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PERIODIC OCCURRENCE OF COMPLETE INTERSECTION MONOMIAL CURVES

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ABSTRACT. We study the complete intersection property of monomial curves in the family $\Gamma_{\underline{a}+\underline{j}} = \{(t^{a_0+j}, t^{a_1+j}, \ldots, t^{a_n+j}) \mid j \geq 0,\ a_0 < a_1 < \cdots < a_n\}$. We prove that if $\Gamma_{\underline{a}+\underline{j}}$ is a complete intersection for $j \gg 0$, then $\Gamma_{\underline{a}+\underline{j}+\underline{a}_n}$ is a complete intersection for $j \gg 0$. This proves a conjecture of Herzog and Srinivasan on eventual periodicity of Betti numbers of semigroup rings under translations for complete intersections. We also show that if $\Gamma_{\underline{a}+\underline{j}}$ is a complete intersection for $j \gg 0$, then $\Gamma_{\underline{a}}$ is a complete intersection. We also characterize the complete intersection property of this family when $n = 3$.

INTRODUCTION

Given an ascending sequence of positive integers $a_0, \ldots, a_n$, the curve in $\mathbb{A}^{n+1}$ parameterized by $t \to (t^{a_0}, t^{a_1}, \ldots, t^{a_n})$ is called an affine monomial curve since the parametrization is by monomials. We say that a monomial curve defined by a sequence $\underline{a} = (a_0, a_1, \ldots, a_n)$ is a complete intersection (Gorenstein) if $k[t^{a_0}, \ldots, t^{a_n}]$ is a complete intersection (Gorenstein). The minimal number of equations defining monomial curves and the various structures of monomial curves have been fascinating algebraists and geometors for a long time. It is well known that these equations are binomial equations. In fact, the ideal of a monomial curve in $\mathbb{A}^{n+1}$ is a weighted homogeneous binomial prime ideal of height n in the polynomial ring $S = k[x_0, \ldots, x_n]$. In the plane, they are principal ideals and the space monomial curves are either complete intersections generated by two binomials or determinantal ideals generated by the 2×2 minors of a 2×3 matrix [5]. This breaks down even in dimension 4 because there is no upper bound for the number of generators for monomial cuves in $\mathbb{A}^4$ [1]. However, because of the structure theorem of Gorenstein ideals in codimension three as ideals generated by Pfaffians [1], the Gorenstein monomial curves in dimension three are either complete intersections generated by 3 elements or the ideal of 4×4 pfaffians of a 5×5 skew symmetric matrix. Thus, for special classes of monomial curves, the number of generators is bounded. We partition the monomial curves into classes so that two monomial curves are in the same class if their consecutive parameters have the same differences. That is, if $\mathbf{m} = \{m_1, \ldots, m_n\}$ is a sequence of positive integers, $C(\mathbf{m})$ is a class of monomial

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curves defined by $\underline{a} = a_0, \ldots, a_n$ with $\Delta(\underline{a}) = \mathbf{m}$. Herzog and Srinivasan conjecture that the minimal number of generators for the ideal defining the monomial curves in a given class $C(\mathbf{m})$ is bounded. In fact, they conjecture that this is eventually periodic with period $\sum_i m_i$. This conjecture is true for monomial curves defined by arithmetic sequences [4]. In this paper we prove the conjecture in dimension 3 completely and prove it for complete intersections in any dimension. We prove that for $a_0 \gg 0$, the complete intersections in the class $C(\mathbf{m})$ occur periodically with period $\sum_i m_i$. Our proof of the conjecture follows from a criterion for complete intersection extending the one in [2] for monomial curves with high a_0. This generalizes and recovers some results of Adriano Marzullo [8].

Now we state the conjecture precisely: Let $\underline{a} = (a_0, \ldots, a_n)$ be a sequence of positive integers and let j be any positive integer. Let $\underline{a}+(\underline{j})$ denote a sequence $(j+a_0, j+a_1, j+a_1, \ldots, j+a_n)$. Let $\Gamma_{\underline{a}+(\underline{j})}$ denote the monomial curve corresponding to the sequence $\underline{a}+(\underline{j})$ and $I_{\underline{a}+(\underline{j})}$ denote the defining ideal of $\Gamma_{\underline{a}+(\underline{j})}$. Then the strong form of the Herzog-Srinivasan conjecture states that the Betti numbers of $I_{\underline{a}+(\underline{j})}$ are eventually periodic in j. Thus, the conjecture says that within a class of monomial curves associated to increasing sequences $\underline{a}$ with the same $\Delta(\underline{a})$, the Betti numbers of the defining ideals are eventually periodic.

In this paper we prove that for any sequence $\underline{a}$, for large j, if $\underline{a}+(\underline{j})$ is a complete intersection, then $\underline{a}+(\underline{j+a_n})$ is a complete intersection. Since we are proving results for $j \gg 0$, we may as well assume that $a_0 = 0$ and the sequence $\underline{a}+(\underline{j}) = (j, j+a_1, \ldots, j+a_n)$. To be precise, let $CI(\underline{a}) = \{j \mid \Gamma_{\underline{a}+(\underline{j})}$ is a complete intersection curve$\}$. We prove:

Theorem 2.1. *If $\underline{a} = (a_1, \ldots, a_n)$, then $CI(\underline{a})$ is either finite or eventually periodic with period a_n. If for $j \gg 0$, $\underline{a}+(\underline{j})$ is a complete intersection, then there exist $1 \leq s \leq n-1$ and $k \in \mathbb{Z}_+$ such that*

(a) *$j = a_n m$ for some $m \in \mathbb{Z}_+$,*
(b) *$\gcd(a_1, \ldots, a_{s-1}, a_s + a_{s+1}, a_{s+2}, \ldots, a_n) = k \neq 1$ and*
(c) *$a_s \in \left\langle \frac{a_1}{k}, \ldots, \frac{a_{s-1}}{k}, \frac{a_{s+1}}{k}, \ldots, \frac{a_n}{k} \right\rangle$*

and

Corollary 2.2. *For $j \gg 0$, if $\underline{a}+(\underline{j})$ is a complete intersection, then $\underline{a}$ is a complete intersection.*

We also give a criterion for these curves to be a complete intersection in Theorem 2.4.

1. Monomial curves in $\mathbb{A}^3$

Let $\underline{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}_+^{n+1}$ with $a_0 < a_1 < \cdots < a_n$ and $R = k[t^{a_0}, \ldots, t^{a_n}]$, where k is a field of characteristic zero. For the reason explained in the introduction, we will assume here that $a_0 = 0$. We begin by recalling a result characterizing the complete intersection property of the sequence $\underline{a}$. We say that a sequence $\underline{a}$ is a complete intersection sequence if $\Gamma_{\underline{a}}$ is a complete intersection monomial curve. For any sequence $\underline{a}$, let $\langle a_1, \ldots, a_n \rangle := \{\sum_{i=1}^n r_i a_i \mid r_i \in \mathbb{Z}_{\geq 0}\}$ be the semigroup generated by $a_1, \ldots, a_n$.

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Theorem 1.1. *The sequence $\underline{a}$ is a complete intersection if and only if $\underline{a}$ can be written as a disjoint union:*

$$\underline{a} = k_1(b_{i_1}, \ldots, b_{i_r}) \sqcup k_2(b_{i_{r+1}}, \ldots, b_{i_n}),$$

where $a_{i_m} = k_1 b_{i_m}$ for $m = 1, \ldots, r$, $a_{i_m} = k_2 b_{i_m}$ for $m = r+1, \ldots, n$, $\gcd(k_1, k_2) = 1$, $k_1 \notin \{b_{i_{r+1}}, \ldots, b_{i_n}\}$, $k_1 \in \langle b_{i_{r+1}}, \ldots, b_{i_n}\rangle$, $k_2 \notin \{b_{i_1}, \ldots, b_{i_r}\}$, $k_2 \in \langle\{b_{i_1}, \ldots, b_{i_r}\rangle$ and both $(b_{i_1}, \ldots, b_{i_r})$ and $(b_{i_{r+1}}, \ldots, b_{i_n})$ are complete intersection sequences.

We say that the sequence $\underline{a}$ is a complete intersection of type $(r, n-r)$ if it splits as in the above theorem.

Lemma 1.2. *Suppose $j > a_n^2$. If $(j, j+a_1, j+a_2, \ldots, j+a_n)$ is a complete intersection of the type $(m, n+1-m)$, then either $m = 1$ or $m = n$.*

Proof. Suppose $1 < m < n$. Then we have a split of the form

$$(j, j+a_1, \ldots, j+a_n) = k_1(\alpha_1, \ldots, \alpha_m) \sqcup k_2(\alpha_{m+1}, \ldots, \alpha_{n+1}),$$

where $k_1 \in \langle \alpha_{m+1}, \ldots, \alpha_{n+1}\rangle$ and $k_2 \in \langle \alpha_1, \ldots, \alpha_m\rangle$. For each $i = 1, 2$, there exist l and r such that $0 \le l < r \le n$ with k_i divides $j + a_l$ and $j + a_r$, where $a_0 = 0$. It follows that k_i divides $a_r - a_l$ and hence $k_i \le a_n$. The inequalities $a_n^2 < j \le k_1\alpha_i \le a_n\alpha_i$ for $i = 1, \ldots, m$ imply that $\alpha_i > a_n$ for all $i = 1, \ldots, m$. Since $k_2 \in \langle \alpha_1, \ldots, \alpha_m\rangle$, $k_2 > a_n$. This is a contradiction. Therefore $m = 1$ or $m = n$. □

Lemma 1.3. *Suppose $j > a_n^2$ and $\underline{a} + (\underline{j})$ is a complete intersection sequence. Then complete intersection splits of the type*

(1) $\underline{a} + (\underline{j}) = k_1\left(\frac{j}{k_1}\right) \sqcup k_2\left(\frac{j+a_1}{k_2}, \ldots, \frac{j+a_n}{k_2}\right)$,

(2) $\underline{a} + (\underline{j}) = k_1\left(\frac{j}{k_1}, \frac{j+a_1}{k_1}, \ldots, \frac{j+a_{n-1}}{k_1}\right) \sqcup k_2\left(\frac{j+a_n}{k_2}\right)$

are not possible.

Proof. First we prove that a split as in (1) is not possible. Suppose (1) is a complete intersection split of $\underline{a} + (\underline{j})$. By multiplying by an appropriate factor, we obtain

$$\begin{aligned} j &= \alpha_1\frac{j+a_1}{k_2} + \cdots + \alpha_n\frac{j+a_n}{k_2} \\ &= (\alpha_1 + \cdots + \alpha_n)\frac{j}{k_2} + \alpha_1\frac{a_1}{k_2} + \alpha_2\frac{a_2}{k_2} + \cdots + \alpha_n\frac{a_n}{k_2}, \end{aligned}$$

where the α_i's are non-negative integers. Therefore,

$$k_2 j = (\sum_{i=1}^{n} \alpha_i)(j) + \alpha_1 a_1 + \cdots + \alpha_n a_n$$

so that

$$[k_2 - (\alpha_1 + \cdots + \alpha_n)]j = \alpha_1 a_1 + \cdots + \alpha_n a_n.$$

The right hand side consists of a linear combination of non-negative integers, not all of them zero. Hence $k_2 > \sum_{i=1}^{n} \alpha_i$. Since $j > a_n^2$, we get

$$a_n^2 < [k_2 - (\alpha_1 + \cdots + \alpha_n)]j \quad < \quad (\sum_{i=1}^{n} \alpha_i)(a_n) \le k_2(a_n) \le a_n^2,$$

which is a contradiction. Therefore a split of the first kind is not possible.

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Now assume that (2) is a complete intersection split for $\underline{a}+(j)$, for some $j > a_n^2$. After multiplying with an appropriate factor we get

$$j + a_n = \alpha_1 \frac{j}{k_1} + \cdots + \alpha_n \frac{j + a_{n-1}}{k_1},$$

where the α_i's are non-negative integers. Therefore

$$a_n = \left(\sum_{i=1}^{n} \alpha_i - k_1 \right) \frac{j}{k_1} + \alpha_2 \frac{a_1}{k_1} + \alpha_3 \frac{a_2}{k_1} + \cdots + \frac{\alpha_n}{k_1} a_{n-1}. \tag{1}$$

If $\sum_{i=1}^{n} \alpha_i < k_1$, then it follows from the above equality that

$$\begin{aligned} a_n &\leq \left(\sum_{i=1}^{n} \alpha_i - k_1 \right) \frac{j}{k_1} + \frac{1}{k_1} a_n \left(\sum_{i=1}^{n} \alpha_i \right) \\ &\leq \left(\sum_{i=1}^{n} \alpha_i - k_1 \right) \frac{j}{k_1} + a_n < a_n, \end{aligned}$$

which is a contradiction.

Now suppose $\sum_{i=1}^{n} \alpha_i > k_1$. It follows from equation (1) that

$$k_1 a_n = \left(\sum_{i=1}^{n} \alpha_i - k_1 \right) j + \alpha_2 a_1 + \cdots + \alpha_n a_{n-1}.$$

Therefore $j \leq k_1 a_n$. As in the proof of Lemma 1.2, we can see that $k_1 \leq a_n$. Therefore, we have $j \leq k_1 a_n \leq a_n^2$, which is a contradiction to the hypothesis that $j > a_n^2$.

If $\sum_{i=1}^{n} \alpha_i = k_1$, then we have

$$\begin{aligned} a_n &= \frac{1}{k_1} (\alpha_2 a_1 + \cdots + \alpha_n a_{n-1}) \\ &< \frac{1}{k_1} \left(\sum_{i=1}^{n} \alpha_i \right) a_{n-1} = a_{n-1}. \end{aligned}$$

This is again a contradiction. Therefore, all three possibilities lead to a contradiction. Hence a complete intersection split of type (2) is not possible. □

We now prove the periodicity conjecture for monomial curves in $\mathbb{A}^3$. Let $a_1 = a$ and $a_2 = b$. We first prove a characterization for $(j, j + a, j + b)$ to be a complete intersection sequence for $j \gg 0$.

Theorem 1.4. *If $j > \max\{ab, b(b-a)\}$, then $(j, j + a, j + b)$ defines a complete intersection ideal if and only if there exist $(j, b) = k \neq 1$ and non-negative integers α, β such that $k(j + a) = \alpha(j) + \beta(j + b)$. Moreover, in this case, $\alpha + \beta = k$ and $(a, b - a) = s$ with $b = sk$. In particular, if a and $b - a$ are relatively prime, $(j, j + a, j + b)$ is a complete intersection if and only if b divides j.*

Proof. Let $(j, j + a, j + b)$ be a complete intersection sequence. By Theorem 1.1 and Lemma 1.3, we can have only one split possible, namely,

$$(j, j + a, j + b) = \frac{j + a}{k'}(k') \sqcup k \left(\frac{j}{k}, \frac{j + b}{k} \right),$$

where $k' \mid k$, $\gcd\left(\frac{j+a}{k'}, k\right) = 1$ and $\frac{j+a}{k'} \in \left\langle \frac{j}{k}, \frac{j+b}{k} \right\rangle$. Let α, β be non-negative integers such that $k(j + a) = \alpha j + \beta(j + b)$. Since $k \leq b$ and $ab \leq j$, we see

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that $k(j+a) \leq kj + j = j(k+1)$. Therefore $kj + ka = j(\alpha+\beta) + \beta b$ so that $\alpha + \beta \leq k$. If $\alpha + \beta < k$, then the equation $ka = (\alpha+\beta-k)j + \beta b$ would imply that $j \leq \beta b - ka$. Since $\beta \leq b$ and $k \leq b$, this implies that $j \leq b(b-a)$, which contradicts the hypothesis. Therefore, $\alpha + \beta = k$.

Further, in this event, $\alpha a = \beta(b-a)$. Therefore $b = (\alpha+\beta)s$ so that $\alpha(b) = (b-a)(\alpha+\beta) = \alpha s(\alpha+\beta)$. Hence $b - a = \alpha s$ and $a = \beta s$.

If $\gcd(a, b-a) = 1$, then $s = 1$ and hence $\alpha + \beta = k = b$, thereby establishing that b divides j.

The converse is clear. □

We now prove the periodicity conjecture for $n = 2$.

Theorem 1.5. *Let $\underline{a}+(\underline{j}) = (j, j+a, j+b)$ and let $I_{\underline{a}+(\underline{j})}$ denote the defining ideal of the monomial curve (t^j, t^{j+a}, t^{j+b}). If $j \gg 0$, then the Betti numbers of $I_{\underline{a}+(\underline{j})}$ are periodic with period b.*

Proof. Since the ideals I in this case are either complete intersections or height 2 Cohen-Macaulay ideals generated by 3 elements, we simply need to show the periodicity of the number of generators.

By Theorem 1.4, if this is a complete intersection, then $(j, b) = k$, $k(j+a) = \alpha j + \beta(j+b)$, with $\alpha + \beta = k$ and $\gcd(j, b) = k$. Thus, $\alpha(j+b) + \beta(j+2b) = k(j+a) + (\alpha+\beta)b = k(j+a+b)$. Therefore, $(j+b, j+a+b, j+2b)$ also defines a complete intersection.

Conversely, suppose $(j+b, j+a+b, j+2b)$ defines a complete intersection. Since $j \geq \max\{ab, b(b-a)\}$, we have the same α, β, giving the equations as before. Therefore, for $j \geq \max\{ab, b(b-a)\}$, $(j+rb, j+a+rb, j+b+rb)$ is a complete intersection for all r if and only if it is a complete intersection for $(j, j+a, j+b)$. Thus the eventual periodicity is true for $d = 2$. □

2. Monomial curves in $\mathbb{A}^n$ for $n \geq 4$

In this section we prove the periodicity of occurrence of complete intersections in the class $\Gamma_{\underline{a}+(\underline{j})} \subset \mathbb{A}^n$ and characterize complete intersection monomial curves in $\mathbb{A}^4$.

Theorem 2.1. *If $\underline{a} = (a_1, \ldots, a_n)$, then $CI(\underline{a})$ is either finite or eventually periodic with period a_n. If for $j \gg 0$, $\underline{a} + (j)$ is a complete intersection, then there exist $1 \leq s \leq n-1$ and $k \in \mathbb{Z}_+$ such that*

(a) *$j = a_n m$ for some $m \in \mathbb{Z}_+$,*
(b) *$\gcd(a_1, \ldots, a_{s-1}, a_s + a_{s+1}, a_{s+2}, \ldots, a_n) = k \neq 1$ and*
(c) *$a_s \in \left\langle \frac{a_1}{k}, \ldots, \frac{a_{s-1}}{k}, \frac{a_{s+1}}{k}, \ldots, \frac{a_n}{k} \right\rangle$.*

Proof. We first assume that $\gcd(a_1, \ldots, a_n) = 1$. Assume that $CI(\underline{a})$ is not finite. Assume that $\underline{a}+(\underline{j})$ is a complete intersection. Therefore it follows from Lemma 1.2 and Lemma 1.3 that we have the complete intersection split of the form

$$\underline{a} + (\underline{j}) \quad = \quad \frac{j+a_s}{k'}(k') \sqcup k\left(\frac{j}{k}, \frac{j+a_1}{k}, \ldots, \frac{j+a_{s-1}}{k}, \frac{j+a_{s+1}}{k}, \ldots, \frac{j+a_n}{k}\right),$$

which satisfies the conditions in Theorem 1.1. Furthermore, we have the following:

(1) $k' \mid k$ and $k' \neq k$.
(2) $k \mid \gcd(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_n)$.

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(3) Since $k' \mid k$, it divides a_i for $i = 1, \ldots, s-1$ and divides j as well. Therefore, $k' \mid a_s$, and hence $k' \mid a_i$ for all $i = 1, \ldots, n$. This implies that $k' \mid \gcd(a_1, \ldots, a_n) = 1$. Therefore $k' = 1$.

Since $\left(\frac{j}{k}, \frac{j+a_1}{k}, \ldots, \frac{j+a_{s-1}}{k}, \frac{j+a_{s+1}}{k}, \ldots, \frac{j+a_n}{k}\right)$ is a complete intersection sequence (associated to a sequence of length $n-1$), it follows by induction on n that for $j \gg 0$,

$$\frac{j}{k} = \frac{a_n}{k} m,$$

and hence $j = a_n m$, where m is a positive integer. Since $k' = 1$, we have

$$j + a_s \in \left\langle \frac{j}{k}, \frac{j+a_1}{k}, \ldots, \frac{j+a_{s-1}}{k}, \frac{j+a_{s+1}}{k}, \ldots, \frac{j+a_n}{k} \right\rangle,$$

and therefore there exist some non-negative integers $\alpha_1, \ldots, \alpha_n$, not all zero, such that

$$j + \sum_{i=1}^{s} a_i = \alpha_1 \frac{j}{k} + \cdots + \alpha_s \frac{j+a_{s-1}}{k} + \alpha_{s+1} \frac{j+a_{s+1}}{k} + \cdots + \alpha_n \frac{j+a_n}{k}. \tag{2}$$

Claim 1. $\sum_{i=1}^{n} \alpha_i = k$.

Proof of Claim 1. From the above equation, we can write

$$a_s = \left(\sum_{i=1}^{n} \alpha_i - k\right) \frac{j}{k} + \sum_{l=1}^{s} \alpha_l \frac{a_{l-1}}{k} + \sum_{l=s+1}^{n} \alpha_l \frac{a_l}{k}.$$

Suppose $\sum_{i=1}^{n} \alpha_i > k$. If $j > a_n^2$, then $\frac{j}{k} > a_n$, and hence we get that $a_s > a_n$, a contradiction.

Suppose $\sum_{i=1}^{n} \alpha_i < k$. Then we get

$$\begin{aligned} a_t &\leq \left(\sum_{i=1}^{n} \alpha_i - k\right) \frac{j}{k} + \frac{1}{k} \left(\sum_{i=1}^{n} \alpha_i\right) a_n \\ &< \left(\sum_{i=1}^{n} \alpha_i - k\right) \frac{j}{k} + a_n \leq 0, \end{aligned}$$

where the last inequality holds since $\frac{j}{k} \geq \sum_{i=1}^{n} a_i$. This is a contradiction since $a_s > 0$. Therefore, we have shown that neither of the cases

(a) $\sum_{i=1}^{n} \alpha_i > k$,
(b) $\sum_{i=1}^{n} \alpha_i < k$

is possible. Therefore, $\sum_{i=1}^{n} \alpha_i = k$. This completes the proof of the claim. □

Claim 2. $\underline{a} + (\underline{j + a_n}) = (j + a_n, j + a_n + a_1, \ldots, j + 2a_n)$ is a complete intersection sequence.

Proof of Claim 2. We show that this sequence has a complete intersection split similar to that of $\underline{a}$. Choose $\alpha_1, \ldots, \alpha_n$ as in (2). Therefore we have $\sum_{i=1}^{n} \alpha_i = k$

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so that

$$\begin{aligned}\sum_{l=1}^{s}\alpha_l\left(\frac{j}{k}+\frac{a_n}{k}+\frac{a_{l-1}}{k}\right) &+ \sum_{l=s+1}^{n}\alpha_l\left(\frac{j}{k}+\frac{a_n}{k}+\frac{a_l}{k}\right)\\ &= j+a_s+\left(\sum_{i=1}^{n}\alpha_i\right)\frac{a_n}{k}\\ &= j+a_s+a_n.\end{aligned}$$

Therefore,

$$j+a_n+a_s\in\left\langle\frac{j}{k}+\frac{a_n}{k},\dots,\frac{j}{k}+\frac{a_{s-1}}{k}+\frac{a_n}{k},\frac{j}{k}+\frac{a_{s+1}}{k}+\frac{a_n}{k},\dots,\frac{j}{k}+\frac{a_n}{k}+\frac{a_n}{k}\right\rangle,$$

We need to show that this split satisfies all the properties of Theorem 1.1. Since $\gcd(j+a_s,k)=1$ and $k\mid a_n$, $\gcd(j+a_n+a_s,k)=1$. Note that

$$\gcd\left(\frac{a_1}{k},\dots,\frac{a_{s-1}}{k},\frac{a_{s+1}}{k},\dots,\frac{a_n}{k}\right)=1.$$

Since $\left(\frac{j}{k},\frac{j+a_1}{k},\dots,\frac{j+a_{s-1}}{k},\frac{j+a_{s+1}}{k},\dots,\frac{j+a_n}{k}\right)$ is a complete intersection, by induction on n, we get that

$$\left(\frac{j}{k}+\frac{a_n}{k},\dots,\frac{j}{k}+\frac{a_{s-1}}{k}+\frac{a_n}{k},\frac{j}{k}+\frac{a_{s+1}}{k}+\frac{a_n}{k},\dots,\frac{j}{k}+\frac{a_n}{k}+\frac{a_n}{k}\right)$$

is a complete intersection sequence. Therefore, if $CI(\underline{a})$ is infinite, then it is eventually periodic with period a_n. □

Now let $k'=\gcd(a_1,\dots,a_n)$. Assume that $\underline{a}+(\underline{j})$ is a complete intersection. Then we have a split of the form

$$\underline{a}+(\underline{j})=\frac{j+a_s}{k'}\ (k')\sqcup k\left(\frac{j}{k},\dots,\frac{j+a_{s-1}}{k},\frac{j+a_{s+1}}{k},\dots,\frac{j+a_n}{k}\right),$$

where $k=\gcd(a_1,\dots,a_{s-1},a_{s+1},\dots,a_n)$, $k'\mid k$, $k\neq k'$, $\gcd\left(\frac{j+a_s}{k'},k\right)=1$ and $\left(\frac{j}{k},\dots,\frac{j+a_{s-1}}{k},\frac{j+a_{s+1}}{k},\dots,\frac{j+a_n}{k}\right)$ is a complete intersection. Since

$$\gcd\left(\frac{a_1}{k},\dots,\frac{a_{s-1}}{k},\frac{a_{s+1}}{k},\dots,\frac{a_n}{k}\right)=1,$$

we can use Theorem 2.1 to conclude that $\frac{j}{k}=\frac{a_n}{k}m$ for some $m\in\mathbb{Z}_+$. Hence $j=(\sum_{i=1}^{n}a_i)m$. We also have that

$$\left(\frac{j+a_n}{k},\dots,\frac{j+a_n+a_{s-1}}{k},\frac{j+a_n+a_{s+1}}{k},\dots,\frac{j+a_n+a_n}{k}\right)$$

is a complete intersection, since the period is $\frac{a_n}{k}$. This shows that we have a complete intersection split

$$\begin{aligned}\underline{a}+(\underline{j+a_n})=&\frac{j+a_n+a_s}{k'}\ (k')\\ &\sqcup k\left(\frac{j+a_n}{k},\dots,\frac{j+a_n+a_{s-1}}{k},\frac{j+a_n+a_{s+1}}{k},\dots,\frac{j+a_n+a_n}{k}\right).\end{aligned}$$

This implies that $\underline{a}+(\underline{j+a_n})$ is a complete intersection, proving the periodicity as well. □

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As a consequence of the above result, we relate the complete intersection property of $\underline{a}$ and $\underline{a}+(\underline{j})$ for $j \gg 0$.

Corollary 2.2. *For $j \gg 0$, if $\underline{a}+(\underline{j})$ is a complete intersection, then $\underline{a}$ is a complete intersection.*

Proof. First assume that $\gcd(a_1, \ldots, a_n) = 1$. We prove the first statement by induction on n. If $n = 1$, there is nothing to prove, and for $n = 2$, (a_1, a_2) is always a complete intersection. Assume now that $\underline{a} = (a_1, \ldots, a_n)$ with $n \geq 3$ and that $\underline{a}+(\underline{j})$ is a complete intersection for $j \gg 0$. By Theorem 2.1, there exist an s and a k such that

$$\underline{a}+(\underline{j}) = j + a_s \ (1) \sqcup k\left(\frac{j}{k}, \frac{j+a_1}{k}, \ldots, \frac{j+a_{s-1}}{k}, \frac{j+a_{s+1}}{k}, \ldots, \frac{j+a_n}{k}\right)$$

with $\left(\frac{j}{k}, \frac{j+a_1}{k}, \ldots, \frac{j+a_{s-1}}{k}, \frac{j+a_{s+1}}{k}, \ldots, \frac{j+a_n}{k}\right)$ a complete intersection, $\gcd(j+a_s, k) = 1$ and $a_s \in \left\langle \frac{a_1}{k}, \ldots, \frac{a_{s-1}}{k}, \frac{a_{s+1}}{k}, \ldots, \frac{a_n}{k} \right\rangle$. By induction on n, we get that $\left(\frac{a_1}{k}, \ldots, \frac{a_{s-1}}{k}, \frac{a_{s+1}}{k}, \ldots, \frac{a_n}{k}\right)$ is a complete intersection. Hence we have a complete intersection split:

$$\underline{a} = a_s \ (1) \sqcup k\left(\frac{a_1}{k}, \ldots, \frac{a_{s-1}}{k}, \frac{a_{s+1}}{k}, \ldots, \frac{a_n}{k}\right).$$

Therefore $\underline{a}$ is a complete intersection. Now assume that $\gcd(a_1, \ldots, a_n) = k'$. Since $\underline{a}+(\underline{j})$ is a complete intersection for $j \gg 0$, it follows from Theorem 2.1 that $j = a_n m$ for some $m \in \mathbb{Z}_+$. Therefore $k' \mid j$. Let $j' = \frac{j}{k'}$ and $a_i = \frac{a_i}{k'}$. Since $\underline{a}+(j)$ is a complete intersection, so is $(j', j'+a_1', \ldots, j'+a_n')$. By the first part, this implies that $(a_1', \ldots, a_n')$ is a complete intersection, and hence $(a_1, \ldots, a_n)$ too is a complete intersection. □

We now prove a partial converse of the above corollary. It can be seen that a converse statement of Corollary 2.2 is not true; cf. Example 3.3.

Proposition 2.3. *If $n \geq 3$ and $\underline{a}$ is a complete intersection and $k_{i+1}a_i \in \langle a_{i+1}, \ldots, a_n \rangle$, where $k_i = \gcd(a_i, \ldots, a_n)$, then there exists $j \gg 0$ such that $\underline{a}+(\underline{j})$ is a complete intersection.*

Proof. First assume that $\gcd(a_1, \ldots, a_n) = 1$. We prove the assertion by induction on n. Let $n = 3$. Let

$$\underline{a} = a_1 \ (1) \sqcup k\left(\frac{a_2}{k}, \frac{a_3}{k}\right),$$

where $k = \gcd(a_2, a_3)$. Since $a_1 \in \left\langle \frac{a_2}{k}, \frac{a_3}{k} \right\rangle$, we can write $ka_1 = \beta a_2 + \gamma a_3$. Since $a_1 < a_i$ for $i = 2, 3$, $k \geq \beta + \gamma$. Let $\alpha = k - \beta - \gamma$. Then for $j \geq 0$, $(j + a_1) = \alpha\frac{j}{k} + \beta\frac{j+a_2}{k} + \gamma\frac{j+a_3}{k}$. By Theorem 1.4, there exists $j \gg 0$, $j = a_3 m$ such that $\left(\frac{j}{k}, \frac{j+a_2}{k}, \frac{j+a_3}{k}\right)$ is a complete intersection.

Let $a_1 = \alpha_2\frac{a_2}{k} + \cdots + \alpha_n\frac{a_n}{k}$. Since $a_1 < a_i$ for all $i = 2, \ldots, n$, $\sum_{i=2}^{n} \alpha_i \leq k$. Let $\alpha_1 = k - \sum_{i=2}^{n} \alpha_i$. Then for any $j > 0$, we can write

$$j + a_1 = \alpha_1\frac{j}{k} + \alpha_2\frac{j+a_2}{k} + \cdots + \alpha_n\frac{j+a_n}{k}.$$

Since $\frac{a_2}{k_2} \in \left\langle \frac{a_3}{k_2k_3}, \ldots, \frac{a_n}{k_2k_3} \right\rangle$ and $\left(\frac{a_3}{k_2k_3}, \ldots, \frac{a_n}{k_2k_3}\right)$ is a complete intersection, by induction we get that $\left(\frac{j}{k_2}, \frac{j+a_2}{k_2}, \ldots, \frac{j+a_n}{k_n}\right)$ is a complete intersection for some $j \gg 0$.

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Therefore by Theorem 1.1, $\underline{a} + (\underline{j})$ is a complete intersection. If $\gcd(a_1, \ldots, a_n) = k_1 \neq 1$, then we can divide by k_1 to get a complete intersection sequence $\underline{a}'$, apply the first part to obtain a j' such that $\underline{a}' + (\underline{j}')$ is a complete intersection and then multiply by k_1 to conclude that $\underline{a} + (\underline{j})$ is a complete intersection. □

We now characterize the complete intersection sequences when $n = 3$. It is actually possible to formulate a similar result in the general case, but it is highly complicated. Therefore, we stick to the case of $n = 3$.

Theorem 2.4. *Let $\underline{a} + (\underline{j}) = (j, j + a, j + b, j + c)$. Then $\underline{a} + (\underline{j})$ is a complete intersection for $j \gg 0$ if and only if there exist non-negative integers m, k, β, γ such that $j = cm$, $k \neq 1$ and one of the following is satisfied:*

(1a) $\gcd(a, c) = k$ *and*
(1b) $ka = \beta b + \gamma c$

OR

(2a) $\gcd(b, c) = k$,
(2b) $kb = \beta a + \gamma(c)$ *with* $\beta + \gamma \leq k$.

Proof. If $\underline{a}+(\underline{j})$ is a complete intersection sequence for $j \gg 0$, then by Theorem 2.1, one of the two sets of conditions is satisfied. We now prove the converse. First assume that (1a) and (1b) are true. Let $\alpha = k - (\beta + \gamma)$. Using (1a), we can write $j + b = \alpha\left(\frac{j}{k}\right) + \beta\left(\frac{j+a}{k}\right) + \gamma\left(\frac{j+c}{k}\right)$. Note that $\gcd\left(\frac{a}{k}, \frac{c}{k}\right) = 1$. Therefore by Theorem 1.4, we get that $\left(\frac{j}{k}, \frac{j+b}{k}, \frac{j+c}{k}\right)$ is a complete intersection if $j \gg 0$ and $\frac{j}{k} = \frac{c}{k}m$ for some m. Therefore if $j \gg 0$ and $j = cm$, then $\left(\frac{j}{k}, \frac{j+b}{k}, \frac{j+c}{k}\right)$ is a complete intersection. Let $k' = \gcd(a, b, c) = \gcd(k, a)$. Then we can write

$$\underline{a} + (\underline{j}) = \frac{j + a}{k'}(k') \sqcup k\left(\frac{j}{k}, \frac{j + b}{k}, \frac{j + c}{k}\right)$$

with $k' \mid k$, $k' \neq k$, $\gcd\left(\frac{j+a}{k'}, k\right) = 1$ and $\left(\frac{j}{k}, \frac{j+a+b}{k}, \frac{j+a+b+c}{k}\right)$ a complete intersection. Therefore $\underline{a} + (j)$ is a complete intersection. If we assume the second set of conditions, then the proof can be obtained by interchanging the roles of a and b. □

3. Examples

We conclude the article by giving some examples. In the first example, we show the periodicity.

Example 3.1. Let $\underline{a} = (11, 16, 28)$. Let $j = 28m$ for some $m > 1$. Then it can be seen that $28m + 11 = 2(7m) + (7m + 4) + (7m + 7)$ and that $(7m, 7m + 4, 7m + 7)$ is a complete intersection (here we need $m > 1$). Therefore $(28m, 28m + 11, 28m + 16, 28m + 28)$ is a complete intersection sequence.

The next example shows that $CI(\underline{a})$ could be non-empty and finite.

Example 3.2. Let $\underline{a} = (3, 8, 20)$. For $j = 28$, we have $\underline{a} + (\underline{j}) = (28, 31, 36, 48)$, and it can be seen that $\underline{a} + (\underline{j}) = 31\ (1) \sqcup 4(7, 9, 12)$ is a complete intersection split. Therefore $\underline{a} + (j)$ is a complete intersection. Suppose $\underline{a} + (\underline{j})$ is a complete

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intersection for $j \gg 0$. Since $\gcd(3, 20) = 1$, the only possible split for $j \gg 0$ is of the form

$$\underline{a} + (\underline{j}) = \frac{j+3}{k'} \ (k') \sqcup 4 \left(\frac{j}{4}, \frac{j+8}{4}, \frac{j+20}{4} \right),$$

with $\alpha + \beta + \gamma = 4$. This gives us the equation

$$12 = 8\beta + 20\gamma.$$

Since this does not have a non-negative integer solution, we arrive at a contradiction. Therefore, $CI(\underline{a})$ is finite. This example also shows that taking $j > a_n$ is not enough.

The next example shows that the converse of Corollary 2.2 is not always true, even for $j > a_n^2$ and $j = a_n m$.

Example 3.3. Let $\underline{a} = (8, 17, 18)$. Then $\underline{a}$ is a complete intersection sequence. For $j \gg 0$ and $j = 18m$, the only possibility of a complete intersection split is of the form

$$\underline{a} + (\underline{j}) = j + 17 \ (1) \sqcup 2 \left(\frac{j}{2}, \frac{j+8}{2}, \frac{j+18}{2} \right)$$

such that $j + 17 = \alpha \frac{j}{2} + \beta \frac{j+8}{2} + \gamma \frac{j+18}{2}$ with $\alpha + \beta + \gamma = 2$. Therefore, $17 = 4\beta + 9\gamma$ and $\beta + \gamma \leq 2$. This does not have a non-negative integer solution. Therefore, $\underline{a} + (\underline{j})$ cannot be a complete intersection for $j > 18^2$.

References

[1] Henrik Bresinsky and Lê Tuân Hoa, *Minimal generating sets for a family of monomial curves in* $\mathbf{A}^4$, Commutative algebra and algebraic geometry (Ferrara), Lecture Notes in Pure and Appl. Math., vol. 206, Dekker, New York, 1999, pp. 5–14. MR1702095 (2000f:14046)

[2] Charles Delorme, *Sous-monoïdes d'intersection complète de N*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 145–154. MR0407038 (53 #10821)

[3] Philippe Gimenez, Indranath Sengupta, and Hema Srinivasan, *Minimal free resolutions for certain affine monomial curves*, Commutative algebra and its connections to geometry, Contemp. Math., vol. 555, Amer. Math. Soc., Providence, RI, 2011, pp. 87–95, DOI 10.1090/conm/555/10991. MR2882676

[4] P. Gimenez, I. Sengupta and H. Srinivasan, Minimal Graded Free Resolutions for Monomial Curves defined by Arithmetic Sequences, arXiv:1108.3203

[5] Jürgen Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970), 175–193. MR0269762 (42 #4657)

[6] D. P. Patil and I. Sengupta, *Minimal set of generators for the derivation module of certain monomial curves*, Comm. Algebra **27** (1999), no. 11, 5619–5631, DOI 10.1080/00927879908826778. MR1713057 (2000g:13014)

[7] I. Sengupta, Betti numbers of certain affine monomial curves, in EACA-2006 (Sevilla), F.-J. Castro Jiménez and J.-M. Ucha Enríquez, Eds., 171–173. ISBN: 84-611-2311-5

[8] A. Marzullo, On the Periodicity of the First Betti Number of the Semigroup Rings under Translation, Thesis, University of Missouri, 2010.

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