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Relativistic tautochrone

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The path joining two points A and B, which a particle falling from rest in a uniform gravitational field must adopt, so that the time of transit from A to B is independent of the location of A is called the tautochrone. In the nonrelativistic case, the path is known to be a cycloid, a standard method associated with the derivation of this result being, for example, the method of Laplace transforms. For the relativistic case that is studied herein, the methods of fractional calculus are shown to be more useful in the derivation of the exact relativistic tautochrone. This latter derivation is then checked out by the Laplace transform approach for the relativistic problem, from the point of view of consistency. Using the same method, the relativistic tautochrone associated with a charged particle of charge q and mass m falling from rest in an uniform electric field is also worked out. The tautochrone turns out to be an incomplete elliptic function of the second kind $E(\delta,r)$. As an application, the power radiated by the charged particle as it accelerates along the curve is then computed; it is found to be proportional to $(1 - v^2/c^2)^{-2}$, with $v(c)$ the velocity of the particle (light). Finally, an appendix highlights the utility of fractional calculus *vis-á-vis* the approach of Abel for the relativistic tautochrone.

I. INTRODUCTION

The tautochrone problem is a famous problem in the theory of integral equations.[1] It consists in the determination of a curve in the (xy) plane such that the time required for a particle to slide down the curve to its lowest point under gravity is independent of its initial placement on the curve. It can be shown that the curve is a cycloid;[2] two methods commonly used for this purpose being the Laplace transform[3] and the approach given by Abel.[4]

An essential feature of the tautochrone problem mentioned above is that it is nonrelativistic. Therefore, in this paper we investigate the somewhat more difficult problem of a particle falling under the influence of a uniform gravitational field, but with the inclusion of relativistic effects. The calculation will then be repeated here to obtain the relativistic tautochrone associated with the motion of a charged particle of mass m and charge q in a uniform electric field.[5]

The immediate inspiration for this paper comes from a recent work by Goldstein and Bender[6] on the relativistic brachistochrone. The latter, as is well known, is perhaps the most famous problem in the calculus of variations. It consists in the determination of that path joining two points A and B, which a particle falling from rest at A in an uniform gravitational field must follow, so that the time of transit from A to B is a minimum.

As regards the determination of the tautochrone itself, it will turn out, as we shall show later on in this paper, that neither the Laplace transform approach[3] nor the solution offered by Abel[4] is useful in the relativistic case. With the former, for example, the gravitational tautochrone is to be obtained from the inverse Laplace transform of the reciprocal of the beta function $B(x,y)$. We have not found this an easy exercise.[7] We have therefore chosen an alternative route, viz., the methods of the fractional calculus,[8] to get the exact expression for the tautochrone. With this as an input, the correctness of the method is then checked via the Laplace transform approach.

It needs to be pointed out here that fractional calculus has been used[9] to obtain the nonrelativistic tautochrone as well. Its use for the relativistic version has, to the best of our knowledge, not been done before.

This paper is organized as follows. In Sec. II we present the essentials of the method of Laplace transforms for the gravitational tautochrone. This is followed in Sec. III by our exact solution of the problem via the method of fractional derivatives.[8] Some relevant definitions as well as rules associated with the fractional calculus are also summarized here. We end this section with a check on the exact expression for the relativistic tautochrone, using the methods of Sec. II. In Sec. IV the calculation of the electric tautochrone is taken up and the method is checked so as to obtain the correct time of transit for the falling particle. Since the nonrelativistic tautochrone (or the brachistochrone) associated with an electric field is known to be a cycloid, one may ask if something analogous is true for the relativistic case. This question also merits consideration as the relativistic brachistochrone associated with uniform electric (and gravitational) fields has already been worked out by Goldstein and Bender.[6] The issue is taken up in Sec. V; in the same

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section we also discuss from the point of view of classical electrodynamics, an interesting physical application of our results. Specifically, we have in mind a calculation of the power radiated by the charged particle as it accelerates along the tautochrone[10] obtained in Sec. IV. Section VI then concludes the paper with a short discussion and summary. Finally, from the point of view of completeness, as well as to highlight the utility of the method of the fractional calculus, our extension of Abel's approach[4] to the gravitational tautochrone is presented briefly in an appendix.

II. SOLUTION BY LAPLACE TRANSFORMS

We begin with a short discussion of the relativistic kinematics of a particle of mass m falling from rest in a uniform gravitational field. Denoting the acceleration due to gravity by g, the law of conservation of energy leads to the equation

$$mc^2 = mc^2/\sqrt{1-v^2/c^2} + \mathscr{E}, \tag{1}$$

where $\mathscr{E}$ is energy lost by the particle from the force field as it is released from a height h, and is given by[6]

$$\mathscr{E} = mc^2(1-\exp(g(h-x)/c^2)). \tag{2}$$

From Eqs. (1) and (2), we obtain,

$$v(x) = c(1-\exp 2g(x-h)/c^2)^{1/2}. \tag{3}$$

The time of fall is therefore given by

$$\begin{aligned} T &= \int_0^T dt \\ &= -\int_h^0 \frac{d\sigma}{v(x)} \\ &= \frac{1}{c}\int_0^h dx\, \sigma'(x)(1-\exp 2g(x-h)/c^2)^{-1/2}, \end{aligned} \tag{4}$$

with $\sigma'(x) = d\sigma/dx$, σ being the arclength along the path joining the initial and final end points, the latter being chosen as the origin. Since T is a constant for the tautochrone, the Laplace transform of (4) reads as

$$\frac{cT}{\lambda} = \int_0^\infty e^{-\lambda h}\, dh \int_0^h \sigma'(x)dx(1-\xi(h-x))^{-1/2}, \tag{5a}$$

with $\xi(x) = \exp(-2gx/c^2)$. The right-hand side of Eq. (5a) is the Laplace transform of the convolution of $\sigma'(x)$ and $\psi(h-x) = (1-\xi(h-x))^{-1/2}$; Eq. (5a) can therefore be recast as

$$cT/\lambda = f(\lambda)G(\lambda), \tag{5b}$$

with $f(\lambda)$ and $G(\lambda)$ being the Laplace transforms of $\sigma'(x)$ and $\psi(x)$, respectively. In particular, $G(\lambda)$ can be evaluated exactly to get

$$G(\lambda) = \int_0^\infty dt\, e^{-\lambda t}\psi(t) = \frac{c^2}{2g} B\left(\frac{c^2\lambda}{2g}, \frac{1}{2}\right), \tag{6}$$

with $B(x,y)$ being the beta function. Here we have made use of the result[11]

$$\int_0^\infty e^{-\mu x}(1-e^{-x/\beta})^{\nu-1}\, dx = \beta B(\beta\mu,\nu), \tag{7}$$

valid for $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$, $\operatorname{Re}\beta > 0$. To obtain the tautochrone from (5b) and (6), we therefore have to find the inverse Laplace transform of

$$f(\lambda) = (cT/\lambda)[1/G(\lambda)]. \tag{8a}$$

We have not found this exercise easy; in contrast, for the nonrelativistic case, the counterpart of (8) is given by

$$f(\lambda) = T(2g/\pi\lambda)^{1/2}, \tag{8b}$$

with g, the acceleration due to gravity. The difficulties inherent to the evaluation of the tautochrone from (8a) relative to (8b) has prompted us to choose an alternative route, as mentioned in the Introduction. This will be taken up in the following section.

III. SOLUTION BY THE FRACTIONAL CALCULUS

We begin with the most frequently encountered definition of an integral of fractional order via an integral transform called the Riemann–Liouville integral, namely,[12]

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x (x-y)^{-q-1} f(y)dy, \quad q<0. \tag{9a}$$

The extension of Eq. (9a) to $q \geqslant 0$ is made, following Oldham and Spanier,[8] by requiring that in this case

$$\frac{d^q f}{[d(x-a)]^q} = \frac{d^n}{dx^n}\left(\frac{d^{q-n} f}{[d(x-a)]^{q-n}}\right), \tag{9b}$$

where d^n/dx^n effects ordinary differentiation up to nth order. Here n is a positive integer and chosen so large that $q-n<0$. Equations (9a) and (9b) are said to define the qth-order differintegral[13] of $f(x)$ with respect to x. A differintegral of $f(x)$ with respect to an arbitrary function $g(x)$ can be defined[14] by considering the Riemann–Liouville integral,

$$\frac{d^q f(x)}{[d(g(x)-g(a))]^q}=\frac{1}{\Gamma(-q)}\int_a^x \frac{f(y)g'(y)dy}{(g(x)-g(y))^{q+1}}, \tag{10}$$

for $q<0$. Clearly, Eq. (10) leads to (9a) on identifying $g(x)$ with x. Further, Eqs. (9) or (10) show that the differintegral of a constant is not zero.

By rewriting Eq. (4) as

$$cT=i\int_0^h dx\,\sigma'(x)e^{-\rho x}(e^{-2\rho h}-e^{-2\rho x})^{-1/2}$$

$$=-\frac{i}{2\rho}\int_0^h dx\frac{\sigma'(x)e^{\rho x}(-2\rho)e^{-2\rho x}}{(e^{-2\rho h}-e^{-2\rho x})^{1/2}}, \tag{11}$$

with $c^2\rho=g$, we see that Eq. (11) can be written as

$$\Gamma(\tfrac{1}{2})d^{-1/2}\eta(h)/[d\mu(h)]^{-1/2}=2icT\rho, \tag{12}$$

with $\eta(h)=\sigma'(h)\,e^{\rho h}$, and $\mu(h)=e^{-2\rho h}-1$. Equations (11) and (12) define the differintegral of $\eta(h)$ of order $q=-\frac{1}{2}$.

Using the notation $d=d/d\mu(h)$, we convert Eq. (12) to

$$d^{1/2}d^{-1/2}\eta(h)=d^{1/2}(2icT\rho/\sqrt{\pi}), \tag{13}$$

and use the composition rule[15]

$$d^q\,d^Q f=d^{q+Q}f, \tag{14}$$

valid at least for $Q<0$ for general differintegrable $f(h)$. [The reader is urged to turn to Sec. VI for additional comments on (14)]. With Eq. (14), (13) becomes

$$\sqrt{\pi}e^{\rho h}\sigma'(h)=2icT\rho d^{1/2}[1]$$

$$=\frac{2icT\rho}{\sqrt{\pi}}\frac{d}{d\mu(h)}\int_0^h\frac{-2\rho e^{-2\rho x}dx}{(e^{-2\rho h}-e^{-2\rho x})^{1/2}}$$

$$=(2icT\rho/\sqrt{\pi})2[d/d\mu(h)][\mu(h)]^{1/2}$$

$$=(2icT\rho/\sqrt{\pi})(\mu(h))^{-1/2}. \tag{15}$$

Therefore, to obtain the tautochrone we need to integrate the differential equation

$$\sigma'(x)=\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}$$

$$=\frac{2cT\rho}{\pi}e^{-\rho x}(1-e^{-2\rho x})^{-1/2} \tag{16a}$$

With $\pi z=2cT\rho$, we rewrite (16a) as

$$\frac{dy}{dx}=\left[\frac{z^2+1-e^{2\rho x}}{e^{2\rho x}-1}\right]^{1/2} \tag{16b}$$

with the positive sign of the square root chosen for definiteness. Before proceeding to integrate (16b) we emphasize here that, just as for the nonrelativistic case,[9] the technique of fractional calculus has thus proved to be immensely useful toward obtaining an exact expression for the relativistic tautochrone.

Equation (16b) can be integrated exactly using standard methods. We express it first as

$$2\rho y=\int_1^t\frac{ds}{s}\left(\frac{\alpha^2-s}{s-1}\right)^{1/2} \tag{17}$$

with $t=\exp(2\rho x)$, $\alpha^2=z^2+1$. From (17), we get, on carrying through the integration,

$$2\rho y=(\pi/2)(\alpha-1)+\alpha\sin^{-1}w_1(x)$$

$$+\sin^{-1}w_2(x), \tag{18}$$

with

$$z^2w_1(x)=(2+z^2)-2(z^2+1)e^{-2\rho x}$$

and

$$z^2w_2(x)=(2+z^2)-2e^{2\rho x}.$$

Alternatively, one could express the solution in a parametric form by introducing the parameter θ, defined by

$$e^{2\rho x}=1+z^2\cos^2\theta. \tag{19a}$$

Equation (16b) then becomes

$$dy=\tan\theta\,dx$$

$$=-z^2\sin^2\theta\,d\theta/\rho(1+z^2\cos^2\theta)$$

$$=(1/\rho)[1-2(1+z^2)/(a+b\cos 2\theta)]d\theta, \tag{19b}$$

with $a=2+z^2$, $b=z^2$. Integrating (19b), we get

$$\rho y=\theta-\alpha\tan^{-1}[(1/\alpha)\tan\theta]+C,$$

with $\alpha^2=1+z^2$, as before. Since the curve passes through the origin, $C=(\pi/2)(\alpha-1)$, yielding,

$$\rho y=\theta-\frac{\pi}{2}+\alpha\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{\alpha}\tan\theta\right)\right). \tag{19c}$$

Equations (19a) and (19c) together yield the parametric form for the tautochrone. For the rest of this section we propose to check on the derivation for the gravitational tautochrone given above with the help of Eqs. (5b) and (6) in Sec. II. For this purpose we rewrite (5b) as

$$\frac{\lambda c}{2gT} B\left(\frac{c^2\lambda}{2g}, \frac{1}{2}\right) \int_0^\infty dx\, e^{-\lambda x}\sigma'(x) = 1. \qquad (20)$$

Using $\sigma'(x)$ from (16), Eq. (20) becomes

$$\frac{\lambda}{\pi} B\left(\frac{c^2\lambda}{2g}, \frac{1}{2}\right) \int_0^\infty e^{-(\lambda+\rho)x}\, dx(1-e^{-2\rho x})^{-1/2} = 1.$$

As the reader may have anticipated, if the $\sigma'(x)$ given in Eq. (16) is correct, then the above equation should reduce to an identity. With Eq. (7), the integral on the lhs of the above equation becomes

$$\frac{\lambda}{\pi} B\left(\frac{c^2\lambda}{2g}, \frac{1}{2}\right) \frac{1}{2\rho} B\left(\frac{\lambda}{2\rho}+\frac{1}{2}, \frac{1}{2}\right) = 1.$$

On using the definition of the beta functions in terms of the gamma function, the equation is rewritten as

$$(\lambda c^2/2g)\Gamma(c^2\lambda/2g)/\Gamma(c^2\lambda/2g+1) = 1, \qquad (21)$$

which is an identity.

IV. DERIVATION OF THE ELECTRIC TAUTOCHRONE

Consider a particle of charge q and rest mass m falling from a height h under the influence of an electric field of magnitude E. By the law of conservation of energy, we have[6]

$$mc^2 = mc^2/\sqrt{1-v^2/c^2} + qE(x-h). \qquad (22)$$

From (22), we get

$$v = c(1-(1+\alpha(h-x))^{-2})^{1/2}, \qquad (23)$$

with $mc^2\alpha = qE$. Denoting the time of fall by T we then have

$$T = \int_0^T dt = -\int_h^0 \frac{d\sigma}{v(x)} = \frac{1}{c}\int_0^h dx\, \sigma'(x)(1-(1+\alpha(h-x))^{-2})^{-1/2}. \qquad (24)$$

We shall now rework Eq. (24) to the form

$$2\alpha cT = \int_0^h \frac{dx\, \sigma'(x) f'(x)}{(f(h)-f(x))^{1/2}}, \qquad (25)$$

with $f'(x) = df/dx$, $f(x) = \alpha^2(2xh - x^2) + 2\alpha x$, and $\sigma'(x) = d\sigma/dx$, σ being the arclength measured along the tautochrone from the origin. Equation (25) can now be written as a differintegral of $\sigma'(x)$ with respect to $f(x)$, namely,

$$\Gamma\left(\frac{1}{2}\right) \frac{d^{-1/2}\sigma'(h)}{[d(f(h)-f(0))]^{-1/2}} = 2\alpha cT.$$

Since $f(0) = 0$, we rewrite the above equation using the notation $d = d/df(h)$ as

$$d^{-1/2}\sigma'(h) = 2\alpha cT/\sqrt{\pi}. \qquad (26)$$

Thus

$$d^{1/2}d^{-1/2}\sigma'(h) = (2\alpha cT/\sqrt{\pi})d^{1/2}[1]. \qquad (27)$$

As in the Sec. III we shall use the composition rule[15] $d^q d^Q f = d^{q+Q} f$ in (27) and obtain

$$\sigma'(h) = \frac{2\alpha cT}{\sqrt{\pi}} \frac{d}{df(h)}\left[\frac{1}{\sqrt{\pi}}\int_0^h \frac{dx\, f'(x)}{(f(h)-f(x))^{1/2}}\right] = \frac{2\alpha cT}{\pi} 2 \frac{d}{df(h)}(f(h))^{1/2} = 2\alpha cT/(\pi(f(h))^{1/2}). \qquad (28)$$

Equation (28) suggests that for general x,

$$\pi\sigma'(x) = 2\alpha cT(f(x))^{-1/2}. \qquad (29)$$

Equation (29) will now be used to obtain the electric tautochrone. With $\sigma'(x) = (1+(dy/dx)^2)^{1/2}$ and the definition of $f(x)$ given earlier, we obtain, choosing the positive sign of the square root

$$\frac{dy}{dx} = [(\xi^2+\lambda^2)/(\eta^2-\lambda^2)]^{1/2}, \qquad (30)$$

with $\lambda = (1+\alpha(h-x))$, $\eta = (1+\alpha h)$, $\xi^2 = \mu^2 - \eta^2$, and $\pi\mu = 2\alpha cT$. On integrating (30) we get

$$y = \int_0^x dx\left(\frac{\xi^2+\lambda^2}{\eta^2-\lambda^2}\right)^{1/2} = \frac{1}{\alpha}\int_\lambda^\eta d\lambda \left(\frac{\xi^2+\lambda^2}{\eta^2-\lambda^2}\right)^{1/2} = \frac{1}{\alpha}\mu E(\delta,r), \qquad (31a)$$

with $E(\delta,r)$ being the incomplete elliptic integral of the second kind; also, $\delta = \cos^{-1}(\lambda/\eta)$ and $r = \eta/\mu$. In arriving at the last step in (31a), we have used the result[16] (valid for $b>u\geqslant 0$),

$$\int_u^b dx\left(\frac{a^2+x^2}{b^2-x^2}\right)^{1/2}=(a^2+b^2)^{1/2}E(\delta,r) \qquad (31b)$$

with $\delta=\cos^{-1}(u/b)$ and $r=b/(a^2+b^2)^{1/2}$. Thus the relativistic tautochrone associated with a charged particle falling in an uniform electric field E is given as in (31a), by an incomplete elliptic integral of the second kind.

To check on the above procedure, we substitute Eq. (29) into (25) to get the identity

$$\pi=\int_0^h \frac{dx\,f'(x)}{\sqrt{f(x)(f(h)-f(x))}}$$

$$=-\sin^{-1}\frac{-2f(x)+f(h)}{f(h)}\Bigg|_0^h=\pi. \qquad (32)$$

We end this section on a note of caution: namely, the composition rule used in rewriting Eq. (27) as (28) has not been verified explicitly here. As in Sec. III it has been assumed to be valid since $\sigma'(x)$ is differintegrable. We propose to reopen this issue for discussion elsewhere.

V. SOME COMMENTS

There are two parts to this section. To begin with, we compare the tautochrone obtained in (31a) with the brachistochrone worked out by Goldstein and Bender.[6] Later on, we shall compute the power radiated by an accelerated charge as it moves along the tautochrone.

As shown in Ref. 6, the relativistic brachistochrone has two new features relative to its Newtonian counterpart.

(a) There are three kinds of curves, each associated with three intervals over which a constant parameter k can vary, viz. $k^2\in(0,1)$, $k^2=1$, and $k^2>1$.

(b) The curve corresponding to $k^2=1$ is the boundary between the $0<k^2<1$ solutions (which increase without bound in the x direction) and $k^2>1$ solutions; the latter curve back to the y axis just as the cycloid does in the nonrelativistic case. In the tautochrone case however, as seen in Sec. II there is only one curve. Besides, being an incomplete elliptic function of the second kind $E(\delta,r)$ whose amplitude $\delta=\cos^{-1}(1-\alpha x/\eta)$ lies in $(\pi/2,0)$ for $x\in(h,0)$, it is a bounded function of x. Elliptic functions of the second kind are known to be doubly periodic; however, for $\delta\in(\pi/2,0)$, their periodicity properties will not be observed.[17]

There are therefore some qualitative differences between the relativistic versions of the brachistochrone and the tautochrone.

From the point of view of physical applications, we shall now work out briefly below the Larmor formula for the tautochrone. Formally, it is given by[10]

$$P(t)=\tfrac{2}{3}(q^2/4\pi c^3)\left(1-\frac{v^2}{c^2}\right)^{-3}\left(\dot{\mathbf{v}}^2-\left(\frac{\mathbf{v}}{c}\times\dot{\mathbf{v}}\right)^2\right). \qquad (33)$$

Here $P(t)$ is the power radiated as a function of time t and $\mathbf{v}$ and $\dot{\mathbf{v}}$ denote the velocity and acceleration vectors of the charge.

Using Eq. (30) they can be worked out easily. Thus

$$\mathbf{v}=\dot{x}\mathbf{i}+\dot{y}\mathbf{j}=\dot{x}(\mathbf{i}+s\mathbf{j}), \qquad (34a)$$

$$\dot{\mathbf{v}}=\ddot{x}\mathbf{i}+(\dot{x}\dot{s}+s\ddot{x})\mathbf{j}, \qquad (34b)$$

with $(\eta^2-\lambda^2)^{1/2}s=(\xi^2+\lambda^2)^{1/2}$. Calculating the derivatives and substituting the relevant expressions into (33) we obtain

$$P(t)=\frac{2}{3}\frac{q^2\alpha^2}{4\pi c^3\mu^2}\left(1-\frac{v^2}{c^2}\right)^{-3}\left[v^4(\xi^2+\lambda^2)^{-1}+c^4(\eta^2-\lambda^2)\left(1-\frac{v^2}{c^2}\right)^3\right]. \qquad (35)$$

For comparison, we note that the counterpart of Eq. (35) for the relativistic brachistochrone is given by[10]

$$P(t)=\frac{2}{3}\frac{q^2\alpha^2}{4\pi}c\left(1-\frac{v^2}{c^2}\right), \qquad (36)$$

for the $k^2=1$ case, with a similar expression for $k^2>1$ (or <1) case. Clearly there is a dramatic difference between the two expressions for the power radiated as $v\to c$. We ascribe this to the different x dependences for the two curves.

VI. DISCUSSION AND SUMMARY

In this paper we have used the methods of the fractional calculus[8] to derive the relativistic tautochrone for a particle falling from rest in an uniform force field. Despite a successful check on the correctness of the procedure, an obvious weak link remains; namely, the applicability of the composition rule given by Eq. (14) has not been discussed *ab initio*. Indeed, the rule given by (14) is valid if an only if[15] $f-d^{-Q}d^{Q}f=0$. This is not particularly useful when $Q=\frac{1}{2}$; we have therefore taken recourse to the argument[15] in Sec. III, that since f is differintegrable, Eq. (14) is valid for $Q<0$. It has enabled us to express the lhs of Eq. (13) for example, as $\eta(h)$ in the first equality in Eq. (15).

To conclude, we would like to mention that an obvious extension of the calculations done here would be to relax the restriction to a uniform gravitational field imposed in this paper. Specifically, we have in mind the inclusion of the effects of general relativity. Such an exercise is presently in progress and will be reported elsewhere.

APPENDIX: SOLUTION BY ABEL'S METHOD

Herein we present (briefly) an adaptation of the method of Abel[4] for the gravitational tautochrone. Returning to Eq. (4) in the text, we rewrite it as ($c^2\rho = g$),

$$cT=\int_0^x dz\,\sigma'(z)(1-\exp 2\rho(z-x))^{-1/2}. \tag{A1}$$

Thus

$$\begin{aligned}\int_0^u cT\,dx(1-\exp 2\rho(x-u))^{-1/2} &= \int_0^u dx \int_0^x dz\,\sigma'(z)(1-\exp 2\rho(x-u))^{-1/2}\\ &\quad\times(1-\exp 2\rho(z-x))^{-1/2}\\ &= \int_0^u dz\,\sigma'(z)\int_z^u dx(1-\exp 2\rho(x-u))^{-1/2}\\ &\quad\times(1-\exp 2\rho(z-x))^{-1/2}. \end{aligned}\tag{A2}$$

The integral over x in (A2) can be evaluated as follows. Denoting it by I, we get, after some elementary algebra,

$$\sqrt{2}\rho I=e^{a/2}\int_0^a d\lambda(\cosh a-\cosh\lambda)^{-1/2}, \tag{A3}$$

with $\rho(u-z)=a$. Equation (A3) can be calculated using the standard integral[18]

$$\int_0^a \frac{\cosh(\gamma+\frac{1}{2})x\,dx}{(\cosh a-\cosh x)^{\nu+1/2}} = \sqrt{\frac{\pi}{2}}\frac{\Gamma(\frac{1}{2}-\nu)}{\sinh a}P_\gamma^\nu(\cosh a), \tag{A4}$$

valid for $\mathrm{Re}\,\nu<\frac{1}{2}$, $a>0$. Putting $\gamma=-\frac{1}{2}$, $\nu=0$ in (A4) we recover the result for (A3) as

$$\begin{aligned}2\rho I&=\pi e^{a/2}P^0_{-1/2}(\cosh a)\\ &=[4/(1+e^{-a})]K[\tanh(a/2)].\end{aligned}\tag{A5}$$

In (A5), $P^0_{-1/2}(\cosh a)$ denotes the associated Legendre function of the first kind (P_ν^μ) of degree $\mu=0$ and order $\nu=-\frac{1}{2}$. Its relation[19] to the complete elliptic integral $K[\tanh(a/2)]$ has been used in arriving at the second equality in (A5). Thus we obtain after substitution in (A2) the result

$$\int_0^u \rho cT\,dx(1-\exp 2\rho(x-u))^{-1/2} =2\int_0^u dx\,\sigma'(x)\frac{K(\tanh(a/2))}{1+e^{-a}}. \tag{A6}$$

Since $\rho(u-x)=a$, the integrand on the rhs of Eq. (A6) is a function of both u and x; on account of this, we get, on differentiating (A6) with respect to u, the result

$$\begin{aligned}\rho cT\frac{d}{du}\int_0^u dx(1-\exp 2\rho(x-u))^{-1/2} &=\frac{\pi}{2}\sigma'(u)+2\int_0^u dx\,\sigma'(x)\\ &\quad\times\frac{d}{du}\left\{\frac{K[\tanh(a/2)]}{1+e^{-a}}\right\}.\end{aligned}\tag{A7}$$

We now note the following.

(a) The integral on the lhs can be calculated exactly; this, in fact, turns out to be a logarithm function and thus the lhs of Eq. (A7) is known.

(b) On the rhs, however, the integral still involves the unknown $\sigma'(x)$; this renders the method of Abel[4] less useful as far as the determination of $\sigma'(x)$ is concerned for the relativistic tautochrone.

(c) By contrast, for the nonrelativistic case, the counterpart of Eq. (A6) above is[4]

$$\int_0^z T\left(\frac{2g}{(z-y)}\right)^{1/2}dy=\pi\int_0^z \sigma'(y)dy. \tag{A8}$$

Thereby one obtains

$$\pi\sigma'(z)=\frac{d}{dz}\int_0^z T\left(\frac{2g}{(z-y)}\right)^{1/2}dy. \tag{A9}$$

Equation (A9) leads to the nonrelativistic tautochrone, namely the cycloid, quite easily.

Thus the utility of Abel's method for the nonrelativistic problem is more obvious than for the relativistic counterpart. Clearly, therefore, the methods of fractional calculus used herein in the text, turn out to be more fruitful than those of the Laplace transform[3] or that of Abel.[4]

[1] For example, J. A. Cochran, *The Analysis of Linear Integral Equations* (McGraw-Hill, New York, 1972).
[2] A conjecture that the curve is a cycloid is attributed to C. Huygens (see Ref. 1, p. 9).
[3] See, for example, M. R. Spiegel, *Theory and Problems of Laplace*

Transforms (McGraw-Hill, Singapore, 1986).
[4] Here we have in mind Abel's solution for the generalized tautochrone problem (see Ref. 1, p. 7).
[5] We shall use the terms electric (gravitational) tautochrone to describe the relativistic tautochrone associated with the electric (gravitational) field.
[6] H. F. Goldstein and C. M. Bender, J. Math. Phys. **27**, 507 (1986).
[7] To obtain the electric tautochrone it turns out that one has to calculate the inverse Laplace transform of the reciprocal of the modified Bessel function $K_1(x)$.
[8] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic, New York, 1974).
[9] See Ref. 8, pp. 183–186.
[10] For the relativistic brachistochrone, the power radiated by an accelerated charge has been worked out in S. G. Kamath and V. V. Sreedhar, Phys. Rev. A **36**, 2478 (1987).
[11] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980), p. 305, Eq. 3.312.1.
[12] See Ref. 8, p. 49.
[13] A term that has been coined by Oldham and Spanier, Ref. 8. It may be mentioned here that for $q = \frac{1}{2}$, Eqs. (9a) and (9b) lead to the concept of a semiderivative introduced by R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley–Interscience, New York, 1962), Vol. II.
[14] This generalization is due to T. J. Osler, SIAM J. Appl. Math. **18**, 658 (1970).
[15] See Ref. 8, p. 85.
[16] See Ref. 11, p. 276, Eq. 3.169.4.
[17] See, for example, *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1968), p. 594.
[18] See Ref. 11, p. 346, Eq. 3.517.2.
[19] See Ref. 17, p. 337, Eq. 8.13.2.