Relativistic shock formation in the presence of radial entropy gradients

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The nonlinear steepening of relativistic acoustic waves is investigated. The nonlinear evolution of a planar wave in a homentropic flow field is understood well through relativistic simple waves. However, in situations where the wave is nonplanar and the flow field is nonhomentropic, the concept of simple waves cannot be used. In the present paper, effect of entropy gradients on the nonlinear distortion of a spherical wave is analyzed using the wave front expansion technique. It is shown that the behavior of a relativistic wave in nonhomentropic environment is slightly different from the nonrelativistic wave. A closed form solution is obtained for the slope at the wave front. A general criterion for a compression wave to steepen into a shock is obtained. The distortion of compression and rarefaction waves is examined for some well known equations of state. However, the method used in this paper is general and can be easily extended to analyze shock formation in any fluid. Also, expressions for time and location of shock formation are obtained. The effects of gravity or self-gravity are not taken into account in this paper. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1904083]

I. INTRODUCTION

The profile of finite amplitude waves gets distorted due to the nonlinearity of the evolution equation. This can lead to formation of discontinuities in flow quantities. Physically, this is interpreted as shock formation. In reality, however, only those waves that survive dissipation by viscosity and heat conduction can form shocks.

The steepening of relativistic acoustic waves in a homentropic environment has been well understood using simple waves. Simple waves in relativistic fluid dynamics were first studied by Taub[1] who introduced them through the Riemann invariants. Subsequently, they were analyzed in detail in several contexts by Liang.[2] Liang studied the process of shock formation and obtained a general formula for the damping and entropy production rate as functions of shock strength. Anile *et al.*[3] performed a detailed analysis of the formation of strong relativistic shock waves and their subsequent damping by numerically integrating the equations of special relativistic fluid dynamics in one dimension. The breaking of relativistic simple waves in magnetofluid dynamics was studied by Muscato.[4]

The characteristics of a simple wave have constant slopes and the Riemann invariants are constant along these characteristics. However, in the presence of spherical geometry and entropy gradients, the Riemann variants of the system are not constant along the characteristics. As a result, the time of shock formation is underestimated if the effect of the spherical geometry is neglected. In the present paper, the analysis is performed for the case when the effects of spherical geometry and density variation are taken into consideration.

The process of shock formation is very important in astrophysical situations. Hanasz[5] has explained that the Kelvin–Helmholtz instability in relativistic jets leads to nonlinear steepening of acoustic waves and formation of shocks. In some jets the light comes primarily from a regularly spaced series of bright knots. Rees[6] suggested that the knots can be attributed to the steepening of nonlinear acoustic waves in the jet. Liang[2] analyzed the nonlinear evolution of adiabatic perturbations in the early universe using simple waves. Also, in models of a spherical accretion, formation of shocks is an important feature. These waves are affected by the gravitational effects and by the base flow. Also, these waves travel through a region of varying thermodynamic properties due to the presence of density gradients.

In this paper, the effects of density gradients and spherical geometry on the steepening of both inward and outward traveling waves are analyzed. However, in the present analysis, effects of gravity and base flow are not taken into consideration. The results presented in this paper are general and have implications on other areas of physics as well. Also, the results presented in this paper can be used for testing the accuracy of numerical codes.

The steepening of nonrelativistic acoustic waves in the presence of spherical geometry and entropy gradients has been studied by various authors. The effect of spherical geometry was discussed by Appert *et al.*[7] in the context of nucleation of liquids and by Lin and Szeri[8] in the context of sonoluminescence effect in bubbles. Lin and Szeri,[8] Tyagi and Sujith,[9] and Muralidharan and Sujith[10] investigated the steepening of nonrelativistic acoustic waves in the presence of entropy gradients. They used the wave front expansion technique to obtain an evolution equation for the slope at the wave front. In the present paper, the analysis is further generalized to describe the case of nonlinear distortion of relativistic acoustic waves in the presence of density gradients. Also the effect of spherical geometry can be analyzed using this method.

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The rest of this paper is organized as follows. In Sec. II, the relativistic fluid dynamic equations were manipulated to obtain the equations in the characteristic form. A closed form solution for the evolution of slope at the wave front was obtained in Sec. III using the method of wave front expansion. Also, expressions for the time and location of shock formation were derived. Some examples highlighting the effects of spherical geometry and entropy gradients are presented in Secs. IV and V. Hereafter, the system of units is chosen such that the speed of light $c=1$.

II. GOVERNING EQUATIONS

The relativistic fluid dynamics equations describing the radial flow of a perfect gas are the following:[11]
Baryon conservation

$$\left(\frac{\partial n\gamma}{\partial t}+\frac{\partial(nu\gamma)}{\partial r}+\frac{2nu\gamma}{r}\right)=0, \tag{1}$$

Euler's equation

$$\frac{\partial(w\gamma^2u)}{\partial t}+\frac{\partial(w\gamma^2u^2)}{\partial r}+\frac{2w\gamma^2u^2}{r}+\frac{\partial p}{\partial r}=0, \tag{2}$$

energy equation

$$\frac{\partial(w\gamma^2)}{\partial t}+\frac{\partial(w\gamma^2u)}{\partial r}+\frac{2w\gamma^2u}{r}-\frac{\partial p}{\partial t}=0. \tag{3}$$

In the above equations, n is the baryon number density, $u=dr/dt$ is the spatial velocity, $\gamma=1/\sqrt{1-u^2}$ is the Lorentz factor, $w=e+p$, e is the relativistic internal energy per unit volume (including the rest mass), p is the thermodynamic pressure, and w is the enthalpy per unit volume. The fluid quantities n,p, and w are related by the first law of thermodynamics:

$$Tds=d\left(\frac{w}{n}\right)-\frac{dp}{n}, \tag{4}$$

where s is entropy per baryon and T is the absolute temperature. Using Eqs. (1), (3), and (4), the energy equation can be reduced to

$$\left(\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial r}\right)=0. \tag{5}$$

In general, the above equations are solved in conjunction with an equation of state. A general equation of state of the form $e=e(p,s)$ is assumed. Hence,

$$de=\left(\frac{\partial e}{\partial p}\right)_s dp+\left(\frac{\partial e}{\partial s}\right)_p ds. \tag{6}$$

The derivatives of w in Eqs. (2) and (3) are eliminated using Eqs. (4)–(6) to yield

$$\frac{(1-u^2)}{w\alpha}\left(\frac{\partial p}{\partial t}+\frac{u+\alpha}{1+u\alpha}\frac{\partial p}{\partial r}\right)+\left(\frac{\partial u}{\partial t}+\frac{u+\alpha}{1+u\alpha}\frac{\partial u}{\partial r}\right)+\frac{2u(1-u^2)\alpha}{(1+u\alpha)r}=0, \tag{7}$$

$$-\frac{(1-u^2)}{w\alpha}\left(\frac{\partial p}{\partial t}+\frac{u-\alpha}{1-u\alpha}\frac{\partial p}{\partial r}\right)+\left(\frac{\partial u}{\partial t}+\frac{u-\alpha}{1-u\alpha}\frac{\partial u}{\partial r}\right)-\frac{2u(1-u^2)\alpha}{(1+u\alpha)r}=0, \tag{8}$$

where α is the speed of sound, given by $\alpha^2=(\partial p/\partial e)_s$.

Equations (7) and (8) are equivalent to

$$\frac{d^+J_+}{dt}+\frac{2u\alpha}{(1+u\alpha)r}=0, \tag{9}$$

$$\frac{d^-J_-}{dt}-\frac{2u\alpha}{(1-u\alpha)r}=0, \tag{10}$$

where

$$J_+=\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)+\int\frac{dp}{w\alpha},\quad J_-=\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)-\int\frac{dp}{w\alpha},$$

and d^+/dt and d^-/dt are the derivatives along the characteristics, given by

$$\frac{d^+}{dt}=\frac{\partial}{\partial t}+\left(\frac{u+\alpha}{1+u\alpha}\right)\frac{\partial}{\partial r},\quad \frac{d^-}{dt}=\frac{\partial}{\partial t}+\left(\frac{u-\alpha}{1-u\alpha}\right)\frac{\partial}{\partial r}.$$

In the nonrelativistic limit, the above equations reduce to the characteristic equations obtained by Appert *et al.*[7] and Lin and Szeri.[8] The system of Eqs. (5), (9), and (10) are equivalent to the system (1)–(3). However, in this system all the equations are in the characteristic form. The Riemann variables of the system are J_+,J_-, and s, and the respective characteristic velocities are C_+,C_-, and u, where

$$C_+=\left(\frac{u+\alpha}{1+u\alpha}\right),\quad C_-=\left(\frac{u-\alpha}{1-u\alpha}\right).$$

Any disturbance will propagate along the characteristics. In the case of simple waves, the Riemann variables are constant along the characteristics. However, here the presence of spherical geometry causes the Riemann variable to change along the characteristics. The wave front expansion technique is used to determine the rate of steepening of the leading edge of the wave front. This method neglects the possibility of shock formation in the middle of the wave.[8,10] In the case when the entropy changes rapidly within the wave, the shock can form in the middle of the wave. So, it is assumed that the entropy does not vary rapidly within the wave. Further, this method is limited to a special form of pure compression or rarefaction wave, which has a discontinuity in its derivative. However, it is possible to get a rough estimate of the time and location of shock formation for a wave with continuous profile.[12] This estimate is quite accurate for waves with high frequencies. Tyagi and Sujith[9] showed that the slope at the wave front can be expressed as the product of amplitude and frequency for a sinusoidal wave in the nonrelativistic case, by comparing the solution obtained using wave front expansion technique with high-frequency solution. The effects of spherical geometry and entropy gradients (or temperature gradients) can be determined using this method. It is also possible to obtain solution in closed form for a general equation of state.

III. NONLINEAR STEEPENING OF A WAVE FRONT

A. Evolution equation

Whitham[12] has shown that for a hyperbolic system a discontinuity in the derivatives propagates along the characteristics. In this paper, a wave with compact support having discontinuity in its first derivative is considered. In general, it is difficult to deal with a complex wave having both a compressive and an expansive part. Hence, pure compression and expansion waves will be treated separately in this paper. The equation of the wave front can be written as $r=R(t)$. A flow variable Λ is expanded in powers of ξ at the wave front, where $\xi=r-R(t)$ as

$$\Lambda(\xi,t) = \Lambda_0[R(t)] + \Lambda_1(t)\xi + \Lambda_2(t)\frac{\xi^2}{2} + \cdots, \quad \text{for } \xi < 0, \tag{11}$$

$$\Lambda(\xi,t) = \Lambda_0[R(t)] + \Lambda_0'[R(t)]\xi + \Lambda_0''[R(t)]\frac{\xi^2}{2} + \cdots, \quad \text{for } \xi > 0, \tag{12}$$

where " $'$ " indicates spatial derivative. Λ indicates u,s, or p, and Λ_1,Λ_2, ... denote the corresponding spatial derivatives behind the wave front. $\xi>0$ is the undisturbed quiescent flow field and $\xi<0$ is the disturbed region where the flow is unsteady. The quantities with subscript "0" are the known flow quantities in the quiescent field. The leading edge of an outward traveling wave and an inward traveling wave propagates with velocities C_+ and C_-, respectively. Since $\xi>0$ is undisturbed, $u_0[R(t)]=0$. Hence, the velocity of the wave front is given by

$$\dot{R}(t) = \left(\frac{u \pm \alpha}{1 \pm u\alpha}\right)_{\xi=0} = \pm \alpha_0[R(t)], \tag{13}$$

where " $\dot{}$ " indicates time derivative. Hereafter, "+" corresponds to an outward traveling wave and "−" corresponds to an inward traveling wave. It is known from thermodynamics that $\alpha=\alpha(p,s)$ and $w=w(p,s)$. Hence,

$$\alpha(p,s) = \alpha_0[R(t)] + \left[\left(\frac{\partial\alpha}{\partial p}\right)_s p_1 + \left(\frac{\partial\alpha}{\partial s}\right)_p s_1\right]\xi + \cdots, \tag{14}$$

$$w(p,s) = w_0[R(t)] + \left[\left(\frac{\partial w}{\partial p}\right)_s p_1 + \left(\frac{\partial w}{\partial s}\right)_p s_1\right]\xi + \cdots. \tag{15}$$

The derivatives with respect to t can be obtained using

$$\left[\frac{\partial}{\partial t}\right]_x = \left[\frac{\partial}{\partial t}\right]_\xi + \left[\frac{\partial\xi}{\partial t}\right]_x\left[\frac{\partial}{\partial\xi}\right] = \frac{\partial}{\partial t} - \dot{R}(t)\frac{\partial}{\partial\xi}. \tag{16}$$

Substituting the above power series in Eqs. (5), (7), and (8) and equating the coefficients of ξ^0 we obtain

$$s_0' - s_1 = 0, \tag{17}$$

$$p_0' = 0, \tag{18}$$

$$p_1 \pm w_0 u_1 \alpha_0 = 0. \tag{19}$$

Equating coefficients of ξ^1,

$$\dot{p}_1 + u_1\alpha_0^2(p_0' - p_1) + u_1 p_1 \pm \alpha_1 p_1 \pm w_0\alpha_0[\dot{u}_1 + u_1^2(1-\alpha_0^2) \pm \alpha_1 u_1] + \frac{2u_1 w_0\alpha_0^2}{R(t)} = 0, \tag{20}$$

where α_1 is the coefficient of ξ^1 in the expression for $\alpha(p,s)$ given by

$$\alpha_1 = \left[s_1\left(\frac{\partial\alpha}{\partial s}\right)_p + p_1\left(\frac{\partial\alpha}{\partial p}\right)_s\right].$$

Equations (17)–(20) can be manipulated to yield

$$\dot{u}_1 + \left[(1-\alpha_0^2) + \left(\frac{\partial\alpha}{\partial p}\right)_s w_0\alpha_0\right]u_1^2 \pm \left[\left(\frac{\partial\alpha}{\partial s}\right)_p s_0' + \frac{w_0'\alpha_0 + w_0\alpha_0'}{2w_0} + \frac{\alpha_0}{R(t)}\right]u_1 = 0. \tag{21}$$

Using the fact that $p_0'=0$, the above equation can be reduced to

$$\dot{u}_1 + \frac{\varsigma}{2}u_1^2 \pm \left[\frac{w_0'\alpha_0 + 3w_0\alpha_0'}{2w_0} + \frac{\alpha_0}{R(t)}\right]u_1 = 0, \tag{22}$$

where ς is the relativistic compressibility parameter given by

$$\varsigma = 2\left[(1-\alpha_0^2) + \left(\frac{\partial\alpha}{\partial p}\right)_s w_0\alpha_0\right]. \tag{23}$$

The fluid is said to be thermodynamically normal if $\varsigma>0$ and it is said to be thermodynamically anomalous if $\varsigma<0$.[13]

In the nonrelativistic limit and in the absence of spherical geometry, the above equation reduces to the equation obtained by Muralidharan and Sujith[10] for a general equation of state. The coefficient of u_1 becomes zero for a simple wave. Hence, for a simple wave, Eq. (22) reduces to

$$\dot{u}_1 + \frac{\varsigma}{2}u_1^2 = 0. \tag{24}$$

It is easier to deal with the position of wave front $R(t)$ as an independent variable instead of t. Therefore, Eq. (22) can be rewritten as

$$\frac{du_1}{dy} + \left(\frac{w_0'\alpha_0 + 3w_0\alpha_0'}{2w_0\alpha_0} + \frac{1}{y}\right)u_1 \pm \frac{\varsigma}{2\alpha_0}u_1^2 = 0. \tag{25}$$

Here, $y=R(t)$ is the position of the wave front. The above nonlinear equation can be transformed into a linear differential equation in $1/u_1$:

$$\frac{d}{dy}\left(\frac{1}{u_1}\right) - \left(\frac{w_0'}{2w_0} + \frac{3\alpha_0'}{2\alpha_0} + \frac{1}{y}\right)\frac{1}{u_1} = \pm\frac{\varsigma}{2\alpha_0}. \tag{26}$$

If r_0 denotes the initial position of the wave front, then the solution to the above equation with an initial slope of $u_1(r_0)$ can be written as

$$\frac{1}{u_1(y)} = \frac{1}{u_1(r_0)}\frac{IF(r_0)}{IF(y)} \pm \frac{1}{IF(y)}\int_{r_0}^{y} \frac{\varsigma(y)}{2\alpha_0(y)} IF(y)dy, \quad (27)$$

where $IF(y)$ is the integrating factor given by

$$IF(y) = \exp\left[-\int\left(\frac{w_0'\alpha_0 + 3w_0\alpha_0'}{2w_0\alpha_0} + \frac{1}{y}\right)dy\right] = y^{-1}[w_0(y)]^{-1/2}[\alpha_0(y)]^{-3/2}. \quad (28)$$

Equation (27) describes the nonlinear evolution of a wave front moving into a gas governed by a general equation of state in the presence of entropy (or density) gradients. The slope at the wave front in the case of nonhomentropic environment is determined by specifying a spatial variation of the thermodynamic properties of the undisturbed medium. A shock forms when $|u_1(y)| \to \infty$. In a homentropic environment, the thermodynamic properties of the undisturbed medium α_0, p_0, e_0, and w_0 do not vary with y, since $p_0' = 0$. The integrating factor $IF(y)$ is a constant for a simple wave. Hence, in the case of planar geometry, only compression waves (negative slope) can steepen into a shock in thermodynamically normal fluid, while rarefaction waves (positive slope) can steepen into a shock only in a thermodynamically anomalous medium. Also, for the shock to be stable, the mechanical stability criterion obtained by Bugaev and Gorenstein[13] must be satisfied. Bugaev and Gorenstein[13] showed that compression shocks are stable in thermodynamically normal medium and rarefaction shocks are stable in thermodynamically anomalous medium. In this paper, the compressibility parameter is assumed to be positive.

B. Shock formation

In this section, expressions for time and location of shock formation are obtained for both inward and outward traveling disturbances.

1. Inward traveling disturbance

In the case of an inward traveling disturbance, $|u_1(y)| \to \infty$ at the center. Hence, an inward traveling disturbance forms a shock either before reaching the center or at the center. Further, if it forms a shock before reaching the center, the location of shock formation is given by

$$\frac{1}{u_1(r_0)} IF(r_0) = \int_{r_0}^{y_s} \frac{\varsigma(y)}{2\alpha_0(y)} IF(y)dy, \quad (29)$$

where y_s is the shock formation location. The time of shock formation can be obtained from

$$t_s = -\int_{r_0}^{y_s} \frac{dy}{\alpha_0(y)}. \quad (30)$$

From Eq. (29) the following assertions can be made.

(i) Under the compressibility assumption $\varsigma > 0$, only compression waves can steepen into shocks.

(ii) Only those compression waves with slopes greater than a minimum value can steepen into shocks before reaching the center. The condition for this is given by

$$\frac{|u_1(r_0)|}{IF(r_0)} > \frac{1}{\max\left|\int_{r_0}^{y_s} \frac{\varsigma(y)}{2\alpha_0(y)} IF(y)dy\right|} = \frac{1}{\int_0^{r_0} \frac{\varsigma(y)}{2\alpha_0(y)} IF(y)dy}. \quad (31)$$

If the improper integral in this equation diverges, then all compression waves will steepen into shocks. It is clear from Eq. (27) that the compression waves, which do not blow up before reaching the center, along with all expansion waves develop into infinitesimal shocks at the center. However, these infinitesimal shocks may not occur in reality due to the action of diffusion close to the center. The effect of diffusion cannot be neglected when the wave front is close to the origin. This has been explained by Lin and Szeri[8] for the nonrelativistic case.

2. Outward traveling disturbance

It is clear from Eq. (27) that an outward traveling disturbance will steepen into a shock only if it is compressive. It is also clear that for a compression wave to steepen into a shock,

$$\frac{|u_1(r_0)|}{IF(r_0)} > \frac{1}{\max\left|\int_{r_0}^{y_s} \frac{\varsigma(y)}{2\alpha_0(y)} IF(y)dy\right|} = \frac{1}{\int_{r_0}^{\infty} \frac{\varsigma(y)}{2\alpha_0(y)} IF(y)dy}. \quad (32)$$

If y_s is the location of shock formation, then the time of shock formation is given by

$$t_s = \int_{r_0}^{y_s} \frac{dy}{\alpha_0(y)}. \quad (33)$$

The results derived in this section can be better understood through the following examples.

IV. HOMENTROPIC ENVIRONMENT

In a homentropic environment, the entropy is a constant. Hence, in the undisturbed medium the equation of state can be written as $e_0 = e_0(p_0)$. Since p_0 is a constant when there is no base flow [Eq. (18)], α_0 and ς do not vary with y. The slope at the wave front is obtained from Eq. (27) as

$$\frac{1}{u_1(y)} = y\left[\frac{1}{r_0 u_1(r_0)} \pm \frac{\varsigma(r_0)}{2\alpha_0(r_0)} \ln\left(\frac{y}{r_0}\right)\right]. \quad (34)$$

Since the velocity of the wave front is independent of y, it is possible to write the wave front position y as

$$y = r_0 \pm \alpha_0 t.$$

Equation (34) can be nondimensionalized as follows:

$$\tilde{t} = \frac{\alpha_0 t}{r_0}, \quad \tilde{u}_1(\tilde{t}) = r_0 u_1(t).$$

Hence, Eq. (34) can be written as

$$\frac{1}{\tilde{u}_1(\tilde{t})} = (1 \pm \tilde{t})\left[\frac{1}{\tilde{u}_1(0)} \pm \frac{\mathrm{s}(0)}{2\alpha_0}\ln(1 \pm \tilde{t})\right]. \tag{35}$$

Therefore, in a homentropic flow field, every compression wave will steepen into a shock. The time of shock formation t_s is given by

$$\tilde{t}_s = \exp\left(\frac{-2\alpha_0(0)}{\tilde{u}_1(0)\mathrm{s}(0)}\right) - 1, \tag{36}$$

for an outward traveling compressive disturbance,

$$\tilde{t}_s = 1 - \exp\left(\frac{2\alpha_0(0)}{\tilde{u}_1(0)\mathrm{s}(0)}\right), \tag{37}$$

for an inward traveling compressive disturbance. Time of shock formation is hence found to be lesser in a fluid with greater compressibility parameter.

The distortion of a wave front depends both on its initial slope and the radial distance. Every inward traveling compression wave front steepens throughout its evolution period before it turns into a shock [see Fig. 1(a)]. However, for waves with very small initial slope, the location of shock formation is closer to the center where diffusive effects become important.

Figure 1(b) shows that an outward traveling compression wave front with a very small initial slope tends to relax initially. This is due to the effect of spherical geometry. However after a large distance, the geometrical relaxation effect becomes negligible and at this point, the wave behaves like a planar wave. It is known that every planar wave (compressive) will steepen into a shock. Hence, a compression wave with very small initial slope tends to relax until it reaches a minimum value and then it steepens to form a shock. However, the shock formation distance is extremely large. In reality, for such extremely large shock formation distances, the wave will get damped before it steepens into a shock, due to dissipative effects. Compression waves with large initial slope steepen throughout their evolution period.

Figure 2 shows that even inward traveling expansion wave fronts steepen to form infinitesimal shocks at the center. This phenomenon is not observed in the Cartesian geometry. However, these infinitesimal rarefaction shocks are not stable. As it was mentioned earlier, the effects of diffusion become important near the origin and hence, such a shock may not occur in reality.[8] Also, rarefaction shock waves can become unstable in thermodynamically normal medium.

A planar right running wave can be considered as the limiting case of an outward traveling spherical disturbance with $r_0 \to \infty$. Hence, from Eq. (34), the time of shock formation for a planar wave is obtained as

$$t_s = \frac{-2}{\mathrm{s}(0)u_1(0)}. \tag{38}$$

On comparing Eqs. (36) and (38) it can be deduced that the time of breaking of an outward traveling spherical compres-

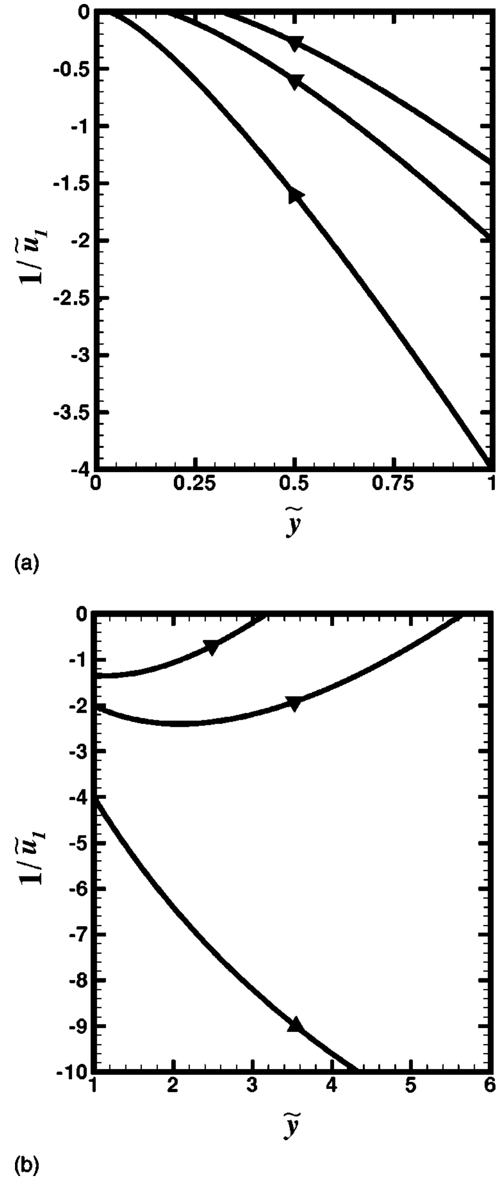


FIG. 1. (a) shows the evolution of inward traveling compression waves with different initial slopes as a function of wave front position. The compressibility parameter is taken as 4/3. (b) shows the evolution of outward traveling compression waves with different slopes as a function of wave front position. The compressibility parameter is taken as 4/3.

sion wave is more than that of a planar compression wave. The behavior of a left running planar wave is similar to that of a right running wave.

The time of breaking of a simple (planar) wave for a barotropic fluid with equation of state $p=(\eta-1)e$ is given by[2,4]

$$t_B = \min\left(-\frac{(1+f(\xi)\alpha)^2}{(1-\alpha^2)f'(\xi)}\right), \tag{39}$$

where $\xi = x - C_+ t$, $f(\xi)$ is the initial profile of the wave, and $\alpha = \sqrt{\eta - 1}$. Since it is assumed that there is no base flow, $f(0)=0$. As was mentioned earlier, it is assumed that the shock forms at the leading edge of the wave front. Hence, if it is assumed that the shock forms at the leading edge of the wave in the above analysis, the minimum value of t_B will

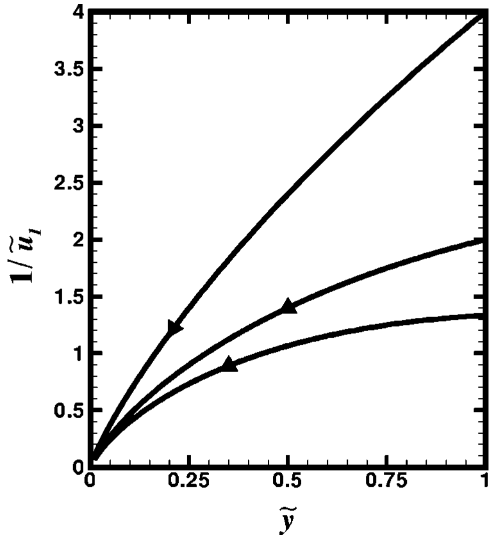


FIG. 2. The evolution of inward traveling rarefaction waves with different initial slopes as a function of wave front position. Rarefaction waves also steepen due to the effect of spherical geometry leading to the formation of infinitesimal shocks at the center. The compressibility parameter is taken as 4/3.

occur at $\xi=0$. Under these assumptions, Eq. (39) will reduce to

$$t_B = \frac{1}{(1-\alpha^2)|f'(0)|}. \tag{40}$$

Hence Eq. (38) agrees with the result obtained using simple waves [Eq. (40)]. When the initial profile is sinusoidal, the shock forms in the middle of the wave. However, the time scales obtained using wave front expansion technique and simple waves are similar.[12]

This example highlights the effect of spherical geometry on the steepening of the compression waves. This method can also be used to understand the effects of variation in the compressibility parameter due to density gradients.

V. NONHOMENTROPIC ENVIRONMENT

In this section, steepening of acoustic waves in a polytropic gas and Synge[14] gas are analyzed. Both these type of gases have positive compressibility parameter.

A. Polytropic gas

In this example, the effect of density gradients on the nonlinear steepening of the wave front is analyzed. For this purpose, let us consider the class of fluids that obey the polytropic equation of state with index η,

$$p = K(s)n^{\eta}, \quad 1 \leqslant \eta \leqslant 2.$$

This state equation is a reasonable approximation to the average thermodynamic properties of stellar material and its simple expression makes it useful for analytical calculations. For a polytropic gas,

$$e = mn + \frac{p}{\eta - 1} \tag{41}$$

and

$$\alpha^2 = \frac{\eta(\eta-1)p}{(\eta-1)mn + \eta p} = \frac{\eta p}{w}, \tag{42}$$

where m is the rest mass. The variables in the quiescent medium α_0, p_0, e_0, and w_0 also satisfy the above relation. The compressibility parameter of the undisturbed medium can be obtained using Eq. (23) as

$$\varsigma = (\eta + 1) - 3\alpha_0^2. \tag{43}$$

Thus, the slope at the wave front can be obtained by substituting the above relations in Eq. (27),

$$\frac{1}{u_1(y)} = \frac{1}{u_1(r_0)}\frac{IF(r_0)}{IF(y)} \pm \frac{1}{IF(y)}\int_{r_0}^{y}\left(\frac{\eta+1}{2\alpha_0} - \frac{3\alpha_0}{2}\right)IF(y)dy. \tag{44}$$

The above equation can be nondimensionalized as follows:

$$\tilde{y} = \frac{y}{r_0}, \quad \tilde{u}_1(\tilde{y}) = r_0 u_1(y), \quad \widetilde{IF}(\tilde{y}) = \frac{IF(y)}{IF(r_0)},$$

$$\tilde{n}(\tilde{y}) = \frac{n_0(y)}{n_0(r_0)}.$$

Using the fact that $p_0'=0$ (see the Appendix), Eq. (44) can be written as

$$\frac{1}{\tilde{u}_1(\tilde{y})} = \frac{\tilde{y}\sqrt{\alpha_0(\tilde{y})}}{\tilde{u}_1(1)\sqrt{\alpha_0(1)}} \pm \tilde{y}\Pi(\tilde{y})\sqrt{\alpha_0(\tilde{y})}, \tag{45}$$

where

$$\Pi(\tilde{y}) = \int_1^{\tilde{y}}\left(\frac{\eta+1}{2\sqrt{\alpha_0^3(\tilde{y})}} - \frac{3\sqrt{\alpha_0(\tilde{y})}}{2}\right)\frac{d\tilde{y}}{\tilde{y}}.$$

If $n_{\min}$ is the minimum value of $\tilde{n}(\tilde{y})$, then

$$\Pi(\tilde{y}) > \left(\frac{(\eta+1)(an_{\min}+b)^{3/4}}{2} - \frac{3}{2(an_{\min}+b)^{1/4}}\right)\int_1^{\tilde{y}}\frac{d\tilde{y}}{\tilde{y}}, \tag{46}$$

where $b=1/(\eta-1)$ and $a=1/\alpha_0^2(1)-b$. Hence, $\Pi(\tilde{y})$ diverges (the magnitude becomes infinite) both when $\tilde{y}\to\infty$ and $\tilde{y}\to 0$. Hence, every outward traveling compression wave will steepen into a shock and every inward traveling compression wave will steepen into a shock before reaching the center. In the nonrelativistic limit ($b/a\to 0$), $\Pi(\tilde{y})$ may not diverge, if $n_{\min}$ is zero. The difference in the behavior of relativistic acoustic waves and nonrelativistic acoustic waves can be understood through the following example.

To analyze the effect of density gradients for which $n_{\min}$ is zero, it is assumed that the density of the quiescent field varies as a power of the radius, i.e., $\tilde{n}(\tilde{r})=\tilde{r}^N$. $N>0$ represents an increasing density field, while $N<0$ represents a decreasing density field. In this kind of density variation, $n_{\min}$ is zero. The wave velocity α_0 is given by [Eq. (A6)]

$$\alpha_0(\tilde{y}) = \frac{1}{\sqrt{a\tilde{n}(\tilde{y}) + b}} = \frac{1}{\sqrt{a\tilde{y}^N + b}}. \tag{47}$$

A decreasing density field approximately represents a gaseous object, such as a nebula, with a highly dense core. In such a field, the density tends to zero at extremely large distances. The presence of an increasing density field is not common in astrophysical situations; however, there can be other physical situations where the density field is an increasing function of the distance. Also, the contrasts in the behavior of inward traveling relativistic waves and nonrelativistic waves can be understood through this example, since the density near the origin is zero.

In the assumed density field, $\Pi(\tilde{y})$ is given by

$$\Pi(\tilde{y}) = \bar{\Pi}(\tilde{y}) - \bar{\Pi}(1),$$

where

$$\bar{\Pi}(\tilde{y}) = \frac{2(\eta+1)}{3N\sqrt{\alpha_0^3(\tilde{y})}} + \frac{(2-\eta)b^{3/4}}{N}\ln\left(\frac{1-b^{1/4}\sqrt{\alpha_0(\tilde{y})}}{1+b^{1/4}\sqrt{\alpha_0(\tilde{y})}}\right) + \frac{2(2-\eta)b^{3/4}}{N}\tan^{-1}\left(\frac{\sqrt{\alpha_0(\tilde{y})}}{b^{1/4}}\right). \tag{48}$$

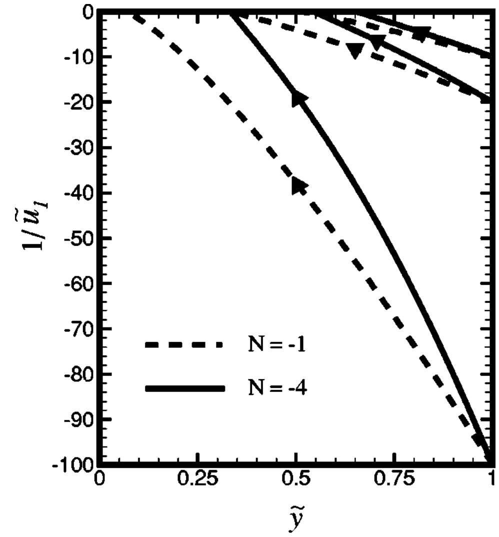
The slope at the wave front can be determined from Eq. (45). In the nonrelativistic limit, $\Pi(\tilde{y})$ reduces to

$$\Pi(\tilde{y}) = \frac{2(\eta+1)}{3N}a^{3/4}(\tilde{y}^{3N/4} - 1). \tag{49}$$

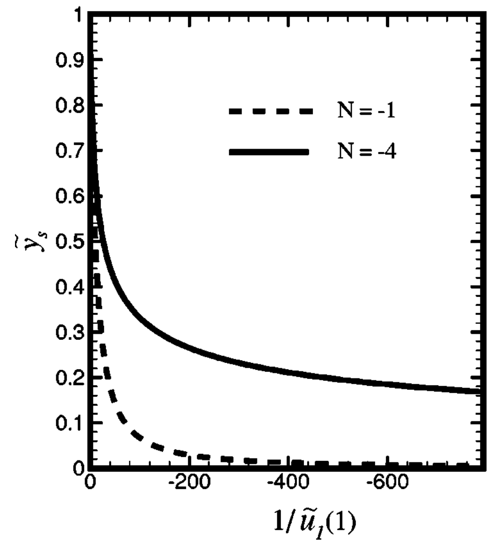
1. Decreasing density field ($N<0$)

It is clear from Eq. (48) that $\Pi(\tilde{y})$ diverges both when $\tilde{y}\to\infty$ and $\tilde{y}\to 0$ in a density field decreasing in the radial direction ($N<0$). Hence, all inward traveling compression waves steepen into shocks before reaching the center. Figure 3(a) shows the evolution of an inward traveling compression wave front. The variation of location of shock formation with initial slope is shown in Fig. 3(b).

It is also seen from Eq. (48) that all outward traveling compression waves steepen into shocks. However, in the nonrelativistic limit, $\Pi(\tilde{y})$ is finite as $\tilde{y}\to\infty$ [Eq. (49)]. Therefore, only those outward traveling compression waves which have slope greater than a nonzero minimum slope can steepen into shocks. This can be explained as follows: In a steep decreasing density gradient, a relativistic compression wave with small initial slope tends to relax initially [see Fig. 4(a)]. However, the same compression wave in a region with a mild density gradient steepens throughout its evolution. This shows that the rate of steepening of a compression wave depends strongly on the environment (density gradients). At large distances, the effect of spherical geometry and the density becomes negligible. When density is negligible, the fluid will behave like a barotropic fluid with equation of state $p = (\eta-1)e$ and hence the speed of sound tends to a finite value. Hence, at large distances, the compression wave behaves like a simple wave and thus it steepens into a shock. In the nonrelativistic limit, the speed of sound tends to infinity at large distances in a decreasing density field, unlike the relativistic case. Hence, the fluid behaves like an incompressible fluid and the wave cannot get distorted any further. Though every relativistic compression wave steepens into a shock [Eq. (46)], it is not physically meaningful since the time of shock formation for a compression with small initial slope is extremely large. Figure 4(a) shows the effect of decreasing density gradients on the slope of the outward traveling compression waves for different initial slopes. Figure 4(b) shows the variation of location of shock formation with initial slope in the presence of decreasing density gradients. However, when the initial slope of the compression wave front is large, it is not affected much by the density gradients.



(a)



(b)

FIG. 3. (a) shows the variation of the slopes of inward traveling compression wave fronts with different initial slopes in decreasing density gradients. The density is assumed to decrease as r^N, where r is the radial distance. The initial speed of sound is 0.1. The polytropic index $\eta=4/3$. (b) shows the variation of the shock formation distance of inward traveling compression waves with the inverse of initial slope in the presence of the same decreasing density fields as in (a). The initial conditions and the polytropic index are also the same as in (a). Compression waves with large initial slopes steepen immediately.

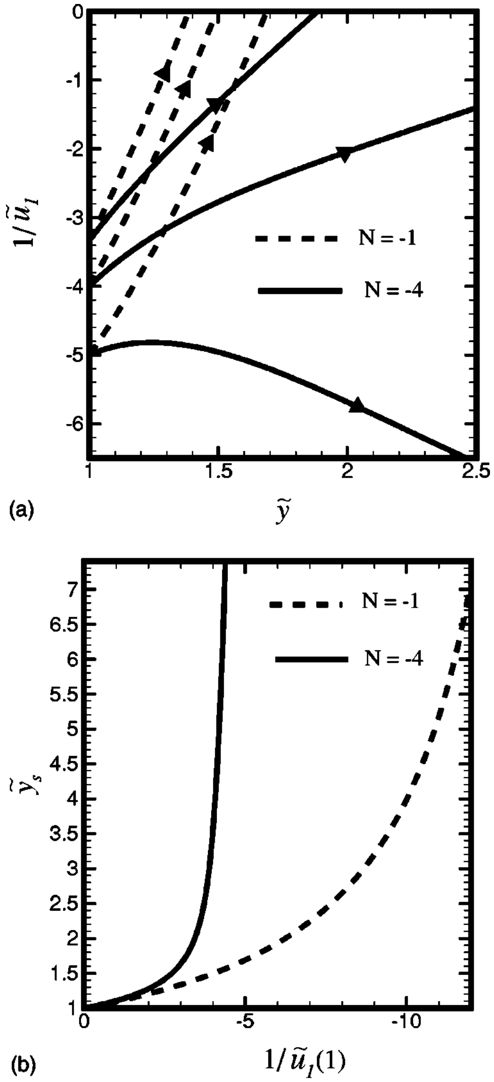


FIG. 4. (a) shows the variation of the slopes of outward traveling compression wave fronts with different initial slopes in the same decreasing density fields as in Fig. 3(a). The initial conditions and the polytropic index are also the same as in Fig. 3(a). In the density field with a larger gradient, compression waves with small initial slopes tend to relax initially and they start steepening after some distance. The region where these compression waves steepen is not shown in the figure. (b) shows the variation of the shock formation distance of outward traveling compression waves with the inverse of initial slope in the presence of the same decreasing density fields as in Fig. 3(a). The initial conditions and the polytropic index are the same as in Fig. 3(a). The shock formation distance increases steeply for compression waves with small initial slopes.

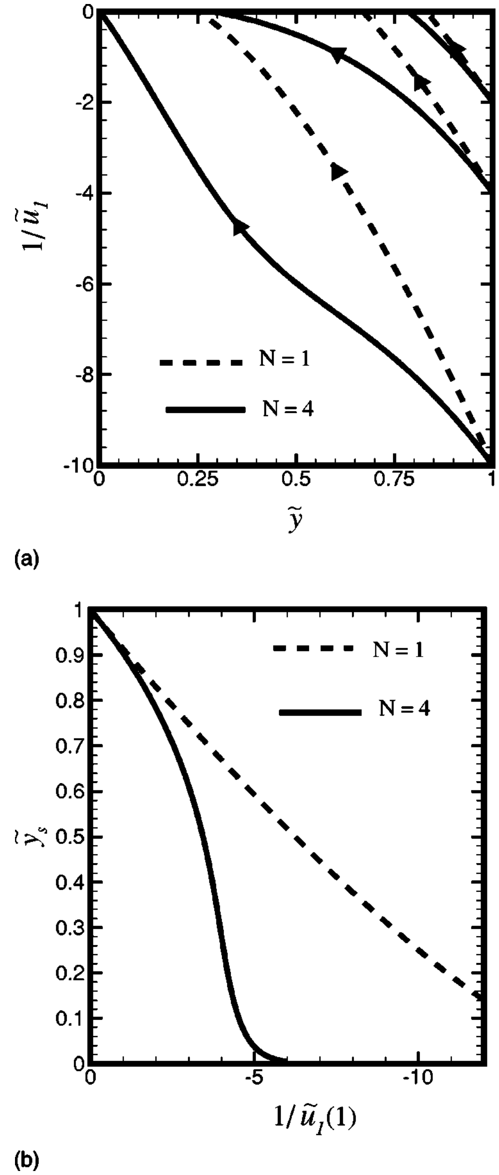
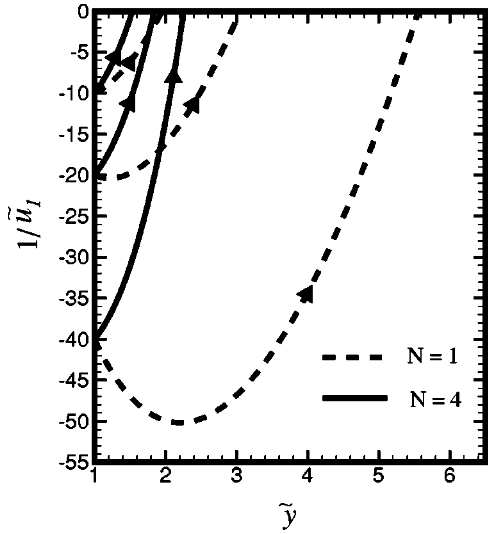


FIG. 5. (a) shows the variation of the slopes of inward traveling compression wave fronts with different initial slopes in an increasing density gradient. The density is assumed to increase as r^N, where r is the radial distance. The initial speed of sound is 0.1. The polytropic index $\eta=4/3$. (b) shows the variation of the shock formation distance of inward traveling compression waves with the inverse of initial slope in the presence of the same increasing density fields as in (a). The initial conditions and the polytropic index are the same as in (a). The shock forms very close to the center for compression waves with small initial slopes.

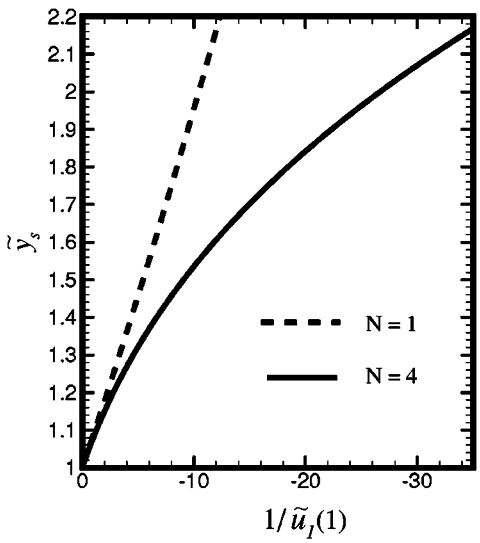
It is clear from Fig. 4(b) that the shock formation occurs at a large distance for a compression wave with small initial slope.

2. Increasing density field ($N>0$)

In the presence of a density field increasing in the radial direction, the inward traveling waves move into a region of decreasing density gradients. Since $\Pi(\tilde{y})$ diverges as $\tilde{y}\rightarrow 0$, in an increasing density field, every inward traveling compression wave will steepen into a shock before reaching the center. Figure 5(a) shows the distortion of an inward traveling compression wave front. Though compression waves with small initial slope develop into shocks when they are close to the center, the shock formation occurs before they reach the center [see Fig. 5(b)]. However, in the nonrelativistic limit, since $\Pi(\tilde{y})$ reaches a finite value as $\tilde{y}\rightarrow 0$, only those compression waves which are sufficiently steep can form a shock before reaching the center. The others form infinitesimal shocks at the center along with the expansion waves. This is because the density tends to zero as the wave moves closer to the center. The fluid in this region behaves like an incompressible fluid (for nonrelativistic fluids). Hence, the wave cannot get distorted further by the density gradients. However, they form infinitesimal shocks at the center due to the effect of geometry.



(a)



(b)

FIG. 6. (a) shows the variation of the slopes of outward traveling compression wave fronts with different initial slopes in the same increasing density fields as in Fig. 5(a). (b) shows the variation of the shock formation distance of outward traveling compression waves with the inverse of initial slope in the presence of the same increasing density fields as in Fig. 5(a).

Figure 6(a) shows the evolution of outward traveling compressive disturbances. All outward traveling compression waves steepen into a shock. In a region with mild density gradient a compression wave front with small initial slope tends to relax initially. This is because the geometrical relaxation effect dominates initially and after some distance the effect of increasing density gradient dominates. Figure 6(b) illustrates the dependence of the location of shock formation on the initial slope for outward traveling compression waves.

B. Synge gas

For a monatomic perfect gas, Synge[14] derived the following relations for the thermodynamic quantities:

$$p = nkT, \tag{50}$$

$$w = mnG(z) = pzG(z), \tag{51}$$

where $z=m/kT$, T is the absolute temperature, k is the Boltzmann constant, and $G(z)=K_3(z)/K_2(z)$, $K_n(z)$ are the modified Bessel's functions. For such a gas the speed of sound is given by

$$\alpha^2 = \frac{G'(z)/G(z)}{z(G'(z)+1/z^2)}. \tag{52}$$

The variables in the quiescent medium, α_0, p_0, e_0, and w_0 also satisfy the above relations. The compressibility parameter of the undisturbed medium can be obtained using Eq. (23) as

$$\varsigma(z_0) = \left[2(1-\alpha_0^2) + 2\alpha_0\left(\frac{d\alpha_0}{dz_0}\right)\frac{G(z_0)}{G'(z_0)}\right]. \tag{53}$$

Substituting for α_0 and w_0, the integrating factor [Eq. (28)] is obtained as

$$IF(y) = \frac{1}{y[p_0 z_0 G_0(z_0)\alpha_0^3]^{1/2}}. \tag{54}$$

Substituting for ς and $IF(y)$ in Eq. (27) from the above relations and using the fact that $p_0'=0$ we get

$$\begin{aligned}\frac{1}{u_1(y)} &= \frac{1}{u_1(r_0)}\frac{y\sqrt{z_0(y)G[z_0(y)]\alpha_0^3[z_0(y)]}}{r_0\sqrt{z_0(r_0)G[z_0(r_0)]\alpha_0^3[z_0(r_0)]}} \\ &\quad \pm y\sqrt{z_0(y)G[z_0(y)]\alpha_0^3[z_0(y)]} \\ &\quad \times \int_{r_0}^{y}\phi[z_0(y)]\frac{dy}{y},\end{aligned} \tag{55}$$

where

$$\begin{aligned}\phi(z_0) = \varsigma(z_0)yIF(y) &= \left[\left(\frac{1}{\alpha_0}-\alpha_0\right)+\left(\frac{d\alpha_0}{dz_0}\right)\frac{G(z_0)}{G'(z_0)}\right] \\ &\quad \times [z_0 G(z_0)\alpha_0^3]^{-1/2}.\end{aligned} \tag{56}$$

The behavior of the quantity $\phi(z)$ dictates the effect of density gradients alone on shock formation. Since $\phi(z_0)$ is positive for all z, the integral in Eq. (55) satisfies the following inequality:

$$\int_{r_0}^{y}\phi[z_0(y)]\frac{dy}{y} > \min\{\phi[z_0(y)]\}\int_{r_0}^{y}\frac{dy}{y}. \tag{57}$$

It is clear from Eq. (56) that the integral in Eq. (57) diverges both when $y\to 0$ and $y\to\infty$, if the minimum value of $\phi(z_0)$ is not zero. Figure 7 shows that $\phi(z_0)$ is an increasing function of z_0. Since $\phi(z_0)$ is nonzero as $z_0\to 0$, the integral in Eq. (55) will diverge as $y\to 0$ and as $y\to\infty$. Therefore, (i) every inward traveling compression will steepen into a shock before reaching the center. (ii) Every outward traveling wave steepens to form a shock.

To trace the complete evolution of the wave front, the integral in Eq. (55) has to be evaluated.

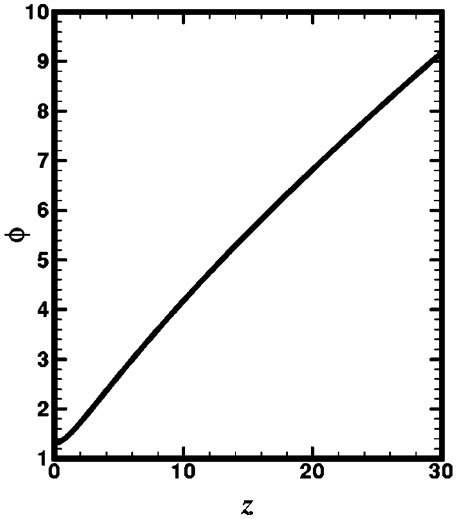


FIG. 7. The variation of $\phi(z)$ as a function of z.

VI. CONCLUSIONS

The steepening of relativistic compression and rarefaction waves is studied using the wave front expansion technique. A closed form solution for the steepening of the leading edge of the wave front propagating into a gas governed by a general equation of state is obtained. Also expressions for time and location for shock formation are obtained. The effects of variation in the relativistic compressibility parameter (due to entropy gradients) and spherical geometry are also analyzed. The following inferences are made from the above analysis.

(a) Every outward propagating compression wave in a homentropic environment will steepen into a shock. However, for waves with very small initial slope, the shock formation distance is extremely large and it is not physically meaningful.
(b) All inward traveling compressive disturbances form shocks before reaching the center. The inward traveling expansion waves form infinitesimal shocks at the center. This is not observed in Cartesian geometry. However, these infinitesimal shocks are unstable and diffuse out quickly.
(c) In the case of polytropic gases, all inward and outward traveling compression waves steepen into shocks. However, according to nonrelativistic fluid mechanics, some compression waves do not steepen into shocks. The above result was analyzed for both increasing and decreasing density gradients.
(d) All compression waves steepen into a shock even in the case of a Synge gas.

The results that were obtained from the above analysis agree with the classical results obtained using relativistic simple waves. Also, the results obtained from this analysis can be used to check the accuracy of numerical codes for relativistic fluid dynamics.

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APPENDIX

Using Eqs. (42)–(44), the integrating factor can be obtained from Eq. (28) as

$$IF(y) = \frac{1}{y(\eta p_0 \alpha_0)^{1/2}}. \tag{A1}$$

Since $p_0' = 0$ [Eq. (18)],

$$p_0(y) = p_0(r_0). \tag{A2}$$

Substituting the above relations into Eq. (44), the expression for the slope at the wave front simplifies to

$$\frac{1}{u_1(y)} = \frac{1}{u_1(r_0)}\left(\frac{y[\alpha_0(y)]^{1/2}}{r_0[\alpha_0(r_0)]^{1/2}}\right) \pm y[\alpha_0(y)]^{1/2}\int\left(\frac{\eta+1}{2\alpha_0^{3/2}} - \frac{3\alpha_0^{1/2}}{2}\right)\frac{dy}{y}. \tag{A3}$$

Using the nondimensional quantities defined after Eq. (44), the above expression can be rewritten as

$$\frac{1}{\tilde{u}_1(\tilde{y})} = \frac{\tilde{y}\sqrt{\alpha_0(\tilde{y})}}{\tilde{u}_1(1)\sqrt{\alpha_0(1)}} \pm \tilde{y}\Pi(\tilde{y})\sqrt{\alpha_0(\tilde{y})}, \tag{A4}$$

where

$$\Pi(\tilde{y}) = \int_1^{\tilde{y}}\left(\frac{\eta+1}{2\sqrt{\alpha_0^3(\tilde{y})}} - \frac{3\sqrt{\alpha_0(\tilde{y})}}{2}\right)\frac{d\tilde{y}}{\tilde{y}}. \tag{A5}$$

The wave front speed α_0 can be written in terms of the nondimensional quantity n_0 as

$$\alpha_0(\tilde{y}) = \frac{1}{\sqrt{a\tilde{n}(\tilde{y}) + b}}, \tag{A6}$$

where $b = 1/(\eta - 1)$ and $a = 1/\alpha_0^2(1) - b$.

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