Approach. *Transportation Research Part B,* Vol. 45, No. 10, 2011, pp. 1660–1679.
6. Zhou, X., and T. Xing. Reformulation and Solution Algorithms for Absolute and Percentile Robust Shortest Path Problems. *IEEE Transactions on Intelligent Transportation Systems,* Vol. 14, No. 2, 2013, pp. 943–954.
7. Boyles, S.D. *Operational, Supply-Side Uncertainty in Transportation Networks: Causes, Effects, and Mitigation Strategies.* PhD dissertation. University of Texas, Austin, 2009.
8. Polychronopoulos, G.H., and J.N. Tsitsiklis. Stochastic Shortest Path Problems with Recourse. *Networks,* Vol. 27, 1996, pp. 133–143.
9. Cheung, R.K. Iterative Methods for Dynamic Stochastic Shortest Path Problems. *Naval Research Logistics,* Vol. 45, 1998, pp. 768–789.
10. Waller, S.T., and A.K. Ziliaskopoulos. On the Online Shortest Path Problem with Limited Arc Cost Dependencies. *Networks,* Vol. 40, No. 4, 2002, pp. 216–227.
11. Hall, R. The Fastest Path Through a Network with Random Time-Dependent Travel Times. *Transportation Science,* Vol. 20, No. 3, 1986, pp. 182–188.
12. Miller-Hooks, E., and H. Mahmassani. Path Comparisons for A Priori and Time-Adaptive Decisions in Stochastic, Time-Varying Networks. *European Journal of Operational Research,* Vol. 146, No. 1, 2003, pp. 67–82.
13. Pretolani, D. A Directed Hypergraph Model for Random Time Dependent Shortest Paths. *European Journal of Operational Research,* Vol. 123, No. 2, 2000, pp. 315–324.
14. Carrion, C., and D. Levinson. Value of Travel Time Reliability: A Review of Current Evidence. *Transportation Research Part A,* Vol. 46, No. 4, 2012, pp. 720–741.
15. Sivakumar, R., and R. Batta. The Variance-Constrained Shortest Path Problem. *Transportation Science,* Vol. 28, No. 4, 1994, pp. 309–316.
16. Sen, S., R. Pillai, S. Joshi, and A. Rathi. A Mean-Variance Model for Route Guidance in Advanced Traveler Information Systems. *Transportation Science,* Vol. 35, No. 1, 2001, pp. 37–49.
17. Nikolova, E., M. Brand, and D.R. Karger. Optimal Route Planning Under Uncertainty. *ICAPS,* Vol. 6, 2006, pp. 131–141.
18. Mirchandani, P.B., and H. Soroush. Optimal Paths in Probabilistic Networks: A Case with Temporary Preferences. *Computers and Operations Research,* Vol. 12, No. 4, 1985, pp. 365–381.
19. Frank, H. Shortest Paths in Probabilistic Graphs. *Operations Research,* Vol. 17, No. 4, 1969, pp. 583–599.
20. Mirchandani, P. Shortest Distance and Reliability of Probabilistic Networks. *Computers and Operations Research,* Vol. 3, No. 4, 1976, pp. 347–355.
21. Seshadri, R., and K.K. Srinivasan. Algorithm for Determining Most Reliable Travel Time Path on Network with Normally Distributed and Correlated Link Travel Times. In *Transportation Research Record: Journal of the Transportation Research Board, No. 2196,* Transportation Research Board of the National Academies, Washington, D.C., 2010, pp. 83–92.
22. Nie, Y., and X. Wu. Shortest Path Problem Considering On-Time Arrival Probability. *Transportation Research Part B,* Vol. 43, No. 6, 2009, pp. 597–613.
23. Srinivasan, K.K., A.A. Prakash, and R. Seshadri. Finding Most Reliable Paths on Networks with Correlated and Shifted Log–Normal Travel Times. *Transportation Research Part B,* Vol. 66, 2014, pp. 110–128.
24. Chen, B.Y., W.H. Lam, A. Sumalee, and H. Shao. An Efficient Solution Algorithm for Solving Multi-Class Reliability-Based Traffic Assignment Problem. *Mathematical and Computer Modelling,* Vol. 54, No. 5, 2011, pp. 1428–1439.
25. Chen, B.Y., W.H. Lam, A. Sumalee, and Z.-l. Li. Reliable Shortest Path Finding in Stochastic Networks with Spatial Correlated Link Travel Times. *International Journal of Geographical Information Science,* Vol. 26, No. 2, 2012, pp. 365–386.
26. Dial, R.B. A Probabilistic Multipath Traffic Assignment Model Which Obviates Path Enumeration. *Transportation Research,* Vol. 5, 1971, pp. 83–113.

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