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Radial Basis Function Based Gridfree Scheme for Interface Capturing: Preliminary Results

Susmita Agarwal[a,*], Y. V. S. S. Sanyasiraju[a]

[a]*Department of Mathematics, IIT Madras, Chennai 600036, India*

Abstract

In this paper, we propose a Radial Basis Function (RBF) based grid free scheme for capturing the interface which is passively advected by the underlying velocity field and present some preliminary results. The level set function is used to capture the interface and reinitialization algorithm is applied to maintain its signed distance property. The presented scheme is validated with two examples.

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Keywords: Radial Basis Function; Level Set Method; Distance Function

1. Introduction

Gridfree schemes are widely used to solve partial differential equations over grid based methods due to the high cost of grid generation and also complexity in the domain. One such gridfree scheme is based on the Radial Basis Functions (RBF) which is becoming very popular in recent years due to its better conditioning and flexibility in handling the non-linearities. However, if collocation type schemes are used, the collocation matrices of the globally supported RBFs are highly dense and ill-conditioned. Further, the problem of ill-conditioning becomes compounded when the number of centres is increased. Some RBF based local schemes, for Burgers equation and incompressible Navier-Stokes equations were proposed by Chandhini and Sanyasiraju [1] and Sanyasiraju and Chandhini [2], respectively.

In the present work, the local RBF based grid free scheme is suitably used to capture the moving interface. Moving interface problems occur in many physical phenomenon. Many methods, namely SLIC algorithms [3], the particle level set methods [4], etc., are available in the literature to capture the moving interface. These methods are mainly based on either Volume of Fluid (VOF) or Level Set Methods. Osher and Sethian [5] introduced the level set methods to capture the interfaces moving with curvature dependent speed. In this paper the interface is identified by the zero of the level set function. Several authors including Sussman et al. [6] and Enright et al.[4] took the level set function as a signed distance to the interface. As noted by Sussman et al. [6], during the computation, the level set function no longer remain a distance function and so a reinitialization technique is applied to maintain the signed distance

* Corresponding author. Tel.: +91-988-411-2506 ; fax: +91-442-257-4602.
E-mail address: agarwal.susmita89@gmail.com

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property. The rest of the paper is arranged as follows. Section 2 describes the numerical methods used in this article. In, Section 3, numerical examples are presented. Conclusions are given in the Section 4.

2. Numerical Method

2.1. The Local RBF Scheme

A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be a radial basis function if there exist a univariate function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(\bar{x}) = \varphi(r)$ where $r = \|\bar{x}\|$. RBFs are mainly used to approximate multivariate function from a given set of data.

For a given set of distinct nodes $\bar{x}_i$ in $\mathbb{R}^d$ and corresponding function values u_i, the RBF interpolant $s(\bar{x})$ is given by

$$s(\bar{x}) = \sum_{j=1}^{n} \lambda_j \varphi(\|\bar{x} - \bar{x}_j\|) + \sum_{j=1}^{l} \gamma_j p_j(\bar{x}) \tag{1}$$

where $\{p_j\}_{j=1}^{l}$ is a basis for π_m^d (space of all d variate polynomial of degree $< m$, m is the order of φ). Eq. (1) satisfies the interpolation conditions

$$s(\bar{x}_i) = u(\bar{x}_i), \quad i = 1, 2, \ldots, n \tag{2}$$

along with the orthogonality condition,

$$\sum_{j=1}^{n} \lambda_j q(\bar{x}_j) = 0 \tag{3}$$

where $q \in \pi_m^d$.

Using the interpolation condition (2) and the orthogonality condition (3) in eq.(1) we get a linear system,

$$\begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} \tag{4}$$

where $\Phi_{ij} = \varphi(\|\bar{x}_i - \bar{x}_j\|)$ and $P_{ij} = p_j(\bar{x}_i)$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, l$.

Now, consider the Lagrange representation of the RBF interpolant,

$$u(\bar{x}) \approx s(\bar{x}) = \sum_{j=1}^{n} \psi_j(\bar{x}) u(\bar{x}_j) \tag{5}$$

where ψ_j satisfies the cardinal conditions

$$\psi_j(\bar{x}_i) = \delta_{ij}. \tag{6}$$

The closed form of ψ_j can be given by

$$\psi_j(\bar{x}) = \frac{|A_j|}{|A|} \tag{7}$$

where matrix $A = \begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix}$ and the matrix A_j is same as matrix A except that the j^{th} row of A is replaced by vector B, which is given by

$$B = [\varphi(\|\bar{x} - \bar{x}_1\|), \varphi(\|\bar{x} - \bar{x}_2\|), \ldots, \varphi(\|\bar{x} - \bar{x}_n\|), q_1(\bar{x}), q_2(\bar{x}), \ldots, q_l(\bar{x})]. \tag{8}$$

Let $\mathcal{L}$ be a given differential operator. To approximate $\mathcal{L}u$ at a given node $\bar{x}_i$, consider a set S_i, of n_i neighbouring nodes of $\bar{x}_i$. The approximation of $\mathcal{L}u(\bar{x}_i)$ is given by

$$\mathcal{L}u(\bar{x}_i) \approx \sum_{j=1}^{n_i} c_j u(\bar{x}_j). \tag{9}$$

By applying the operator $\mathcal{L}$ on both sides of eq.(5) and comparing it with eq.(9) we get

$$c_j = \mathcal{L}(\psi_j(\bar{x})). \tag{10}$$

c_j can also be obtained, as in [7], by solving the linear system

$$\begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix}\begin{pmatrix} c \\ \mu \end{pmatrix} = \mathcal{L}(B(\bar{x}_i)) \tag{11}$$

where B is as given in eq.(8) and μ is a dummy vector.

The RBF based gridfree scheme is used to solve the evolution equation of the level set function which is described in the following section.

2.2. Level Set Method

Consider the interface Γ, which is a closed curve or surface enclosing an open domain Ω and moving with the time in the computational domain. Γ is identified with the zero of the level set function ϕ, which has the following properties

$$\phi(\bar{x}) \begin{cases} < 0, \bar{x} \in \Omega \\ > 0, \bar{x} \in \bar{\Omega}^T \\ = 0, \bar{x} \in \Gamma \end{cases} \tag{12}$$

The interface Γ moves with the underlying velocity field $\bar{u}$, which may depend on position, time or topology of the interface. The interface is evolved according to the equation

$$\phi_t + \bar{u} \cdot \nabla\phi = 0. \tag{13}$$

The level set function ϕ is taken to be the sign distance to the interface such that the property (12) is satisfied and $|\nabla\phi| = 1$.

As noted by Sussman et al. [6], ϕ doesnt remain as the sign distance function during the computations, as time progresses. To overcome this situation a reinitialization algorithm is applied. This algorithm uses the fact that for a distance function ϕ, $|\nabla\phi| = 1$. Therefore the equation

$$\phi_\tau + \text{sgn}(\phi_0)(|\nabla\phi| - 1) = 0 \tag{14}$$

can be solved to steady state to maintain the sign distance property of ϕ. ϕ_0 is some function which is same as ϕ at the interface and $\text{sgn}(\phi_0)$ is numerically approximated signum function, which is given by

$$\text{sgn}(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 + \varepsilon^2}} \tag{15}$$

where ε is the smoothening parameter. Sussman et al., [6], Russo and Smereka, [8] used $\varepsilon = \Delta x$, throughout the domain for their computations. Recently Liu et al., [9] improved the value of ε, which is given as,

$$\begin{cases} \varepsilon_{i+1}^2 = \phi_i^2 \\ \varepsilon_i^2 = \phi_{i+1}^2, \text{ if } \phi_i \cdot \phi_{i+1} < 0 \end{cases} \quad \text{if } \phi_i \cdot \phi_{i+1} < 0, \ \varepsilon_i = 0, \text{ otherwise} \tag{16}$$

so that the interface doesnt move out of a given cell.

The level set evolution equation, eq.(13) is solved using third-order TVD Runge-Kutta method:

$$\begin{aligned} \phi^{(1)} &= \phi^{(0)} - dt\mathcal{L}(\phi^{(0)}) \\ \phi^{(2)} &= \frac{3}{4}\phi^{(0)} + \frac{1}{4}\phi^{(1)} - \frac{1}{4}dt\mathcal{L}(\phi^{(1)}) \\ \phi^{(3)} &= \frac{1}{3}\phi^{(0)} + \frac{2}{3}\phi^{(2)} - \frac{2}{3}dt\mathcal{L}(\phi^{(2)}) \end{aligned} \tag{17}$$

where $\mathcal{L}(\phi) = \bar{u} \cdot \nabla\phi$. To calculate $\mathcal{L}(\cdot)$, the RBF based grid free scheme, as described in the previous section, is used.

To solve the reinitialization equation, consider its discretized form

$$\phi_i^{n+1} = \phi_i^n - d\tau \ \mathrm{sgn}(\phi_0)_i G(\phi)_i \tag{18}$$

where

$$G(\phi)_i = \begin{cases} \sqrt{max(a_+^2, b_-^2) + max(c_+^2, d_-^2)} - 1, \ \text{if } \phi_0 > 0 \\ \sqrt{max(a_-^2, b_+^2) + max(c_-^2, d_+^2)} - 1, \ \text{if } \phi_0 < 0 \end{cases} \tag{19}$$

where a and b are the approximation of ϕ_x and c and d are the approximation of ϕ_y in the upwind direction, which are determined by the sign of ϕ_0. Also $h_+ = \max(h, 0)$ and $h_- = \min(h, 0)$. a, b, c and d are calculated using the RBF based grid free scheme.

3. Results and Validation

The *Multiquadric* function, $\varphi(r) = \sqrt{1 + (\epsilon r)^2}$, which is an infinitely smooth RBF, has been used to solve the examples with $\epsilon = 0.01$.

For the examples presented here, ϕ_0 in eq.(15) is taken as ϕ. The smoothening parameter ε is taken as given in eq.(16).

3.1. Example 1: Rigid body Rotation of Zalesaks Disk [10]

Initially the interface is given by the disk, which is a slotted circle centred at $(50, 75)$ with a radius of 15, and with a slot of width 5 and length 25 unites. The disk is inside a square of length 100 units. The constant vorticity velocity field is given by

$$u = \frac{\pi}{314}(50 - y); \ v = \frac{\pi}{314}(x - 50) \tag{20}$$

so that the disk completes one revolution in every 628 time units. The computational domain is discretized in 300×300 nodes. The time step, for the evolution equation, eq.(13) is taken as $dt = 0.01$ and that for the reinitialization equation, eq.(14) is taken as $d\tau = 0.00005$.

Fig. 1 illustrate the numerical solution obtained, by applying the method described in this paper, after one revolution(blue) and the initial position of the disk(red). It can be seen that the numerical method presented here captures the interface accurately except at the four corners of the slot of the disk due to diffusion error.

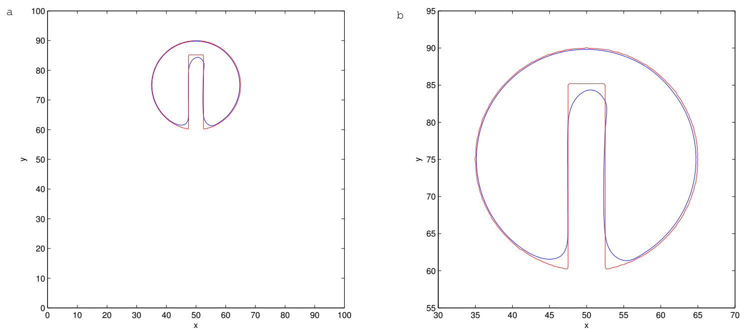

Fig. 1. Fig. 1. Numerical solution after one revolution (blue) and initial disk (red) (a) in complete computational domain; (b) the enlarged figure.

3.2. Example 2: Single Vortex

A circle of radius 0.15, is placed inside a unit square, with centre at (0.5, 0.75). An initial velocity field is defined in the domain by the stream function,

$$\Psi = \pi^{-1} \sin^2(\pi x) \sin^2(\pi y). \tag{21}$$

For this example, 256×256 nodes have been taken in the computational domain. The time step, for the evolution equation, eq.(13) is taken as $dt = 0.00005$ and that for the reinitialization equation, eq.(14) is taken as $d\tau = 0.000005$.

The velocity field starches out the circle into a long thin fluid element which is illustrated in Fig. 2 at time $T = 1$, 3 and 5 respectively. It can be seen from the presented results that the numerical method is able to maintain the long thin fluid element.

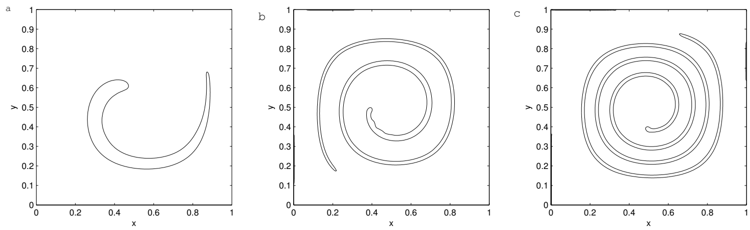

Fig. 2. Fig. 2. Vortex obtained at (a) T=1; (b) T=3; (c) T=5.

4. Conclusion

We proposed a gridfree scheme to capture the moving interface which is based on RBF and uses the level set function to capture the interface. Structured point distribution is used for the examples presented here, however one can also use unstructured point distribution. As we can see in the Example 1, the presented method captures the interface accurately except the four corners, however the very sharp and the thin fluid element of the second problem has been captured by the method very accurately.

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