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Revisiting Ramakrishnan's Approach to Relativity

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The conditions under which the Velocity Addition Theorem (VAT) is formulated by Ramakrishnan gave rise to doubts about the uniqueness of the theorem. These conditions are rediscussed with reference to their algebraic and experimental implications.

1. INTRODUCTION

Accommodating light speed as the upper bound for a Newtonian sum of velocities, Ramakrishnan [1] interpreted space-like regions as implying the existence of independent particles. It was also pointed out that the existence of independent particles should not be interpreted as mediated by a tachyon, since this amounts to a spontaneous splitting of one particle into two. The relativistic VAT employed to draw these significant conclusions was formulated in the beginning of [1] and Levinson [2] later pointed out that this formulation does not lead to a unique outcome. But the formulation itself being not the core of the paper, his interpretations of space-like regions should be unaffected if the formulation can be made unique. This has been achieved by using group theoretic arguments [3]. In the meanwhile one of us [4] demonstrated that if tachyons are included in Levinson's approach as a sufficient condition, his derivation would uniquely select the VAT out of the several choices. This shows that either the existence of tachyons is not a necessary condition or if tachyons exist, they are not the mediators of the "independent particles" referred to by Ramakrishnan. Now tachyons being still speculative, we propose to supplement the postulates in [1] minimally within the framework of special relativity which will make the VAT unique and restore the conclusions of [1]. For our discussion we will adopt the notation of [2].

2. UNIQUENESS OF VAT

Let

$$w = \frac{u - v}{1 - g(u, v)} \tag{2.1}$$

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where $g(u, v)$ is such that

$$u, v < 1 \qquad \text{implies} \qquad w < 1 \tag{2.2}$$

As

$$u, v \to 0 \qquad g(u, v) \to 0 \tag{2.3}$$

As

$$u, v \to 1 \qquad g(u, v) \to v, u \tag{2.4}$$

$$g(u, v) = g(v, u) = g(-u, -v) \tag{2.5}$$

[Ramakishnan himself has enumerated only the conditions (2.2)–(2.4) and (2.5) is imposed by Levinson from a general symmetry consideration].

Further let $h(x)$ satisfy $h(1) = 1$ and $0 \leqslant h(x) < 1$, $0 \leqslant x < 1$. Let $[1 - h(x)]/(1 - x) \leqslant A$, $0 \leqslant x < 1$ for some A. Let $F(x, r)$ be defined for $|x| \leqslant 1$, $0 \leqslant r \leqslant 2$ and let

$$F(x, r) \geqslant 0 \qquad\qquad x \geqslant 0$$

$$F(x, r) < \min\left(2, \frac{1}{4A^2}\right) \qquad x < 0.$$

Then Levinson proved that

$$g(u, v) = uv - uvF(uv, u^2 + v^2)(1 - h(u^2))(1 - h(v^2))$$

satisfies (2.2)–(2.5) and not only $g(u, v) = uv$ as was supposed in [1].

We now supplement these conditions by the postulate of a unique inversion of (2.1) viz.,

(*) The unique inverse of (2.1) is given by replacing u by w and w by u and v by $-v$.

From this it follows that the most general form of $g(u, v)$ is

$$g(u, v) = \alpha(v) + u\beta(v) \tag{2.6}$$

since otherwise it will not be possible to solve (2.1) uniquely for u in terms of w and v for all values of w and $v < 1$. Now $g(u, v) = uv$ follows uniquely since the conditions (2.3) and (2.4) when imposed in (2.6) give $\alpha(v) = 0$ and $\beta(v) = v$.

3. Uniqueness of the Lorentz transformation (LT)

The condition (2.3) appears to give rise to another type of ambiguity discussed in [5][1] which is however removed by the specific form of (2.1). In order to estimate the crucial nature of the form of (2.1) in relation to (2.3), we will

[1] We thank Prof. A. Ramakrishnan for bringing it to our notice.

discuss the problem from a different angle and we do it in Minkowski representation.

The velocity vector is *defined* as

$$U = \left(\frac{dx}{d\tau}, \frac{idt}{d\tau}\right)$$

where τ is the invariant proper time since such a definition ensures the invariance of U^2 in the "simplest" way. Thereby the spatial and temporal parts of U become

$$U = \left(\frac{u}{(1-u^2)^{1/2}}, \frac{i}{(1-u^2)^{1/2}}\right), \qquad U^2 = -1. \tag{3.1}$$

Further the momentum vector is *defined* as

$$P = m_0 U, \qquad P^2 = -m_0{}^2 \tag{3.2}$$

where m_0 is the invariant proper mass. Also the force is defined as

$$F = \frac{dP}{d\tau} \tag{3.3}$$

so that the ordinary 3-force is given by

$$\mathbf{f} = \frac{d\mathbf{P}}{dt} = \frac{d}{dt}\left(\frac{m_0 \mathbf{u}}{(1-u^2)^{1/2}}\right). \tag{3.4}$$

This 3-force developes two components one parallel to $\mathbf{u}$ and another parallel to $d\mathbf{u}/dt$ giving rise to two effective masses, a transverse mass $m_0(1-u^2)^{-1/2}$ and a longitudinal mass $m_0(1-u^2)^{-3/2}$. For consistency with the definition of momentum we pass a rule that only the tranverse mass is to be retained as *the* mass [6]. These ambiguities naturally give rise to inconistencies in a relativistic formulation of statics [7] and elsewhere.

At first sight it appears that this chain of definitions can be set right with a different definition of the velocity vector. For e.g., one may take

$$U = (u, i(1+u^2)^{1/2}) \tag{3.5}$$

which also ensures $U^2 = -1$ and in which the spatial part coincides with the Newtonian velocity. This will at once ensure a 4-momentum in which the spatial part coincides with the Newtonian momentum. Thus (3.5) leaves the spatial part of 4-vectors unaltered from their Newtonian counterparts, but introduces changes only in the temporal part so as to protect the relativistic invariants.

While all this is gratifying from a conservative point of view, the changes that

(3.5) introduces into the conventional LT is disturbing. The new LT for one dimensional motion would be

$$\begin{aligned} \xi' &= (1 + u^2)^{1/2}\, \xi + u\eta \\ \eta' &= (1 + u^2)^{1/2}\, \eta + u\xi \end{aligned} \tag{3.6}$$

where $\xi = x$ or p and $\eta = t$ or E respectively. This gives $(p'/E')_{p=0} = u(1 + u^2)^{-1/2}$ and not u. If one requires that $(p'/E')_{p=0} \to u$ for $u \to 0$ only in the limit as in (2.3), the transformation (3.6) will continue to give the correct velocity. Indeed any transformation

$$\begin{aligned} \xi' &= (1 + f^2(u))^{1/2}\, \xi + f(u)\, \eta \\ \eta' &= (1 + f^2(u))^{1/2}\, \eta + f(u)\, \xi \end{aligned} \tag{3.7}$$

where $f(u) = -f(-u)$ to meet the demands of (*) and $f(u) \to u$ as $u \to 0$ in the limit would also be a LT. Therefore what distinguishes the conventional LT is that $(P'/E')_{p=0} = u$ for any $u < 1$ and not only in the limit as $u \to 0$. In the velocity space this requirement is precisely met in ref. [1] by the specific form of (2.1) where the constraint is taken as $w \to u - v$ (and not $\to 0$ as misquoted in Equation 1b of ref. [4]) as $u, v \to 0$.

4. A Remark

The algebraic nature of the splitting of U as in (3.5) implies a relative velocity $u\,(1 + u^2)^{-1/2}$ and not u as in (3.1). Mathematically this rules out any other splitting than (3.1). But a generally unnoticed situation in high energy experimental relativity which continue to cause concern about it is the following [8, 9]: For obvious reasons the velocity of a high energy particle is not measured directly by the ratio of the distance covered to the transit time as in classical mechanics. But whatever value one obtains for the velocity by any other method, it is seldom used in solving problems of high energy physics since even a trivial error in its value leads to very large differences for the energy and momentum because of the factor $(1 - u^2)^{-1/2}$ in (3.1). Therefore the problems are stated in terms of energy and the invariant relation

$$E^2 - P^2 = m_0^{\,2} \tag{4.1}$$

is used to find the momentum. The velocity is only derived from the expression $P/E = u$ which is itself a consequence of (3.1). In this sense, there does not exist a high energy experiment which constitues an independent and unrestricted verification of the concept that the velocity continues to be u even when it is large but <1. A possible situation might be that when a charged particle is accelerated it is enveloped by a cloud of radiation which causes it to retard; therefore it is not unlikely that an observer within the cloud may record a velocity different from (actually less than) the one recorded by an observer outside. However the two velocities should be naturally related and therefore we could expect the velocity within the cloud to be a function of u which tends to u as u tends to 0. This situation necessitates the use of different values of $f(u)$ in (3.7) for different regions of space as delineated by Rosen [10].

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