Reliability analysis using high dimensional model representation for mixed uncertain variables

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Reliability analysis using high dimensional model representation for mixed uncertain variables

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Abstract. The reliability estimation of a structural system depends on the type of uncertain parameters considered. If all the uncertain parameters are defined as random variables, then the reliability can be estimated by using probability theory. However, when modeling uncertain variables with limited information as intervals with upper and lower bounds, the entire range of these bounds should be explored. When dealing with the combination of both random and interval variables, as every combination of interval variables demands one probabilistic based analysis, the computational cost involved in estimating the reliability of the system increases exponentially. Therefore in this paper, high dimensional model representation (HDMR) is used to approximate the failure function accurately, and fast Fourier transform techniques are applied to solve the convolution integral. The proposed methodology demonstrates the improvement in computational efficiency and accuracy in the estimation of the reliability bounds using numerical examples.

1. Introduction

Structural reliability estimation involves determination of the probability that a structural response exceeds a threshold limit, defined by a limit state/performance function influenced by several random parameters. Symbolically, the reliability problem can be stated as the evaluation of multidimensional integral

$$\begin{aligned} P_S &= 1 - P_F \\ &\equiv 1 - P(g(\mathbf{x}) \le 0) = 1 - \int_{g(\mathbf{x}) \le 0} p_{\mathbf{x}}(\mathbf{x}) \quad d\mathbf{x} \end{aligned} \tag{1}$$

where $\mathbf{x} = \{x_1, x_2, ..., x_N\}$ represent the *N*–dimensional random variables of the model under consideration; $g(\mathbf{x})$ is the limit state/performance function, such that $g(\mathbf{x}) \le 0$ represents the failure domain; and $p_{\mathbf{x}}(\mathbf{x})$ is the joint probability density function of the input random variables.

Monte Carlo simulation can be used to deal with this multi-fold integration. However, it requires a large number of samples to accurately estimate the small order of structural failure probabilities. To

reduce the computational cost, several algorithms were developed that make use of surrogate representations of the failure surface and compute the failure probability [1]. The above multi-dimensional PDF integral is also represented using convolution integral as described in basic probability and statistics literature [2, 3]. For a failure surface that is a linear combination of random variables this convolution integral can be evaluated using Fast Fourier transforms (FFT). In order to use FFT, the limit state function must be available as a separable closed-form expression.

In the presence of both random as well as fuzzy variables, the computational cost involved in the development of the membership function of reliability increases because multiple reliability analyses, which by itself is a computationally expensive procedure, is required at each confidence level [4]. This is because the entire bounds of the fuzzy variables are to be explored to determine the bounds of the reliability at a particular confidence level. Moreover, while dealing with a non-linear function with fuzzy variables, the joint membership function results with over estimation or unbounded results due to dependency problem.

In this paper, high dimensional model representation (HDMR) [5, 6] is used to approximate the failure function accurately, and fast Fourier transform techniques are applied to solve the convolution integral. Transformation techniques are used for fuzzy variables to avoid dependency problem. The methodology presented in this paper provides an accurate estimate of the membership function of the reliability using these transformation techniques along with FFT.

2. High dimensional model representation

In recent years there have been efforts to develop efficient methods to approximate multivariate functions in such a way that the component functions of the approximation are ordered starting from a constant and gradually approaching to multivariance as we proceed along the terms like first-order, second-order, and so on. One such method is HDMR [5, 6]. HDMR is a general set of quantitative model assessment and analysis tools for capturing the high-dimensional relationships between sets of input and output model variables. It is a very efficient formulation of the system response, if higher order variable correlations are weak, allowing the physical model to be captured by the first few lower order terms. Practically for most well-defined physical systems, only relatively low order correlations of the input variables are expected to have a significant effect on the overall response. HDMR expansion utilizes this property to present an accurate hierarchical representation of the physical system.

The degree of accuracy of reliability estimation depends on the accurate representation of the limit state function. Computational complexity for the generation of response surface of implicit limit state function arises due to increase in number of input variables, while using conventional response surface in conjunction with design of experiments. The concept of HDMR expansions is used here for the purpose of approximating the limit state function most accurately and efficiently when the number of input variables is large.

Let the *N*-dimensional vector $\mathbf{x} = \{x_1, x_2, ..., x_N\}$ represent the input variables of the model under consideration, and the response variable as $g(\mathbf{x})$. Since the influence of the input variables on the response variable can be independent and/or cooperative, HDMR expresses the response $g(\mathbf{x})$ as a hierarchical correlated function expansion in terms of the input variables as

$$g(\mathbf{x}) = g_0 + \sum_{i=1}^{N} g_i\left(x_i\right) + \sum_{1\le i_1\le i_2\le N} g_{i_1 i_2}\left(x_{i_1}, x_{i_2}\right) + \sum_{1\le i_1\le \ldots\le i_l\le N} g_{i_1 i_2 \ldots i_l}\left(x_{i_1}, x_{i_2}, ..., x_{i_l}\right) + \cdots + g_{12\cdots N}\left(x_1, x_2, \cdots, x_N\right) \quad (2)$$

where g_0 is a constant term representing the zeroth-order component function or the mean response of $g(\mathbf{x})$. The function $g_i(x_i)$ is a first-order term expressing the effect of variable x_i acting alone, although generally nonlinearly, upon the output $g(\mathbf{x})$. The function $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ is a second-order term, which describes the cooperative effects of the variables x_{i_1} and x_{i_2} upon the output $g(\mathbf{x})$. The

higher order terms give the cooperative effects of increasing numbers of input variables acting together to influence the output $g(\mathbf{x})$. The last term $g_{12\cdots N}(x_1, x_2, \cdots, x_N)$ contains any residual dependence of all the input variables locked together in a cooperative way to influence the output $g(\mathbf{x})$. Once all the relevant component functions in Equation (2) are determined and suitably represented, then the component functions constitute HDMR, thereby replacing the original, computationally expensive method of calculating $g(\mathbf{x})$ by the computationally efficient model. Usually the higher order terms in Equation (2) are negligible such that HDMR with only low-order correlations to second-order amongst the input variables are typically adequate in describing the output behaviour.

Depending on the method adopted to determine the component functions in Equation (2), there are two particular HDMR expansions: ANOVA-HDMR and cut-HDMR. ANOVA-HDMR is useful for measuring the contributions of the variance of individual component functions to the overall variance of the output. On the other hand, cut-HDMR expansion is an exact representation of the output g(x) in the hyperplane passing through a reference point in the variable space.

With the cut-HDMR method, first a reference point $\mathbf{c} = \{c_1, c_2, ..., c_N\}$ is defined in the variable space. In the convergence limit, cut-HDMR is invariant to the choice of reference point **c**. In practice, **c** is chosen within the neighborhood of interest in the input space. The expansion functions are determined by evaluating the input-output responses of the system relative to the defined reference point c along associated lines, surfaces, subvolumes, etc. (i.e. cuts) in the input variable space. This process reduces to the following relationship for the component functions in Equation (2).

$$g_0 = g(\mathbf{c}), \tag{3}$$

$$g_i(x_i) = g(x_i, \mathbf{c}^i) - g_0, \tag{4}$$

$$g_{i_1 i_2}(x_{i_1}, x_{i_2}) = g(x_{i_1}, x_{i_2}, \mathbf{c}^{i_1 i_2}) - g_{i_2}(x_{i_2}) - g_0, \tag{5}$$

where the notation $g(x_i, \mathbf{c}^i) = g(c_1, c_2, ..., c_{i-1}, x_i, c_{i+1}, ..., c_N)$ denotes that all the input variables are at their reference point values except x_i. The g_0 term is the output response of the system evaluated at the reference point **c**. The higher order terms are evaluated as cuts in the input variable space through the reference point. Therefore, each first-order term $g_i(x_i)$ is evaluated along its variable axis through the reference point. Each second-order term $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ is evaluated in a plane defined by the binary set of input variables x_{i_1}, x_{i_2} through the reference point, etc. The process of subtracting off the lower order expansion functions removes their dependence to assure a unique contribution from the new expansion function.

Considering terms up to first-order in Equation (2) yields, respectively,

$$g(\mathbf{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \Re_2, \tag{6}$$

Substituting Equations (3)-(4) into Equation (6) leads to

$$g(\mathbf{x}) = \sum_{i=1}^{N} g(c_1, ..., c_{i-1}, x_i, c_{i+1}, ..., c_N) - (N-1) g(\mathbf{c}) + \Re_2, \tag{7}$$

Now consider first-order approximation of $g(\mathbf{x})$

$$\begin{aligned} \tilde{g}(\mathbf{x}) &\equiv g(x_1, x_2, ..., x_N) \\ &= \sum_{i=1}^{N} g(c_1, ..., c_{i-1}, x_i, c_{i+1}, ..., c_N) - (N-1) g(\mathbf{c}). \end{aligned} \tag{8}$$

Comparison of Equations (7) and (8) indicates that the first-order approximation leads to the residual error $g(\mathbf{x}) - \tilde{g}(\mathbf{x}) = \Re_2$, which includes contributions from terms of two and higher order component functions.

The notion of 0th, 1st, etc. in HDMR expansion should not be confused with the terminology used either in the Taylor series or in the conventional least-squares based regression model. It can be shown that the first-order component function $g_i(x_i)$ is the sum of all the Taylor series terms, which contain and only contain variable x_i. Hence first-order HDMR approximations should not be viewed as first-order Taylor series expansions nor do they limit the nonlinearity of $g(\mathbf{x})$. Furthermore, the approximations contain contributions from all input variables. Thus, the infinite number of terms in the Taylor series is partitioned into finite different groups and each group corresponds to one cut-HDMR component function. Therefore, any truncated cut-HDMR expansion provides a better approximation and convergent solution of $g(\mathbf{x})$ than any truncated Taylor series because the latter only contains a finite number of terms of Taylor series. Furthermore, the coefficients associated with higher dimensional terms are usually much smaller than that with one-dimensional terms. As such, the impact of higher dimensional terms on the function is less, and therefore can be neglected. Compared with the FORM and SORM, which retain only linear and quadratic terms, respectively, first-order HDMR approximation $\tilde{g}(\mathbf{x})$ provides more accurate representation of the original implicit limit state function $g(\mathbf{x})$.

3. Probability density and characteristic functions

The characteristic function, which is the Fourier transform of the marginal density, and the marginal density of a random variable Y are expressed as a pair of Fourier transforms. The formulation of the Fourier transform pairs is as follows:

$$M_Y(\theta) = \int_{-\infty}^{\infty} p_Y(y) e^{2\pi\theta i y} \quad dy, \tag{9}$$

$$p_Y(y) = \int_{-\infty}^{\infty} M_Y(\theta) e^{-2\pi\theta i y} \quad d\theta, \tag{10}$$

where $p_Y(y)$ and $M_Y(\theta)$ are the marginal density and characteristics function of Y, respectively. i denotes the imaginary number defined as $i = \sqrt{-1}$. Equation (9) defines the forward Fourier transform of $p_Y(y)$, whereas Equation (10) defines the inverse Fourier transform of $M_Y(\theta)$. The properties of the characteristic function $M_Y(\theta)$ can be summarized as:

(1) $\left|M_Y(\theta)\right| \le 1$, $M_Y(-\theta) = \overline{M_Y(\theta)}$, where $\left|\bullet\right|$ and $\bar{\bullet}$ are the absolute value and complex conjugate of $\bullet$, respectively.

(2) The characteristic function of a random variable $X = aY + b$ is expressed as

$$M_X(\theta) = e^{2\pi i\theta b} M_Y(a\theta), \tag{11}$$

(3) The characteristic function of a random variable Y, which is the sum of statistically independent random variables, Y_1, Y_2,..., Y_N, is given by the product of the characteristic function of each random variable $M_{Y_1}(\theta)$, $M_{Y_2}(\theta)$,..., $M_{Y_N}(\theta)$ as

$$M_Y(\theta) = M_{Y_1}(\theta) \cdot M_{Y_2}(\theta) \cdot \ldots \cdot M_{Y_N}(\theta), \tag{12}$$

4. Membership function of reliability

Concept of FFT can be applied to the problem if the limit state function is in the form of a linear combination of independent variables and when either the marginal density or the characteristic function of each basic random variable is known. Even if the function of the basic variables is nonlinear, an appropriate transformation of the basic random variables could yield a linear function of independent random variables. The HDMR concepts are used to express the implicit limit state function $g(\mathbf{x})$ that depends on $\mathbf{x}=\{x_1,x_2,...,x_N\}\in\Re^N$ as a linear combination of lower order component functions. The steps involved in the proposed method for the estimation of membership function of reliability are as follows:

(1) If $\mathbf{u}=\{u_1,u_2,...,u_N\}^T\in\Re^N$ is the standard Gaussian variable, let $\mathbf{u}^*=\{u_1^*,u_2^*,...,u_N^*\}^T$ be the MPP or design point, determined by a standard nonlinear constrained optimization. The MPP has a distance β_{HL}, which is commonly referred to as the Hasofer–Lind reliability index. Construct an orthogonal matrix $\mathbf{R}\in\Re^{N\times N}$ whose N–th column is $\alpha^*=\mathbf{u}^*/\beta_{HL}$, i.e., $\mathbf{R}=[\mathbf{R}_1\,|\,\alpha^*]$ where $\mathbf{R}_1\in\Re^{N\times N-1}$ satisfies $\alpha^{*T}\mathbf{R}_1=0\in\Re^{1\times N-1}$. The matrix $\mathbf{R}$ can be obtained, for example, by Gram–Schmidt orthogonalization. For an orthogonal transformation $\mathbf{u}=\mathbf{R}\mathbf{v}$. Let $\mathbf{v}=\{v_1,v_2,...,v_N\}^T\in\Re^N$ represent the rotated Gaussian space with the associated MPP $\mathbf{v}^*=\{v_1^*,v_2^*,...,v_N^*\}^T$. The transformed limit state/performance function $g(\mathbf{v})$ therefore maps the original function into rotated Gaussian space $\mathbf{v}$. First-order HDMR approximation of $g(\mathbf{v})$ in rotated Gaussian space v with $\mathbf{v}^*=\{v_1^*,v_2^*,...,v_N^*\}^T$ as reference point can be represented as follows:

$$\begin{aligned}\tilde{g}(\mathbf{v}) &\equiv g(v_1,v_2,...,v_N)\\ &=\sum_{i=1}^{N} g(v_1^*,...,v_{i-1}^*,v_i,v_{i+1}^*,...,v_N^*)-(N-1)g(\mathbf{v}^*).\end{aligned} \tag{13}$$

In addition to the MPP as the chosen reference point, the accuracy of first-order HDMR approximation in Equation (13) may depend on the orientation of the first N–1 axes. In the present work, the orientation is defined by the matrix $\mathbf{R}$. In Equation (13), the terms $g(v_1^*,...,v_{i-1}^*,v_i,v_{i+1}^*,...,v_N^*)$ are the individual component functions and are independent of each other. Equation (13) can be rewritten as,

$$\tilde{g}(\mathbf{v})=a+\sum_{i=1}^{N} g(v_i,\mathbf{v}^{*i}), \tag{14}$$

where $a=-(N-1)g(\mathbf{v}^*)$.

(2) New intermediate variables are defined as,

$$z_i=g(v_i,\mathbf{v}^{*i}). \tag{15}$$

.The purpose of these new variables is to transform the approximate function into the following form

$$\tilde{g}(\mathbf{v})=a+z_1+z_2+...+z_N. \tag{16}$$

(3) Due to rotational transformation in $\mathbf{v}$-space, component functions z_i in Equation (16) are expected to be linear or weakly nonlinear function of random variables v_i. In this work both linear and quadratic approximation of z_i are considered.

Linear Approximation

Consider a linear approximation: $z_i=b_i+c_iv_i$, where coefficients $b_i\in\Re$ and $c_i\in\Re$ (non-zero) are obtained by least-squares approximations from exact or numerically simulated conditional responses $\{g(v_1^1,v^{*i}),g(v_1^2,v^{*i}),...,g(v_1^n,v^{*i})\}^T$ at n sample points along the variable axis v_i. Then Equation (16) results in

$$\tilde{g}(\mathbf{v}) \equiv a + z_1 + z_2 + ... + z_N = a + \sum_{i=1}^{N} (b_i + c_i v_i). \quad (17)$$

The least-squares approximation is chosen over interpolation, because the former minimizes the error when $n > 2$.

Quadratic Approximation

The linear approximation described in the preceding can be improved by a quadratic approximation: $z_i = b_i + c_i v_i + e_i v_i^2$, where coefficients $b_i \in \Re$, $c_i \in \Re$ and $e_i \in \Re$ (non-zero) are obtained by least-squares approximations from exact or numerically simulated conditional responses $\{g(v_1^1, v^{*i}), g(v_1^2, v^{*i}), ..., g(v_1^n, v^{*i})\}^T$ at n sample points along the variable axis v_i. Then Equation (16) results in

$$\tilde{g}(\mathbf{v}) \equiv a + z_1 + z_2 + ... + z_N = a + \sum_{i=1}^{N} (b_i + c_i v_i + e_i v_i^2). \quad (18)$$

Again the least-squares approximation is chosen over interpolation, because the former minimizes the error when $n > 3$.

(4) This surrogate model is divided into two parts, one containing the terms with random variables ($g_{\mathrm{R}}(\mathbf{z})$) and the other containing fuzzy variables ($g_{\mathrm{F}}(\mathbf{z})$).

(5) In the random variables part, as v_i follows standard Gaussian distribution, marginal density of the intermediate variables z_i can be easily obtained by simple transformation (using chain rule).

$$p_{Z_i}(z_i) = p_{V_i}(v_i) \left| \frac{1}{dz_i / dv_i} \right|. \quad (19)$$

(6) Now the approximation is a linear combination of the intermediate variables z_i. Therefore, the PDF of $g_{\mathrm{R}}(\mathbf{z})$, which is the convolution of the individual marginal density of the intervening variables z_i, can be expressed as follows:

$$p_{\tilde{G}_{\mathrm{R}}}(\tilde{g}_{\mathrm{R}}) = p_{z_1}(z_1) * p_{z_2}(z_2) * ..... * p_{z_N}(z_N). \quad (20)$$

where $p_{\tilde{G}_{\mathrm{R}}}(\tilde{g}_{\mathrm{R}})$ represents PDF of limit state/performance function $g_{\mathrm{R}}(\mathbf{z})$.

(7) Applying FFT on both sides of Eq. (20) leads to

$$\mathrm{FFT}[p_{\tilde{G}_{\mathrm{R}}}(\tilde{g}_{\mathrm{R}})] = \mathrm{FFT}[p_{z_1}(z_1)] \cdot \mathrm{FFT}[p_{z_2}(z_2)] \cdot ... \cdot \mathrm{FFT}[p_{z_N}(z_N)]. \quad (21)$$

By applying inverse FFT on both side of Equation (21), PDF of limit state function $g_{\mathrm{R}}(\mathbf{z})$ is obtained.

(8) From the fuzzy variables part, by using transformation techniques the joint membership function of $g_{\mathrm{F}}(\mathbf{z})$ is obtained. At each confidence level the minimum and the maximum values joint membership function are used to integrate the area under the PDF of $g_{\mathrm{R}}(\mathbf{z})$, and obtain the membership function of reliability. Actually the contribution of fuzzy variables acts as a linear shift in the PDF of $g_{\mathrm{R}}(\mathbf{z})$.

5. Computational effort

When comparing computational efforts by various methods in evaluating the membership functions of reliability, the number of original limit state/performance function evaluations is chosen as the primary comparison tool in this paper. This is because of the fact that the number of function evaluations indirectly indicates the CPU time usage. For direct MCS, number of original function evaluations is the same as the sampling size. While evaluating the membership functions of reliability through direct

MCS, CPU time is more because it involves the number of repeated actual finite-element analysis. In the present work, FFT is used in conjunction with HDMR based approximation. To obtain linear/quadratic approximation of the HDMR component functions, n(= 3, 5, 7 or 9) uniformly distributed sample points $v_i^* - (n-1)/2$, $v_i^* - (n-3)/2$, …, v_i^*, $v_i^* + (n-3)/2$, $v_i^* + (n-1)/2$ are deployed along each of the variable axis v_i through the MPP. Thus the total number of function evaluations required by the present method, in addition to those required for identification of the MPP, is $(n-1)N$.

6. Numerical examples

Four numerical examples involving explicit mathematical function and implicit functions from structural mechanics problems are presented to illustrate the performance of the present method. To evaluate the accuracy and the efficiency of the present method, comparisons of the estimated membership functions of reliability have been made with direct MCS.

6.1. Example 1: Explicit mathematical function

This example considers a hypothetical limit state function of the following form:

$$g(\mathbf{x}) = 8.0 - 0.32(x_1 - 1)^2 x_2^2 - x_2 + x_3^3 - 0.2\sin(x_1 x_3), \tag{22}$$

where x_1 and x_2 are assumed to be independent standard normal variables. The variable x_3 is assumed to be fuzzy variable with its membership function given by the following equation.

$$\mu_{x_3}(x_3) = \begin{cases} (x_3 + 1), & -1 \le x_3 \le 0 \\ (-x_3 + 1), & 0 \le x_3 \le 1. \end{cases} \tag{23}$$

As the limit-state function is available as closed-form expression in terms of uncertain variables, there was no need for constructing approximation of the limit-state function. The MPP was obtained for the limit-state function with the fuzzy variable set to its value at maximum possibility. Using HDMR, the response surface model was constructed in the form of linear combination of independent variables. This model was divided into two parts, one containing the terms with random variables ($g_{\mathrm{R}}(\mathbf{x})$) and the other containing fuzzy variables ($g_{\mathrm{F}}(\mathbf{x})$). The part with the random variables is convoluted to obtain the PDF of $g_{\mathrm{R}}(\mathbf{x})$. The part with the fuzzy variables was used in estimating the joint membership function of $g_{\mathrm{F}}(\mathbf{x})$. The transformation techniques used here to get the joint membership function avoid the dependency problem that generally occurs in fuzzy analysis. Due to the presence of only one fuzzy variable in this example, the joint membership function of the fuzzy variables is nothing but the membership function of the fuzzy intervening variable. The bounds of the joint membership function at a particular confidence level act a linear shift in the PDF of $g_{\mathrm{R}}(\mathbf{x})$. So, at each confidence level, the lower and upper bounds of the joint membership function are used to shift the PDF of $g_{\mathrm{R}}(\mathbf{x})$ and integrate the area in the failure region to obtain the reliability bounds.

Figure 1 shows the comparison of the membership functions obtained by the present method using linear and quadratic approximations with that of Monte Carlo simulation. While direct MCS requires 21×10^5 number of evaluations, the present method using linear and quadratic approximations needs 12 function evaluations only in addition to those required for obtaining the MPP.

6.2. Example 2: Cantilever beam

A cantilever beam subjected to a tip load P is considered in this example. The failure limit state is defined as tip displacement greater than 0.15 in.

$$g(\mathbf{x}) = 0.15 - \frac{4PL^3}{Ebh^3}, \tag{24}$$

where *L*, *b*, and *h* are the length, width and height of the beam respectively. *L* and *h* are considered as log-normally distributed random variables with mean values 30 in. and 2.5093 in., and standard deviations 3 in. and 0.25 in. respectively. The width *b* is considered as a normal random variable with mean value of 0.8359 in. and standard deviation of 0.08 in. The load *P* is treated as a triangular fuzzy variable [60, 80, 100]. The Young's Modulus *E* is 10^7 psi.

Figure 2 shows the comparison of the membership functions obtained by the present method using linear and quadratic approximations with that of Monte Carlo simulation. In this example both linear and quadratic approximations yield the same results. The present method required 16 actual function evaluations whereas MCS needed 21×10^5 number of evaluations.

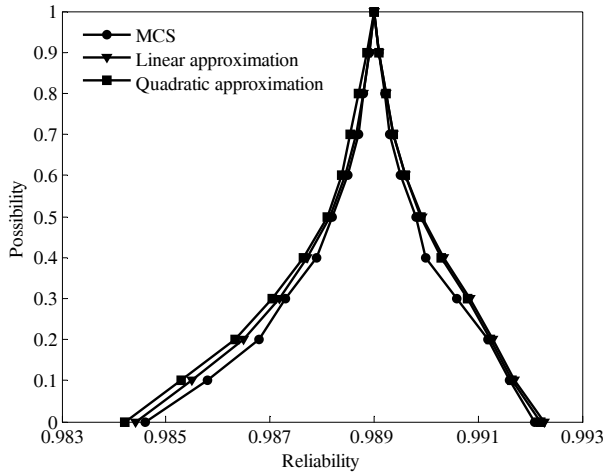

Figure 1. Membership function of reliability (Example 1)

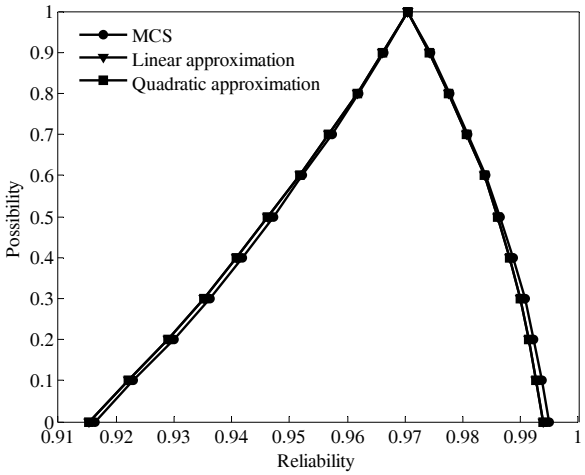

Figure 2. Membership function of reliability (Example 2)

6.3. *Example 3: Soil settlement problem*

The settlement of a point *A* in Figure 3 caused by the construction of a structure can be shown to be primarily caused by the consolidation of the clay layer. Suppose the contribution of settlement due to secondary consolidation is negligible. For normally loaded clay, the settlement *S* is given by:

$$S = \frac{C_c}{1+e_0} H \log \frac{p_0 + \Delta p}{p_0}, \tag{25}$$

where C_c is the compression index of the clay; e_0 is the void ratio of the clay layer before loading; H is the thickness of the clay layer; p_0 is the original effective pressure at point *B* (mid height of the clay layer) before loading; and Δp is the increase in pressure at point *B* caused by the construction of the structure; "log" denotes logarithm to the base 10. Because of the non-uniform thickness and lack of homogeneity of the clay layer, the settlement predicted by the empirical formula could be subject to uncertainty in predicted settlement.

Suppose satisfactory performance requires that the settlement be less than 2.5 in. The details of the uncertain variables are presented in Table 1. The limit state/performance function is defined as:

$$g(\mathbf{x}) = 2.5 - S. \tag{26}$$

Figure 4 shows the comparison of the membership functions obtained by the present method using linear and quadratic approximations with that of Monte Carlo simulation. The present method required 20 actual function evaluations whereas MCS needed 21×10^5 number of evaluations.

Table 1. Properties of the uncertain variables (Example 3)

Uncertain variable	Random variable			Fuzzy variable
	Mean	COV	Distribution type	
C_c	0.396	0.25	Gaussian	-
e_0	1.190	0.15	Gaussian	
H	-	-	-	[151.2, 168, 184.8]
p_0	-	-	-	[3.348, 3.72, 4.092]
Δp	-	-	-	[0.45, 0.5, 0.55]

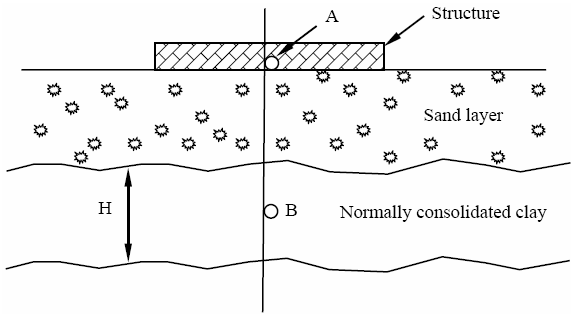

Figure 3. Soil profile (Example 3)

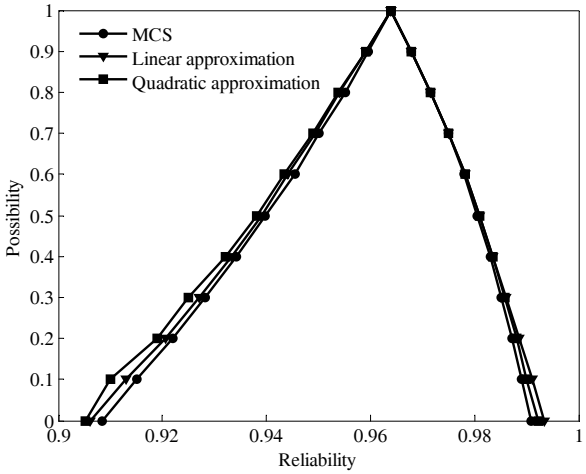

Figure 4. Membership function of reliability (Example 3)

6.4. Example 4: Ten-bar truss

In this example, a ten-bar truss, as shown in Figure 5, was considered to estimate the membership function of its reliability. The following equation shows the criterion for failure, which is the maximum displacement at the tip of the structure

$$g(\mathbf{x}) \;=\; 1.0 - \frac{\Delta_{\text{tip}}(\mathbf{x})}{0.04826}. \tag{27}$$

To demonstrate the applicability of the proposed method for multiple variables, five independent cross sectional areas were considered. The cross-sectional areas of the structure were physically linked as represented by Equation (28). By linking the cross-sectional areas, the number of independent random variables was reduced to five. These random variables were assumed to be normally distributed with mean values of 1.6129×10^{-3} m^2 and a standard deviation of 0.16129×10^{-3} m^2.

$$A_1 = A_2 = x_1,\; A_4 = A_5 = x_2,\; A_3 = A_8 = x_3,\; A_6 = A_7 = x_4,\;\; A_9 = A_{10} = x_5 \;\; . \tag{28}$$

The variations of the forces applied on the structure were modelled using triangular fuzzy membership functions given by Equation (29). The Young's modulus was taken as 70 GPa.

$$\mu_{P_i}(P_i) = \begin{cases} (P_i - 40.338)/4.482, & 40.338 \le P_i \le 44.82, \\ (-P_i + 49.302)/4.482, & 44.82 \le P_i \le 49.302, \end{cases} \quad i = 1,2\,. \tag{29}$$

Since this example is a problem with implicit limit-state function, the MPP was obtained for the limit-state with the fuzzy variables set to their values at maximum possibility. Using HDMR, the response surface model was constructed in the form of linear combination of independent variables. This model was divided into two parts, one containing the terms with random variables ($g_{\text{R}}(\mathbf{x})$) and

the other containing fuzzy variables ($g_F(\mathbf{x})$). The part with the random variables is convoluted to obtain the PDF of $g_R(\mathbf{x})$. The part with the fuzzy variables was used in estimating the joint membership function of $g_F(\mathbf{x})$. The transformation techniques used here to get the joint membership function avoid the dependency problem that generally occurs in fuzzy analysis. The bounds of the joint membership function at a particular confidence level act a linear shift in the PDF of $g_R(\mathbf{x})$. So, at each confidence level, the lower and upper bounds of the joint membership function are used to shift the PDF of $g_R(\mathbf{x})$ and integrate the area in the failure region to obtain the reliability bounds.

Figure 6 shows the comparison of the membership functions obtained by the present method using linear and quadratic approximations with that of Monte Carlo simulation. While direct MCS requires 21×10^5 number of evaluations, the present method using linear and quadratic approximations needs 28 functions evaluations only in addition to those required for obtaining the MPP.

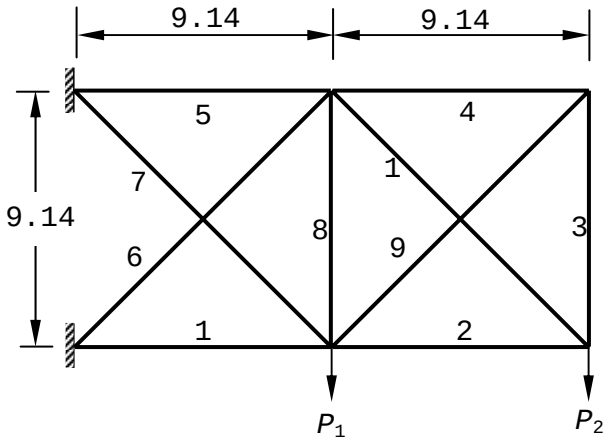

Figure 5. Ten-bar truss (Example 4)

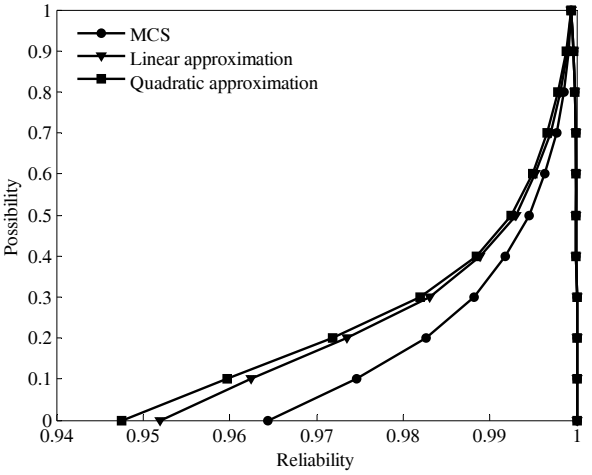

Figure 6. Membership function of reliability (Example 4)

7. Conclusion

An efficient method for reliability analysis to deal with the mixed uncertain variables is presented in this paper. The method involves HDMR approximation in conjunction with FFT for the estimation of reliability bounds. The results of the numerical example indicate that the proposed method provides accurate and computationally efficient estimates of the reliability bounds involving both random and interval variables.

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