ON THE RATIONALITY OF NAGARAJ-SESHADRI MODULI SPACE

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ABSTRACT. We show that each of the irreducible components of moduli of rank 2 torsion-free sheaves with odd Euler characteristic over a reducible nodal curve is rational.

1. INTRODUCTION

Let X be a reducible nodal curve over an algebraically closed field k of characteristic 0 such that it is a union of two smooth irreducible components X_1 of genus $g_1 \geq 2$ and X_2 of genus $g_2 \geq 2$ meeting exactly at one node p. Let $\mathbf{a} = (a_1, a_2)$ be a tuple of positive rational numbers such that $a_1 + a_2 = 1$; we call this a polarisation on X. Let χ be an integer such that $a_1\chi$ is not an integer. In this setting it is a theorem of Nagaraj-Seshadri [6, Theorem 4.1] that the moduli space $M(2,\mathbf{a},\chi)$ of semi-stable rank two torsion-free sheaves on X with Euler characteristic χ is a reduced, connected projective scheme with exactly two irreducible components, and when χ is odd, the moduli space is a union of two smooth varieties M_{12} and M_{21} intersecting transversally along a smooth divisor N.

Let $\xi = (L_1, L_2)$, where L_1 and L_2 are two invertible sheaves on X_1 and X_2 (of suitable degrees) respectively. Then in [6, Section 7] the analogue of a "fixed determinant moduli space" has been defined and we denote it by $M(2,\mathbf{a},\chi,\xi)$. It is shown in ([17], [1]) that when χ is odd and $a_1\chi$ is not an integer, $M(2,\mathbf{a},\chi,\xi)$ is also a reduced, connected projective scheme with exactly two smooth components meeting transversally along a smooth divisor. The main result of this article is the following:

Theorem 1.1. *If $gcd\ (\chi, 2) = 1$, then both the irreducible components of $M(2,\mathbf{a},\chi,\xi)$ are rational. In particular $M(2,\mathbf{a},\chi,\xi)$ is rationally connected.*

Over a smooth projective curve of genus $g \geq 2$, the rationality of the moduli space was first proved by Tjurin [16, Theorem 14] in the rank 2 and odd degree case. When rank and degree are coprime this result was generalized by Newstead [9], [10], King and Schofield [4] in higher order of generalities. It is still not known if the moduli space is rational or not in the non-coprime case, even for rank 2 and degree 0. In the non-smooth case, when the curve is irreducible and has any number of nodal singularities and genus ≥ 2, rationality in the coprime case was proved by Bhosle and Biswas [2, Theorem 3.7]. Over a reducible nodal curve X as described above it has been shown by Basu that each irreducible component of $M(2,\mathbf{a},\chi,\xi)$ is unirational [1, Lemma 2.5]. Motivated by this result we go to the next step i.e. to prove rationality of each of these components. The proof of our result broadly follows the strategy of Newstead [9] but involves several technical difficulties.

It is well known that the moduli space of bundles over curves has a good specialization property, i.e. if a smooth projective curve Y specializes to a projective curve X with nodes

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as the only singularities, then the moduli space of vector bundles M_Y on Y specializes to the moduli space of torsion-free sheaves M_X on X [3], [7], [12], [13]. It is known that rationality of projective varieties does not have a good specialization property, for example a family of cubic surface which is rational specializes to a non-rational surface which is birational to $E \times \mathbb{P}^1$ where E is a cubic curve . Our result shows that in the rank 2 and odd Euler characteristic case the moduli space of vector bundles gives an example of a family of rational varieties specializing to a rationally connected variety with two irreducible rational components. We hope that Theorem 1.1 will be useful in the study of degeneration of higher dimensional smooth projective algebraic varieties.

Further it will be interesting to see if Theorem 1.1 can be generalized to a more general situation i.e if the underlying curve C has more than 2 components together with more than one node. In such a general situation the moduli space of semistable torsion-free sheaves (arbitrary rank) has been constructed by Seshadri (see [Chapter VII, [13]). In particular when C is a tree like curve without any rational components, then the number of components of the moduli space and inequalities involving Euler characteristics has been computed by Montserrat Teixidor I Bigas [14], [15, Theorem 3.2]. It will be interesting to investigate the rationality of each of these components.

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2. A Brief Description of the Moduli Space

In this section, we shall briefly recall some of the results proved in [6] which will be useful in later sections. Let X be a reducible projective nodal curve as before which has two smooth irreducible components X_1 and X_2 meeting at the nodal point p. Any torsion-free sheaf E on X can be identified with a triple $(E_1, E_2, \overrightarrow{T})$ or $(E'_1, E'_2, \overleftarrow{S})$ where E_i's are locally free sheaves over X_i's and $\overrightarrow{T}$ and $\overleftarrow{S}$ are linear maps from $E_1(p)$ to $E_2(p)$ and $E'_2(p)$ to $E'_1(p)$ respectively. In fact in [6, Lemma 2.3], an equivalence is shown between the category of torsion-free sheaves and the category of triples. Let E be a torsion-free sheaf on X identified by the triple $(E_1, E_2, \overrightarrow{T})$ as well as $(E'_1, E'_2, \overleftarrow{S})$. Then we have the following equality of Euler characteristics between them (see[6],Remark 2.11)-

$$\chi(E) = \chi(E_1, E_2, \overrightarrow{T}) = \chi(E'_1, E'_2, \overleftarrow{S}), \tag{2.1}$$

where

$$\chi(E_1, E_2, \overrightarrow{T}) := \chi(E_1) + \chi(E_2) - rk(E_2), \tag{2.2}$$

and

$$\chi(E'_1, E'_2, \overleftarrow{S}) := \chi(E'_1) + \chi(E'_2) - rk(E'_1). \tag{2.3}$$

Let $\mathbf{a} = (a_1, a_2)$ be a polarisation on X with $a_i > 0$ rational numbers and $a_1 + a_2 = 1$. For every non-zero triple $(E_1, E_2, \overrightarrow{T})$, we define

$$\mu((E_1, E_2, \overrightarrow{T})) = \frac{\chi(E_1, E_2, \overrightarrow{T})}{a_1 rk(E_1) + a_2 rk(E_2)}.$$

Definition 2.1. *Let* $(E_1, E_2, \overrightarrow{T})$ *be a triple. We say that the triple* $(E_1, E_2, \overrightarrow{T})$ *is stable (resp. semi-stable) if for every proper subtriple* $(G_1, G_2, \overrightarrow{U})$,
$\mu((G_1, G_2, \overrightarrow{U})) < \mu((E_1, E_2, \overrightarrow{T}))$ *(resp.* $\leq$*).*

When χ is odd and $a_1\chi$ is not an integer the moduli space $M(2, \mathbf{a}, \chi)$ of semistable torsion-free sheaves on X with Euler characteristic χ is a reduced, connected projective variety with the two smooth irreducible components M_{12} and M_{21} intersecting transversally on a smooth projective variety N [6, Theorem 4.1]. The irreducible components M_{12} and M_{21} have the following description in terms of triples:

The first component M_{12} is a smooth projective variety which is a fine moduli space of stable triples $(E_1, E_2, \overrightarrow{T})$ such that E_i's are rank 2 vector bundles over X_i's, and $\overrightarrow{T} : E_1(p) \to E_2(p)$ is a nonzero linear map such that

$$a_1\chi < \chi(E_1) < a_1\chi + 1, \; a_2\chi + 1 < \chi(E_2) < a_2\chi + 2. \tag{2.4}$$

The second component M_{21} also has a similar description in terms of triples. It is a smooth projective variety which is a fine moduli space of stable triples $(E_1', E_2', \overleftarrow{S})$ such that E_i''s are rank 2 vector bundles over X_i's, and $\overleftarrow{S} : E_2'(p) \to E_1'(p)$ is a nonzero linear map such that

$$a_1\chi + 1 < \chi(E_1') < a_1\chi + 2, \; a_2\chi < \chi(E_2') < a_2\chi + 1. \tag{2.5}$$

The intersection $N = M_{12} \cap M_{21}$ can be identified with $P_1 \times P_2$ where P_i's are certain parabolic moduli spaces over X_i's (see [6], Theorem 6.1 for details). In terms of triples, N is given by

$$\{[E_1, E_2, \overrightarrow{T}] \in M_{12} \mid rk(\overrightarrow{T}) = 1\},$$

which can be identified with

$$\{[E_1', E_2', \overleftarrow{S}] \in M_{21} \mid rk(\overleftarrow{S}) = 1\}.$$

In this paper, we are interested in the fixed determinant case. Let $E \in M(2, \mathbf{a}, \chi)$ be identified by the triple $(E_1, E_2, \overrightarrow{T})$ as well as the triple $(E_1', E_2', \overleftarrow{S})$. Let $\chi_i := \chi(E_i)$ and $\chi_i' := \chi(E_i')$, for $i = 1, 2$. If χ_i satisfy the inequalities in (2.4), then $E \in M_{12}$ and if χ_i' satisfy the inequalities in (2.5), then $E \in M_{21}$. Let $J^{d_i}(X_i)$ be the Jacobian of line bundles of degree d_i on X_i, where $d_i = \chi_i - 2(1 - g_i)$, for $i = 1, 2$. Now by [6, Proposition 7.1], there is a well-defined surjective morphism

$$\det : M(2, \mathbf{a}, \chi \neq 0) \longrightarrow J^{d_1}(X_1) \times J^{d_2}(X_2),$$

given by

$$\begin{array}{lll} E & \mapsto & (\Lambda^2(E_1), \Lambda^2(E_2)) \quad \text{if} \quad E \in M_{12}, \quad and \\ E & \mapsto & \Phi((\Lambda^2(E_1')), \Lambda^2(E_2')) \quad \text{if} \quad E \in M_{21}, \end{array}$$

where

$$\Phi : J^{d_1+1}(X_1) \times J^{d_2-1}(X_2) \to J^{d_1}(X_1) \times J^{d_2}(X_2)$$

is an isomorphism defined by

$$(L_1, L_2) \mapsto (L_1 \otimes \mathcal{O}_{X_1}(-p), L_2 \otimes \mathcal{O}_{X_2}(p)).$$

Now let us fix $L_1 \in J^{d_1}(X_1)$ and $L_2 \in J^{d_2}(X_2)$, we write $\xi = (L_1, L_2)$. Then the fixed determinant moduli space $M(2, \mathbf{a}, \chi, \xi)$ is by definition $\det^{-1}(\xi)$. By [6, Proposition 7.2], it is reduced. Let $\det_{12}$ (resp. $\det_{21}$) be the morphism $\det|_{M_{12}}$ (resp. $\det|_{M_{21}}$). For notational convenience we write $M_{12}(\xi)$ (resp. $M_{21}(\xi)$) for $\det_{12}^{-1}(\xi)$ (resp. $\det_{21}^{-1}(\xi)$). Then we have

$$M_{12}(\xi) \quad = \quad \{[E_1, E_2, \overrightarrow{T}] \in M_{12} \mid \Lambda^2(E_1) = L_1,\ \Lambda^2(E_2) = L_2\},$$

and

$$M_{21}(\xi) \quad = \quad \{[E_1', E_2', \overleftarrow{S}] \in M_{21} \mid \Lambda^2(E_1') = L_1 \otimes \mathcal{O}_{X_1}(p), \Lambda^2(E_2') = L_2 \otimes \mathcal{O}_{X_2}(-p))\}.$$

By [1, Proposition 6.5], the fixed determinant moduli space is a connected, projective scheme with exactly two smooth irreducible components $M_{12}(\xi)$ and $M_{21}(\xi)$, meeting transversally along the smooth divisor $N(\xi) = M_{12}(\xi) \cap N$ (which is identified with $M_{21}(\xi) \cap N$). Since χ is assumed to be an odd integer, and $\chi = \chi_1 + \chi_2 - 2$, we can conclude that either χ_1 is odd or χ_2 is odd and not both. (Same argument applies to χ_1' and χ_2' also).

Our aim in this paper is to prove that both $M_{12}(\xi)$ and $M_{21}(\xi)$ are rational. First we prove that $M_{12}(\xi)$ is rational. That the other component $M_{21}(\xi)$ is rational follows from similar arguments.

3. Construction of a stable family

Let i_1 and i_2 be the closed immersions given by $X_1 \to X$ and $X_2 \to X$ respectively. We choose an invertible sheaf L_1 on X_1 such that it is generated by global sections and is of degree $2g_1 - 1$ (see Remark 3.2(a)). Let L_2 be an invertible sheaf on X_2 of degree $2g_2$. Clearly $H^1(X_j, L_j) = 0$, for $j = 1, 2$. Also by [8, Lemma 5.2], L_2 is generated by global sections. Now if $\overrightarrow{\lambda} : L_1(p) \to L_2(p)$ is an isomorphism of vector spaces, then the triple $(L_1, L_2, \overrightarrow{\lambda})$ corresponds to an invertible sheaf L on X and we have

$$\chi(L) = \chi(L_1, L_2, \overrightarrow{\lambda}) = \chi(L_1) + \chi(L_2) - rk(L_2) = g, \tag{3.1}$$

where $g = g_1 + g_2$. Also by [6, Proposition 2.2], we have the following short exact sequence

$$0 \to L \to i_{1_*}(L_1) \oplus i_{2_*}(L_2) \to T_\lambda \to 0, \tag{3.2}$$

where T_λ is supported only at p, and over the residue field $k(p)$, it is a vector space of dimension one. By [6, Lemma 2.3],

$$L = \{(v, w) \in i_{1_*}(L_1) \oplus i_{2_*}(L_2) \mid \overrightarrow{\lambda}(v(p)) = w(p)\}. \tag{3.3}$$

Lemma 3.1. *Let L be as above. Then*

(i) The functor $H^0(X, -)$ applied to (3.2) is exact.

(ii) dim $(H^0(X, L)) = g$ and dim $(H^1(X, L)) = 0$.

(iii) dim $(H^0(X, L^*)) = 0$ and dim $(H^1(X, L^*)) = 3g - 2$, where L^* is the dual of L.

Proof. Applying the functor $H^0(X, -)$ to (3.2), we get the exact sequence

$$0 \to H^0(X, L) \to H^0(X_1, L_1) \oplus H^0(X_2, L_2) \xrightarrow{\beta} H^0(X, T_\lambda). \tag{3.4}$$

(Here we are using the fact that $H^0(X_j, L_j) = H^0(X, i_{j_*}(L_j))$ as i_j's are closed immersions $X_j \to X$). Our aim is to show that the map β is surjective. Since $H^0(X, T_\lambda)$ is one dimensional, it is enough to show that β is a non-zero map. Now, as deg $(L_1) = 2g_1 - 1$ and $H^1(X_1, L_1) = 0$, it follows that dim $(H^0(X_1, L_1)) = g_1$. Similarly dim $(H^0(X_2, L_2)) = g_2 + 1$.

Consider the natural maps

$$\phi_j : H^0(X_j, L_j) \to L_j(p), \tag{3.5}$$

for $j = 1, 2$. As L_j's are generated by global sections, these maps are surjective. Let $(v_1, w_1) \in H^0(X_1, L_1) \oplus H^0(X_2, L_2)$ be such that $v_1(p) \neq 0$, $w_1(p) \neq 0$ and $\overrightarrow{\lambda}(v_1(p)) = w_1(p)$. Then by (3.3), $(v_1, 0)$ does not belong to $H^0(X, L)$, hence $\beta(v_1, 0) \neq 0$ in $H^0(X, T_\lambda)$. This proves (i).

(ii) is a direct consequence of (i).

Since pull-back operation commutes with tensor product, we have $i_j^*(L^*) = L_j^*$, for $j = 1, 2$. As deg $(L_1) = 2g_1 - 1$ and deg $(L_2) = 2g_2$, we have $H^0(X_j, L_j^*) = 0$ for $j = 1, 2$. Now by (3.2), (3.4) (applying to L^*) we get $H^0(X, L^*) = 0$. Hence,

$$\begin{aligned} \dim\,(H^1(X, L^*)) &= -\chi(L^*) \\ &= -\chi(L_1^*) - \chi(L_2^*) + 1 \\ &= 3g - 2. \end{aligned}$$

This proves (iii). □

Remark 3.2. *(a) In the above Lemma, we assume L_1 on X_1 to be globally generated and of degree $2g_1 - 1$. To see the existence of such an invertible sheaf L_1, one can take a degree one invertible sheaf $\mathscr{L}_1$ on X_1 such that $H^0(X_1, \mathscr{L}_1) = 0$ and define L_1 to be $\omega_{X_1} \otimes \mathscr{L}_1$, where ω_{X_1} is the canonical sheaf on X_1. The existence of such an $\mathscr{L}_1$ is clear because the genus $g_1 \geq 2$.*

(b) Let $q = 3g - 2$. Then by fixing a basis of $H^1(X, L^*)$, we can identify it with k^q. We have the natural k^*−action on k^q and

$$W = \{(a_1, a_2, \cdots, a_q) \in k^q \mid a_1 \neq 0\}$$

is clearly an invariant open subset of k^q under the k^*− action.

Let $A := \{(a_1, a_2, \cdots, a_q) \in W \mid a_1 = 1\}$ (Clearly A is Zariski closed and every orbit of the $k^* - action$ on W meets A in exactly one point).

Since the maps ϕ_j's mentioned in (3.5) are surjective, we have dim $(ker(\phi_1)) = g_1 - 1$ and dim $(ker(\phi_2)) = g_2$. Let $\{v_2, \cdots, v_{g_1}\}$ be a basis of $ker(\phi_1)$ and $\{w_2, \cdots, w_{g_2+1}\}$ be a basis of $ker(\phi_2)$. These bases can be extended to the bases $\{v_1, v_2, \cdots, v_{g_1}\}$ and $\{w_1, w_2, \cdots, w_{g_2+1}\}$ of $H^0(X_1, L_1)$ and $H^0(X_2, L_2)$ respectively where v_1 and w_1 are as in the proof of the Lemma 3.1. It is also clear from (3.3) that $(v_1, w_1), (v_2, 0), \cdots, (v_{g_1}, 0), (0, w_2), \cdots, (0, w_{g_2+1})$ will form a basis for $H^0(X, L)$.

Suppose $(0, 0) \neq (v, w) \in H^0(X, L)$. Then we have

$$\begin{aligned} (v, w) &= \alpha_1(v_1, w_1) + \alpha_2(v_2, 0) + \cdots + \alpha_{g_1}(v_{g_1}, 0) \\ &\quad + \beta_2(0, w_2) + \cdots + \beta_{g_2+1}(0, w_{g_2+1}), \end{aligned}$$

where α_i's and β_j's are scalars and at least one of them is non-zero.

We know that every non-zero section (v, w) defines a non-zero map $\mathscr{O}_X \to L$. Further, this map is injective if and only if both v and w are non-zero which is true if and only if at least one $\alpha_i \neq 0$ and at least one $\beta_j \neq 0$ or $\alpha_1 \neq 0$. Let

$$C' = \{(v, w) := \phi \in H^0(X, L) \mid \phi : \mathscr{O}_X \hookrightarrow L \text{ injective}\}.$$

Clearly C' is a non-empty open subset in $H^0(X,L)$.

Lemma 3.3. *(cf. [5]) Let L be as above. Then there exists a vector space V and a universal extension*

$$0 \to \mathcal{O}_{X\times V} \to \tilde{\mathcal{E}} \to \pi^*(L) \to 0 \tag{3.6}$$

of bundles over $V \times X$ (where $\pi : V\times X \to X$ is the projection map), such that there is a natural isomorphism

$$\alpha : V \to H^1(X,L^*)$$

where for each $v \in V$, $\alpha(v)$ is the element corresponding to the restriction of the extension (3.6) *to* $\{v\}\times X$.

Remark 3.4.
(1) *Suppose $\tilde{\mathcal{E}}$ is as in Lemma 3.3 and $v \in H^1(X,L^*)$ is such that dim $(H^0(X,\tilde{\mathcal{E}}_v)) = 1$. Then one can easily see that for any $w \in H^1(X,L^*)$, $\tilde{\mathcal{E}}_v \cong \tilde{\mathcal{E}}_w$ if and only if v and w are in the same orbit under the natural action of k^* on $H^1(X,L^*)$.*
(2) *When X is smooth the above lemma was proved in* [11, Proposition 3.1, pp. 19-20].

Lemma 3.5. *Let L_1 be as above. Then there exists an extension*

$$0 \to \mathcal{O}_{X_1} \to E_1 \to L_1 \to 0, \tag{3.7}$$

for which dim $(H^0(X_1,E_1) = 1$, and such an E_1 is stable.

Proof. The existence of such an extension on X_1 can be seen as a special case of [9, Lemma 5], and stability of the bundle E_1 can be seen as a special case of [9, Lemma 6]. □

Lemma 3.6. *Let L_2 be as above. Then there exists an extension*

$$0 \to \mathcal{O}_{X_2} \to E_2 \to L_2 \to 0, \tag{3.8}$$

for which dim $(H^0(X_2,E_2) = 2$, and such an E_2 is semi-stable.

Proof. Suppose $e_2 \in H^1(X_2,L_2^*)$ and (3.8) is the corresponding extension. Then it is clear that $\chi(E_2) = 2$ and therefore dim $(H^0(X_2,E_2)) \geq 2$.

Suppose $\phi_2 \in H^0(X_2,L_2)$ is any non-zero section. Then we have an injective morphism

$$\phi_2 : \mathcal{O}_{X_2} \hookrightarrow L_2. \tag{3.9}$$

Tensoring (3.9) by the canonical sheaf ω_{X_2} and applying the global section functor, we get the map

$$H^0(X_2,\omega_{X_2}) \hookrightarrow H^0(X_2,L_2\otimes\omega_{X_2}).$$

Taking dual and using the duality theorem, we get the map

$$H^1(X_2,L_2^*) \xrightarrow{\tilde{\phi_2}} H^1(X_2,\mathcal{O}_{X_2}).$$

Clearly $\tilde{\phi_2}$ is onto. This implies

$$\dim\,(ker(\tilde{\phi_2})) \;=\; \dim\,(H^1(X_2,L_2^*)) - g_2 \;>\; 0. \tag{3.10}$$

Applying the sheaf functors $\mathcal{H}om(L_2,-)$ and $\mathcal{H}om(\mathcal{O}_{X_2},-)$ to (3.8) and taking the long exact sequence, we get the following commutative diagram -

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Hom(L_2,\mathcal{O}_{X_2}) & \longrightarrow & Hom(L_2,E_2) & \longrightarrow & Hom(L_2,L_2) & \longrightarrow & H^1(X_2,L_2^*) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\tilde{\phi_2}} & & \\
0 & \longrightarrow & H^0(X_2,\mathcal{O}_{X_2}) & \longrightarrow & H^0(X_2,E_2) & \longrightarrow & H^0(X_2,L_2) & \longrightarrow & H^1(X_2,\mathcal{O}_{X_2}) & \longrightarrow & \cdots
\end{array}$$

From this diagram, it is clear that ϕ_2 lifts to a section on E if and only if $\tilde{\phi_2}(e_2) = 0$. This fact is proved in [11, Lemma 3.1], in greater generality. Also

$$\begin{aligned}\dim\,(H^0(X_2,E_2)) &= \dim\,(H^0(X_2,\mathcal{O}_{X_2})) + \dim\,(ker(H^0(X_2,L_2)) \to H^1(X_2,\mathcal{O}_{X_2})) \\ &= 1 + \dim\,(ker(H^0(X_2,L_2)) \to H^1(X_2,\mathcal{O}_{X_2})). \qquad (3.11)\end{aligned}$$

So

$$\dim\,(H^0(X_2,E_2)) > 2 \quad \Leftrightarrow \quad \dim\,(ker(H^0(X_2,L_2)) \to H^1(X_2,\mathcal{O}_{X_2})) > 1.$$

Suppose there exists an $e_2 \in \ker\tilde{\phi_2}$ (with (3.8) as the corresponding extension) such that ϕ_2 is the only section (up to scalar multiplication) which lifts to E_2. Then we are done.

Suppose this is not the case with any non-zero section $\phi_2 \in H^0(X_2,L_2)$. This means for every non-zero section $\phi_2 \in H^0(X_2,L_2)$ and every $e_2 \in \ker\tilde{\phi_2}$ (with (3.8) as the corresponding extension) there are at least two linearly independent sections that lift to the corresponding bundle E_2.

Let

$$Y = \{(e_2,\phi_2) \mid \phi_2 \neq 0 \textit{ and } \tilde{\phi_2}(e_2) = 0\} \subset H^1(X_2,L_2^*) \times H^0(X_2,L_2).$$

This implies

$$\begin{aligned}\dim\,(Y) &= \dim\,(H^0(X_2,L_2)) + \dim\,(H^1(X_2,L_2^*)) - g_2 \\ &= \dim\,(H^1(X_2,L_2^*)) + 1.\end{aligned}$$

(The last equality is true because $\dim\,(H^0(X_2,L_2)) = g_2 + 1$).

Now if $e_2 \in p_1(Y)$ (where p_1 is the first projection map from $H^1(X_2,L_2^*) \times H^0(X_2,L_2)$), then $\dim\, p_1^{-1}(e_2) \cap Y \geq 2$ because $e_2 \in p_1(Y)$ implies the corresponding bundle E_2 has at least two linearly independent lifts from $H^0(X_2,L_2)$ according to our assumption. So

$$\begin{aligned}\dim\, p_1(Y) &\leq \dim\,(Y) - 2 \\ &= \dim\,(H^1(X_2,L_2^*)) - 1.\end{aligned}$$

This implies that there exists an extension $e_2' \in H^1(X_2,L_2^*)$ which is not in $p_1(Y)$. So if E_2' is the bundle corresponding to e_2', then by equation (3.11), $\dim\,(H^0(X_2,E_2')) = 1$. But this is a contradiction as $\dim\,(H^0(X_2,E_2)) \geq 2$ for every extension in $H^1(X_2,L_2^*)$.

This proves that there exists a non-zero section $\phi_2 \in H^0(X_2,L_2)$ and an extension $e_2 \in \ker\tilde{\phi_2}$ such that ϕ_2 is the only non-zero section (up to scalar multiplication) which lifts to the corresponding bundle E_2. So by equation (3.11), $\dim\,(H^0(X_2,E_2)) = 2$.

Now to prove that such an E_2 is semi-stable, let G_2 be a line sub-bundle of E_2. We want to prove $\deg\,(G_2) \leq \frac{\deg\,(E_2)}{2} = \frac{2g_2}{2} = g_2$. Suppose $\deg\,(G_2) > g_2$, then $\chi(G_2) > 1$ and $\dim\, H^0(X_2,G_2) > 1$. But $\dim\,(H^0(X_2,E_2)) = 2$ and $G_2 \subset E_2$. So $\dim\, H^0(X_2,G_2) = 2$. This implies the map $\mathcal{O}_{X_2} \to E_2$ in the extension (3.8) factors through G_2. This forces G_2 to be isomorphic to $\mathcal{O}_{X_2}$. This implies $\deg\,(G_2) = 0$, which contradicts the fact that $\deg\,(G_2) > g_2$.

This proves that E_2 is semi-stable. □

Now by [6, Proposition 2.2], we have the following exact sequence-

$$0 \to L^* \to i_{1_*}(L_1^*) \oplus i_{2_*}(L_2^*) \to T_{\lambda^*} \to 0,$$

where $(L_1^*, L_2^*, \overrightarrow{\lambda^*})$ is the triple corresponding to L^*. By taking the long exact sequence corresponding to this and observing that $H^0(X, L^*) = H^0(X_1, L_1^*) = H^0(X_2, L_2^*) = 0$, we get -

$$0 \to H^0(X, T_{\lambda^*}) \to H^1(X, L^*) \xrightarrow{\gamma} H^1(X_1, L_1^*) \oplus H^1(X_2, L_2^*) \to 0. \tag{3.12}$$

Let $e_1 \in H^1(X_1, L_1^*)$ be an extension as in Lemma 3.5 and $e_2 \in H^1(X_2, L_2^*)$ be an extension as in Lemma 3.6. Let the corresponding extensions be (3.7) and (3.8) respectively. Then E_1 is stable, and E_2 is semi-stable. Since the map γ in the exact sequence (3.12) is surjective, given $(e_1, e_2) \in H^1(X_1, L_1^*) \oplus H^1(X_2, L_2^*)$, there exists an $e \in H^1(X, L^*)$ such that $\gamma(e) = (e_1, e_2)$. Let

$$0 \to \mathscr{O}_X \to E \to L \to 0 \tag{3.13}$$

be the extension corresponding to e. Then $E|_{X_1} = E_1$, $E|_{X_2} = E_2$, and so the triple corresponding to E will look like $(E_1, E_2, \overrightarrow{T})$, where $T : E_1(p) \to E_2(p)$ is an isomorphism at the node p. Since E_1 and E_2 are semi-stable and T has full rank, by [1, Lemma 2.3], the triple $((E_1, E_2, \overrightarrow{T}))$ is semi-stable. So the corresponding vector bundle E is semi-stable. Since $\chi(E) = 1$ and $a_1\chi$ is not an integer, semi-stability coincides with stability. So E is stable. Thus we have produced an extension of the form (3.13) in $H^1(X, L^*)$ such that E is stable.

Remark 3.7. *From the above arguments, it is clear that there exists an extension in $H^1(X, L^*)$ of the form (3.13) such that the corresponding bundle E is stable. Since stability is an open condition, the set $B = \{v \in H^1(X, L^*) \mid \tilde{\mathscr{E}}_v \ is\ stable\}$ is a non-empty k^*- invariant open set in $H^1(X, L^*)$, where $\tilde{\mathscr{E}}$ is as in Lemma 3.3. Since $H^1(X, L^*)$ is irreducible, $B \cap W$ is a non-empty k^*- invariant open set in $H^1(X, L^*)$, where W is as in Remark 3.2(b). This implies $B \cap A \neq \emptyset$, where A is as defined in Remark 3.2(b). Let $S = B \cap A$. Then S is a non-empty open subset of the affine space A consisting of stable rank two locally free sheaves $\tilde{\mathscr{E}}_s$.*

4. Rationality

We are now in a position to state and prove the main proposition -

Proposition 4.1. *Let χ be an odd integer and a_1, a_2 be rational numbers such that $0 < a_1 < a_2 < 1$ and $a_1 + a_2 = 1$. Suppose $a_1\chi$ is not an integer and χ_1 and χ_2 are integers such that*

$$a_1\chi < \chi_1 < a_1\chi + 1,\ a_2\chi + 1 < \chi_2 < a_2\chi + 2. \tag{4.1}$$

Let $L = (L_1, L_2, \overrightarrow{\lambda})$ be an invertible sheaf on X such that deg $(L_1) = \chi_1 - 2(1 - g_1)$ and deg $(L_2) = \chi_2 - 2(1 - g_2)$. Then there exists a non-empty open subset S of an affine space and a locally free sheaf $\mathscr{E}$ of rank two on $S \times X$ such that

(i) dim $(S) = 3g - 3$,

(ii) $\mathscr{E}_s \cong \mathscr{E}_t \Leftrightarrow s = t$,

(iii) for all $s \in S$, $\mathscr{E}_s$ is stable and $\Lambda^2(\mathscr{E}_s) = L$.

Proof. We prove the Proposition by considering the following two different cases.

Case 1 : Suppose χ_1 is odd and χ_2 is even and χ_1, χ_2 satisfies (4.1). In order to prove the Proposition for this case, we first assume $\chi = 1$. This implies $\chi_1 = 1$, $\chi_2 = 2$, deg $(L_1) = 2g_1 - 1$, deg $(L_2) = 2g_2$ and L_2 is globally generated. We further assume L_1 is globally generated and prove the Proposition in this case. Given such an L, we can apply Lemma 3.3 and get an extension $\tilde{\mathscr{E}}$ on $H^1(X, L^*) \times X$ as in (3.6). Let $q :=$ dim $(H^1(X, L^*)) = (3g - 2)$. From Remark

3.2(b), we obtain an affine space A which is closed in $H^1(X, L^*)$ and is of dimension $3g-3$. Let S be the open subset of A defined in Remark 3.7.

Let $\mathscr{E}' = \tilde{\mathscr{E}}\mid_{S\times X}$. By Remark 3.7, $\mathscr{E}'_s$ is stable and $\Lambda^2(\mathscr{E}'_s) = L$, $\forall\ s \in S$. Since S is open in A, dim $(S) = 3g-3$. By the choice of A, and Remark 3.4, it is clear that $\mathscr{E}'_s \cong \mathscr{E}'_t \Leftrightarrow s = t$. This proves the proposition for the case $\chi = 1$ and L_1 globally generated.

Now to prove the proposition for arbitrary odd χ_1 and even χ_2, let deg $(L_1) = 2l_1 - 1$, and deg $(L_2) = 2l_2$, where l_1, l_2 are integers. Let M be an invertible sheaf on X given by the triple $(M_1, M_2, \overrightarrow{\delta})$ such that deg $(M_1) = l_1 - g_1$, deg $(M_2) = l_2 - g_2$. Since the pull-back operation commutes with tensor product, the triple corresponding to the invertible sheaf $L\otimes M^{-2}$ will be $(L_1 \otimes M_1^{-2}, L_2 \otimes M_2^{-2}, \overrightarrow{\gamma})$, where $\overrightarrow{\gamma}$ is the corresponding map at p induced by $\overrightarrow{\lambda}$ and $\overrightarrow{\delta}$. Clearly deg $(L_1 \otimes M_1^{-2}) = 2g_1 - 1$, and deg $(L_2 \otimes M_2^{-2}) = 2g_2$. We can also assume that $L_i \otimes M_i^{-2}$ is generated by global sections for $i = 1, 2$, by choosing an appropriate M_1 (see Remark 4.2).

Now corresponding to this invertible sheaf $L \otimes M^{-2}$, we have proved above that there exists a locally free sheaf $\mathscr{E}'$ on $S\times X$ satisfying all the required properties. Let $\mathscr{E} := \mathscr{E}' \otimes p_X^*(M)$. So $\mathscr{E}_s = \mathscr{E}'_s \otimes M$. Clearly $\Lambda^2(\mathscr{E}_s) = L$ for every $s \in S$.

We now claim that $\mathscr{E}_s$ is stable. Let $\mathscr{E}'_s = (E_1, E_2, \overrightarrow{T})$. Then it is clear that $\chi(E_1) = 1$ and $\chi(E_2) = 2$. Also it is clear that the triple corresponding to $\mathscr{E}_s$ is $(E_1 \otimes M_1, E_2 \otimes M_2, \overrightarrow{T} \otimes \overrightarrow{\delta})$. Now since $\mathscr{E}'_s$ is stable, by [6, Theorem 5.1], E_1 and E_2 are semi-stable. So $E_j \otimes M_j$ is semi-stable, for $j = 1, 2$. Since $\overrightarrow{T} \otimes \overrightarrow{\delta}$ has full rank, by [1, Lemma 2.3],

$$semi-stability\ of\ E_j \otimes M_j \quad \Rightarrow \quad \mathscr{E}_s\ is\ semi-stable.$$

Since $\chi(\mathscr{E}_s)$ is odd and $a_1\chi$ is not an integer, semi-stability of $\mathscr{E}_s$ implies it is stable for every $s \in S$.

Case 2 : Suppose χ_1 is even and χ_2 is odd. Let L'_1 be an invertible sheaf on X_1 of degree $2g_1$ and L'_2 be an invertible sheaf on X_2 of degree $2g_2 - 1$ such that it is globally generated. Let $L' = (L'_1, L'_2, \overrightarrow{\lambda'})$, where λ' is a non-zero scalar. Then one can prove all results of section (3) by replacing L in those results by L' (The proofs are similar).
Now to prove the Proposition in this case, one first proves the existence of S and a locally free sheaf $\mathscr{E}'$ of rank two on $S\times X$ such that $\Lambda^2({\mathscr{E}'}_s) = L'$, $\forall s \in S$ as in Case (1), and then tensoring $\mathscr{E}'$ by a suitable invertible sheaf $M' = (M'_1, M'_2, \overrightarrow{\delta})$ as in Case(1), one gets a locally free sheaf $\mathscr{E}$ of rank two on $S\times X$ satisfying all the required properties. □

Remark 4.2. *Let L_1 be an invertible sheaf on X_1 of degree $2l_1 - 1$, where l_1 is any integer. Then the invertible sheaf $\omega_{X_1} \otimes \mathscr{L}_1 \otimes L_1^{-1}$ (where ω_{X_1} and $\mathscr{L}_1$ are as in Remark 3.2(a), is of degree $2(g_1 - l_1)$. So there exists an invertible sheaf N_1 of degree $(g_1 - l_1)$ such that $N_1^2 = \omega_{X_1} \otimes \mathscr{L}_1 \otimes L_1^{-1}$. Let $M_1 = N_1^{-1}$. Then $L_1 \otimes M_1^{-2} = \omega_{X_1} \otimes \mathscr{L}_1$ which is globally generated and is of degree $2g_1 - 1$.*

4.1. **Proof of Theorem 1.1.** Let χ and L be as in Proposition 4.1. Then there exists a non-empty open subset S of an affine space and a rank two locally free sheaf $\mathscr{E}$ on $S\times X$ such that properties (i), (ii) and (iii) of Proposition 4.1 are satisfied. Since $M_{12}(\xi)$ is a fine moduli space, the sheaf $\mathscr{E}$ on $S\times X$ induces a morphism

$$f : S \to M_{12}(\xi).$$

By (ii) of Proposition 4.1, f is injective. Since S and $M_{12}(\xi)$ are of the same dimension and we are in characteristic zero, this implies that f is birational. So $M_{12}(\xi)$ is rational.

Now, we briefly outline the proof of rationality of the other component $M_{21}(\xi)$.

Suppose L_1 and L_2 are as in Proposition 4.1. Then clearly deg $(L_1 \otimes \mathscr{O}_{X_1}(p)) = \chi'_1 - 2(1-g_1)$ and deg $(L_2 \otimes \mathscr{O}_{X_2}(-p)) = \chi'_2 - 2(1-g_2)$, where χ'_1 and χ'_2 are integers satisfying the inequalities

$$a_1\chi + 1 < \chi'_1 < a_1\chi + 2, \; a_2\chi < \chi'_2 < a_2\chi + 1. \tag{4.2}$$

Let $\hat{L} = (L_1 \otimes \mathscr{O}_{X_1}(p), L_2 \otimes \mathscr{O}_{X_2}(-p), \overleftarrow{\lambda})$, where λ is a non-zero scalar. Then one can show exactly as in Proposition 4.1 that there exists a non-empty open subset $\hat{S}$ of an affine space and a rank two locally free sheaf $\hat{\mathscr{E}}$ on $\hat{S} \times X$ such that

(i) dim $(\hat{S}) = 3g - 3$,

(ii) $\hat{\mathscr{E}}_s \cong \hat{\mathscr{E}}_t \Leftrightarrow s = t$,

(iii) for all $s \in \hat{S}$, $\hat{\mathscr{E}}_s$ is stable and $\Lambda^2(\hat{\mathscr{E}}_s) = \hat{L}$.

Now since $M_{21}(\xi)$ is a fine moduli space, the sheaf $\hat{\mathscr{E}}$ on $\hat{S} \times X$ induces a morphism

$$g : \hat{S} \to M_{21}(\xi).$$

Also the fact that $\hat{\mathscr{E}}_s \cong \hat{\mathscr{E}}_t \Leftrightarrow s = t$, implies g is injective. Since $\hat{S}$ and $M_{21}(\xi)$ are of same dimension and we are in characteristic zero, it implies that g is birational. So $M_{21}(\xi)$ is rational. This proves Theorem 1.1.

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