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Abstract. The master equation for dichotomous diffusion (DD) (the integral of a random telegraph process) is the well-known telegrapher's equation, which is converted to the Klein–Gordon equation by a simple transformation. After a brief recapitulation of the solution and of the analogy between DD and the Dirac equation in one spatial dimension, we consider *velocity*-biased DD. The corresponding master equation and its solution are presented. It is shown that these may be interpreted physically in terms of a Lorentz transformation to a frame moving with a boost velocity equal to the mean drift velocity of the diffusing particle. The modifications that arise in the connection with the Dirac equation are also exhibited. The correspondence between the rest mass of the Dirac particle and the frequency of direction reversal in the DD is shown to be modified precisely by the time dilatation correction to the latter quantity.

Contents

1. Introduction

The two main paradigms of diffusive processes as applied to physical situations are the Wiener process (standard Brownian motion) and dichotomous diffusion (DD). The former, first studied by Einstein [1] and Smoluchowski [2] in the physical context of the Brownian motion of colloidal particles, can be obtained as the integral of white noise (a stationary, Gaussian, delta-correlated Markov process). The latter is the integral of a stationary dichotomous Markov process (also called a random telegraph process). It can also be regarded as the continuum version of a 'persistent' random walk in which the walker remembers the direction of the previous step while taking the next step, thereby making the random walk a non-Markovian one. Such a random walk was originally proposed [3, 4] as a means of incorporating the effects of a finite velocity or momentum for a diffusing particle, in contrast to ordinary Brownian motion which formally implies an infinite velocity for the particle. Since then, DD has been extensively investigated, both in its own right and in the context of various applications [5]–[11]. Several interesting aspects of the more general problem of dichotomous flows have also been studied (see, for instance, [12]–[14]).

Although the position of the diffusing particle in DD is a non-Markov process, its position and velocity together constitute a Markov process. This leads to a differential equation of second order (in time), the so-called telegrapher's equation, for the probability density of the position of the particle in either of its two velocity states (and hence for the total positional probability density as well). A simple change of variables converts this to the Klein–Gordon equation, suggesting an analogy with relativistic quantum mechanics, similar to the well-known correspondence [15] between the Schrödinger equation for a non-relativistic free particle and the ordinary diffusion equation for Brownian motion. Such a correspondence was pointed out several years ago [16] between the coupled first-order equations obeyed by the probability distributions in the two velocity states in DD, and the Dirac equation in $(1 + 1)$ dimensions. Subsequently, the analogy between stochastic differential equations and quantum mechanical wave equations has been developed further in the context of stochastic quantum mechanics. This includes studies of relativistic stochastic processes associated with the Klein–Gordon equation [17], prescriptions to pass from random walks and Brownian motion to the Schrödinger and Dirac equations [18]–[21], the connection between the one-dimensional Dirac equation and certain Markovian random walks involving an extra time dimension [22], and an extension to nonlinear Dirac and diffusion equations [23].

In this paper, we call attention to an interesting extension of this correspondence that emerges in the case of biased DD, the bias being in velocity or momentum space rather than position space [24, 25]. Starting with a persistent random walk with unequal frequencies of direction reversal out of the left-moving and right-moving states of the walker, we pass to the continuum limit of velocity-biased DD. The associated master equation (and its solution) are shown to be interpretable as unbiased DD viewed from a Lorentz-transformed frame, the boost velocity of which is just the mean drift velocity induced by the bias. Moreover, the mean frequency of direction reversal is also modified by precisely the associated time dilatation factor. The other modifications that the quantum mechanical analogy undergoes are also brought out and discussed.

2. DD

We begin with a brief recapitulation of DD. A possible starting point is the simple stochastic differential equation corresponding to DD. Alternatively, we could begin with the difference equations corresponding to a persistent random walk, and pass to the continuum limit. As this is more convenient for our present purposes, we adopt this approach.

Consider a random walk on a linear chain with sites labelled by m, in discrete time n. The walk is spatially unbiased, but at each time step the walker has a probability p of jumping in the same direction as his previous step, and a probability $q = 1 - p$ of reversing direction. If $\alpha_R(m, n)$ (respectively, $\alpha_L(m, n)$) is the probability that the walker is at site m after n steps in the right-moving (respectively, left-moving) state, the recursion relations satisfied by α_R and α_L are [5]

$$\begin{aligned} \alpha_R(m, n) &= p\alpha_R(m - 1, n - 1) + q\alpha_L(m - 1, n - 1), \\ \alpha_L(m, n) &= q\alpha_R(m + 1, n - 1) + p\alpha_L(m + 1, n - 1). \end{aligned} \tag{1}$$

If we choose the symmetric initial conditions $\alpha_R(m, 0) = \alpha_L(m, 0) = \frac{1}{2}\delta_{m,0}$, we also have the symmetry property $\alpha_R(-m, n) = \alpha_L(m, n)$. DD is the continuum limit of this random walk, attained by setting $n = t/\tau$, where τ^{-1} is the mean jump rate, $m = x/a$, where a is the lattice spacing, and then passing to the limits $\tau \to 0$, $a \to 0$, $q \to 0$ such that $\lim(a/\tau) = c$, the speed of the diffusing particle, while $\lim(q/\tau) = \lambda$, its mean rate of direction reversal. The probability densities $P_{R,L}(x, t) = \lim(2a)^{-1}\alpha_{R,L}(m, n)$ then satisfy the coupled first-order differential equations

$$(\partial_t + c\partial_x + \lambda)P_R = \lambda P_L, \qquad (\partial_t - c\partial_x + \lambda)P_L = \lambda P_R. \tag{2}$$

Eliminating P_R (or P_L) in equations (2), we find that each of these quantities satisfies the so-called telegrapher's equation, namely,

$$(\partial_t^2 + 2\lambda\partial_t - c^2\partial_x^2)P_{R,L} = 0. \tag{3}$$

For simplicity, let us consider the solution on the infinite line $-\infty < x < \infty$ with natural boundary conditions, with symmetric initial conditions

$$P_R(x, 0) = P_L(x, 0) = \tfrac{1}{2}\delta(x). \tag{4}$$

The additional initial conditions required for the solution of the second-order equation (3) are immediately obtained by using (4) in equations (2), to get

$$\left(\frac{\partial P_R}{\partial t}\right)_{t=0} = -\tfrac{1}{2}c\delta'(x), \qquad \left(\frac{\partial P_L}{\partial t}\right)_{t=0} = \tfrac{1}{2}c\delta'(x). \tag{5}$$

With these initial data, the general solution [26] of the telegrapher's equation reduces to the following explicit expressions for P_R and P_L:

$$\begin{aligned} P_{R,L}(x, t) = \tfrac{1}{2}\mathrm{e}^{-\lambda t}\{&\delta(x \mp ct) + (\lambda/2c)[\theta(x + ct) - \theta(x - ct)] \\ &\times [I_0(\lambda\xi/c) \pm (x \pm ct)\xi^{-1}I_1(\lambda\xi/c)]\}, \end{aligned} \tag{6}$$

where $\xi = (c^2t^2 - x^2)^{1/2}$ and I_r is the modified Bessel function of order r. We note that $P_R(-x, t) = P_L(x, t)$ for any $t \geqslant 0$, as expected for the symmetric initial conditions considered. Each of the densities $P_{R,L}$ is normalized to $\frac{1}{2}$. The total probability density $P = P_R + P_L$ is normalized to unity, and is given by

$$P(x, t) = \tfrac{1}{2}\mathrm{e}^{-\lambda t}\{\delta(x + ct) + \delta(x - ct) + \lambda[\theta(x + ct) - \theta(x - ct)] \\ \times [(1/c)I_0(\lambda\xi/c) + (t/\xi)I_1(\lambda\xi/c)]\}. \tag{7}$$

As already mentioned, it is well known that the telegrapher's equation (3) may also be obtained [6]–[8] as the master equation for DD by starting with the corresponding stochastic differential equation

$$\dot{x} = c\zeta(t), \tag{8}$$

where $\zeta(t)$ is a stationary dichotomous Markov process taking the values ± 1, with zero mean and autocorrelation $\langle\zeta(0)\zeta(t)\rangle = \exp(-2\lambda t)$. A 'formula of differentiation' [27] can then be used to obtain the partial differential equation obeyed by the characteristic function of the positional probability distribution, and thence the master equation for $P(x, t)$ itself. The method may also be extended [28, 29] to arrive at the master equations for various generalizations of DD [30]–[32], including one involving a multi-state Markov process for the velocity [33], but we do not go into these aspects here.

It is easily checked that the mean displacement $\langle x(t)\rangle = 0$, while the mean-squared displacement is given by

$$\langle x^2(t)\rangle = \int_{-\infty}^{\infty} x^2 P(x, t)\,\mathrm{d}x = (c^2/2\lambda^2)(2\lambda t - 1 + \mathrm{e}^{-2\lambda t}). \tag{9}$$

Thus $\langle x^2\rangle \simeq c^2t^2$ for $t \ll \lambda^{-1}$, while $\langle x^2\rangle \to c^2t/\lambda$ for $t \gg \lambda^{-1}$. As is well known, the telegrapher's equation goes over into the usual diffusion equation, and $P(x, t)$ goes over to the standard Gaussian solution of the latter, in the 'diffusion limit' $c \to \infty, \lambda \to \infty$ such that $c^2/(2\lambda) \to$ a finite diffusion constant D.

The correspondence between DD and relativistic quantum mechanics arises as follows [16]: setting

$$P_{R,L}(x, t) = \mathrm{e}^{-\lambda t}\psi_{1,2}(x, t) \tag{10}$$

in equation (3), we find that each of the 'components' ψ_1 and ψ_2 obeys the Klein–Gordon equation

$$(\partial_t^2 - c^2\partial_x^2 - \lambda^2)\psi_{1,2} = 0, \tag{11}$$

which suggests a formal connection with relativistic wave equations. In terms of $\psi_{1,2}$, the coupled equations (2) for $P_{R,L}$ read

$$(\partial_t + c\partial_x)\psi_1 = \lambda\psi_2, \qquad (\partial_t - c\partial_x)\psi_2 = \lambda\psi_1. \tag{12}$$

On the other hand, the Dirac equation for a free relativistic particle of rest mass m_0 is

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - \frac{m_0 c}{\hbar}\right)\psi = 0, \tag{13}$$

where c stands for the speed of light in free space, as usual. In $(1+1)$ dimensions, we have $\partial_\mu = (c^{-1}\partial_t, \partial_x)$, and in the Weyl representation,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{14}$$

Therefore equation (13) reduces to

$$(\partial_t + c\partial_x)\psi_1 = (m_0 c^2/\mathrm{i}\hbar)\psi_2, \qquad (\partial_t - c\partial_x)\psi_2 = (m_0 c^2/\mathrm{i}\hbar)\psi_1. \tag{15}$$

But these equations are precisely those in equations (12) provided that, in the latter, we identify the speed c of the diffusing particle with the speed of light, and the frequency λ of its direction reversal with the rest mass of the Dirac particle according to [16]

$$\lambda = m_0 c^2/\mathrm{i}\hbar. \tag{16}$$

The factor i that occurs in equation (16) is of course a reflection of the familiar analytic continuation to imaginary time that occurs in the passage from the diffusion equation to the Schrödinger equation (or from the heat kernel to the quantum mechanical propagator) in the stochastic interpretation of quantum mechanics [15]. In the light of the analogy above, a Dirac particle in one dimension may be regarded as moving back and forth on a line at the speed of light (Zitterbewegung!) with random reversals of direction. The components of its wave function are related to the probability density of the (classical) diffusing particle according to the correspondence

$$P_{R,L} \longleftrightarrow \mathrm{e}^{\mathrm{i}m_0 c^2 t/\hbar}\psi_{1,2}. \tag{17}$$

In one spatial dimension, of course, there is no spin, and the components $\psi_{1,2}$ pertain to helicity amplitudes.

3. Velocity-biased DD

We now turn to biased DD, where the bias is caused by a difference in the *frequencies of direction reversal* when the particle is in the left- and right-moving states, respectively [8, 24, 25], rather than a spatial bias produced, say, by an applied field. The bias is therefore effectively in velocity or momentum space. Going back to the discrete-time random walk, let p_R (respectively, p_L) be the probability that the walker continues in the same direction as the previous step when he is in the right-moving (left-moving) state; the corresponding probabilities for direction reversal are $q_R = 1 - p_R$ and $q_L = 1 - p_L$, respectively. The recursion relations (1) are then replaced for this biased random walk by

$$\begin{aligned} \alpha_R(m,n) &= p_R\alpha_R(m-1,n-1) + q_L\alpha_L(m-1,n-1), \\ \alpha_L(m,n) &= q_R\alpha_R(m+1,n-1) + p_L\alpha_L(m+1,n-1). \end{aligned} \tag{18}$$

The continuum counterpart of the random walk is obtained as before, in the limit $\tau \to 0$ and $q_{R,L} \to 0$ such that $\lim q_R/\tau = \lambda_R$, $\lim q_L/\tau = \lambda_L$ and $\lambda_r \neq \lambda_L$. Equations (2) are replaced by

$$(\partial_t + c\partial_x + \lambda_R)P_R = \lambda_L P_L, \qquad (\partial_t - c\partial_x + \lambda_L)P_L = \lambda_R P_R. \tag{19}$$

As before, we are interested in solutions on the infinite line with natural boundary conditions. The most natural initial conditions correspond to treating the diffusion as an on-going equilibrium renewal process; since the mean residence times in the right- and left-moving states are λ_R^{-1} and λ_L^{-1} respectively, these initial conditions are [8]

$$P_R(x,0) = \left(\frac{\lambda_L}{\lambda_L + \lambda_R}\right)\delta(x), \qquad P_L(x,0) = \left(\frac{\lambda_R}{\lambda_L + \lambda_R}\right)\delta(x). \tag{20}$$

As in the unbiased case, we may eliminate P_L (or P_R) from the coupled equations (19) to obtain the second-order equation

$$[\partial_t^2 + (\lambda_L + \lambda_R)\partial_t + c(\lambda_L - \lambda_R)\partial_x - c^2\partial_x^2]P_{R,L} = 0. \tag{21}$$

The additional initial conditions to be satisfied by the solutions of equation (21) follow from equations (19) and (20), and are

$$(\partial_t P_R)_{t=0} = -c\left(\frac{\lambda_L}{\lambda_L + \lambda_R}\right)\delta'(x), \qquad (\partial_t P_L)_{t=0} = c\left(\frac{\lambda_R}{\lambda_L + \lambda_R}\right)\delta'(x), \tag{22}$$

in place of equations (5). It is convenient to define the average reversal rate

$$\lambda = \tfrac{1}{2}(\lambda_L + \lambda_R) \tag{23}$$

and a certain velocity

$$v = \left(\frac{\lambda_L - \lambda_R}{\lambda_L + \lambda_R}\right)c. \tag{24}$$

Therefore $|v| \leqslant c$. The velocity v has, as we shall see, the significance of a mean drift velocity: $v > 0$ implies a drift to the right, while $v < 0$ implies a drift to the left. In terms of the parameters λ and v, the master equation (21) reads

$$(\partial_t^2 + 2\lambda\partial_t + 2\lambda v\partial_x - c^2\partial_x^2)P_{R,L} = 0. \tag{25}$$

The 'equilibrium' initial conditions (20) imply that the normalizations of P_R and P_L remain independent of time, being equal to $\lambda_L/(\lambda_L + \lambda_R)$ and $\lambda_R/(\lambda_L + \lambda_R)$ for all $t \geqslant 0$. The total normalized probability density $P = P_R + P_L$ also satisfies equation (25). Its first moment, the mean displacement, works out to

$$\langle x(t)\rangle = vt. \tag{26}$$

Hence $\mathrm{d}\langle x(t)\rangle/\mathrm{d}t = v$, justifying the identification of v with the mean drift velocity. The mean-squared displacement can also be found, without solving for P explicitly, by observing that it satisfies the ordinary differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}}{\mathrm{d}t}\right)\langle x^2(t)\rangle = 2c^2 + 4v^2\lambda t. \tag{27}$$

The variance of the position is then found to be

$$\mathrm{Var}[x(t)] = \langle x^2(t)\rangle - \langle x(t)\rangle^2 = \left(\frac{c^2 - v^2}{2\lambda^2}\right)(2\lambda t - 1 + \mathrm{e}^{-2\lambda t}). \tag{28}$$

This is to be compared with equation (9) for the unbiased case.

The probability density $P(x, t)$ satisfies equation (25) with the initial conditions

$$P(x, 0) = \delta(x), \qquad \left(\frac{\partial P}{\partial t}\right)_{t=0} = -v\delta'(x). \tag{29}$$

To solve the differential equation, we note that if we set

$$P(x, t) = \psi(x, t)\exp\left[-\lambda\left(t - \frac{xv}{c^2}\right)\right] \tag{30}$$

in equation (25), then $\psi(x, t)$ satisfies the Klein–Gordon equation

$$(\partial_t^2 - c^2\partial_x^2 - \lambda'^2)\psi = 0 \tag{31}$$

with a modified constant λ' related to the mean reversal rate λ by

$$\lambda' = \lambda\left(1 - \frac{v^2}{c^2}\right)^{1/2}. \tag{32}$$

This yields, after some simplification, the solution

$$\begin{aligned} P(x, t) = \frac{1}{2}\exp\left[-\lambda\left(t - \frac{xv}{c^2}\right)\right]\Bigg\{&\left(1 - \frac{v}{c}\right)\delta(x + ct) + \left(1 + \frac{v}{c}\right)\delta(x - ct) \\ &+ \lambda'[\theta(x + ct) - \theta(x - ct)] \\ &\times\left[\frac{1}{c}\left(1 - \frac{v^2}{c^2}\right)^{1/2} I_0(\lambda'\xi/c) + \frac{1}{\xi}\left(t + \frac{xv}{c^2}\right)I_1(\lambda'\xi/c)\right]\Bigg\}, \end{aligned} \tag{33}$$

where $\xi^2 = c^2t^2 - x^2$ as before. It is evident that this expression reduces, in the unbiased case ($v = 0$), to the solution given in equation (7).

We now turn to the important question of the physical interpretation of the solution above for $P(x, t)$, and of the particular exponential factor occurring in equation (30). Let us begin with the master equation (25) and make a Lorentz transformation (a boost) to a frame moving with a velocity v, *treating c as the speed of light in vacuum*. Then, denoting the new coordinates by (x', t') and writing $(1 - v^2/c^2)^{-1/2} = \gamma$ as usual, we have

$$x' = \gamma(x - vt), \qquad t' = \gamma(t - xv/c^2), \tag{34}$$

so that

$$\frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right), \qquad \frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right). \tag{35}$$

Therefore, continuing to denote the probability density in the new space-time coordinates as $P(x', t')$, equation (25) becomes, in the moving frame,

$$(\partial_{t'}^2 + 2\lambda'\partial_{t'} - c^2\partial_{x'}^2)P = 0, \tag{36}$$

where, as already found in equation (32), $\lambda' = \lambda/\gamma$ is precisely the average frequency of direction reversal in the *moving* frame, allowing for time dilatation. The transformation $P = \mathrm{e}^{-\lambda' t'}\psi$ then converts equation (36) to the Klein–Gordon equation for ψ, just as the transformation $P = \mathrm{e}^{-\lambda t}\psi$ did in the unbiased case. We note that $\partial_t^2 - c^2\partial_x^2 = \partial_{t'}^2 - c^2\partial_{x'}^2$, and that $\lambda' t' = \lambda(t - xv/c^2)$, which explains the origin of the exponential factor in equation (30). We may also verify that if we start with the expression in curly brackets in equation (7) (the solution for unbiased DD) and make a Lorentz transformation to a frame moving with velocity v, then, apart from an overall factor of γ^{-1}, we obtain precisely the expression in curly brackets in equation (33) (the solution for biased DD), in terms of the new coordinates. In carrying out this verification, it is helpful to exploit the fact that the combination $[\theta(x + ct) - \theta(x - ct)]$ is an indicator function that is unity inside the forward light cone and zero elsewhere, and that is invariant under orthochronous boosts. The extra factor γ^{-1} arises because the probability density has the dimensions of $(\text{length})^{-1}$, and is compensated for by a factor γ arising from the measure $\mathrm{d}x$ under the transformation, such that the normalization of the total probability is preserved.

Finally, we show how the connection between DD and the Dirac equation in $(1 + 1)$ dimensions is modified in the velocity-biased case under discussion. As in equation (30), we relate $P_{R,L}$ to the spinorial components $\psi_{1,2}$ by setting

$$P_{R,L} = \psi_{1,2}\exp\left[-\lambda\left(t - \frac{xv}{c^2}\right)\right]. \tag{37}$$

Then, since each of the components $\psi_{1,2}$ satisfies the Klein–Gordon equation (31), the reversal frequency $\longleftrightarrow$ mass identification now reads

$$\lambda' = \lambda\left(1 - \frac{v^2}{c^2}\right)^{1/2} = \frac{m_0c^2}{\mathrm{i}\hbar}, \tag{38}$$

rather than $\lambda = m_0c^2/\mathrm{i}\hbar$ as in the unbiased case (equation (16)). Interestingly (and remarkably) enough, this means that the original frequency of direction reversal λ is connected to the so-called 'kinematic' mass m, rather than the rest mass itself, for we now have

$$\lambda = \frac{mc^2}{\mathrm{i}\hbar}, \qquad \text{where} \quad m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{39}$$

Although each of the components $\psi_{1,2}$ individually satisfies the Klein–Gordon equation, the coupled first-order equations satisfied by these objects do not correspond directly to the Dirac equation: substituting equation (37) in the coupled equations (19) for P_R and P_L, we get

$$(\partial_t + c\partial_x)\psi_1 = \lambda(1 + v/c)\psi_2, \qquad (\partial_t - c\partial_x)\psi_2 = \lambda(1 - v/c)\psi_1. \tag{40}$$

On the other hand, the coefficient on the right-hand side is the same ($= m_0c^2/\mathrm{i}\hbar$) in both the equations in the case of the Dirac equation (equation (15)). This is easily taken care of by writing, instead of equation (37),

$$P_R = \varphi_1(1+v/c)^{1/2}\exp[-\lambda(t-xv/c^2)], \qquad P_L = \varphi_2(1-v/c)^{1/2}\exp[-\lambda(t-xv/c^2)], \tag{41}$$

so that the spinor $(\varphi_1\,\varphi_2)^{\mathrm{T}}$ obeys the Dirac equation (15) on making the identification specified in equation (38).

For completeness, we point out that there is an alternative (and perhaps less significant) way in which a connection between velocity-biased DD and a relativistic wave equation may be made. One may retain the quantities $\psi_{1,2}$ as defined in equation (37), *continue* to identify λ itself with $m_0c^2/\mathrm{i}\hbar$ as in the case of unbiased DD, and ask what sort of wave equation (40) represents. Now, in the Weyl representation (equation (14)), we have $\gamma^0 = \sigma_x$ and $\gamma^1 = -\mathrm{i}\sigma_y$, in terms of the Pauli matrices. The standard Dirac equation $(\mathrm{i}\gamma^\mu\partial_\mu - m_0c/\hbar)\psi = 0$ can therefore be written in this representation as

$$\left(\mathrm{i}\sigma_x\partial_t + c\sigma_y\partial_x - \frac{m_0c^2}{\hbar}\right)\psi = 0. \tag{42}$$

Multiplying throughout by σ_x, we therefore get the coupled set of equations (15), namely,

$$(\partial_t + c\sigma_z\partial_x)\psi = \left(\frac{m_0c^2}{\mathrm{i}\hbar}\right)\sigma_x\psi. \tag{43}$$

On the other hand, equations (40) can be written (with λ set equal to $m_0c^2/\mathrm{i}\hbar$) in the form

$$(\partial_t + c\sigma_z\partial_x)\psi = \left(\frac{m_0c^2}{\mathrm{i}\hbar}\right)\left(\sigma_x + \frac{v}{c}\mathrm{i}\sigma_y\right)\psi. \tag{44}$$

To express this in terms of the γ matrices, we multiply both sides of this equation by σ_x. But in $(1+1)$ dimensions the counterpart of the usual γ^5 matrix is simply $\gamma^0\gamma^1$, which is just σ_z in the Weyl representation. Therefore equation (44) reduces to

$$\left[\mathrm{i}\gamma^\mu\partial_\mu - \frac{m_0c}{\hbar}\left(1 - \frac{v}{c}\gamma^5\right)\right]\psi = 0, \tag{45}$$

suggestive of a 'helicity-projected' form of the original Dirac equation.

4. Concluding remarks

We have considered a persistent random walk with different frequencies of direction reversal in the two velocity states of the walker. In the continuum limit, this leads to a form of biased DD that generalizes the usual DD, and enables an extension to be made of the analogy between the coupled master equations for DD and the Dirac equation for a relativistic particle in one spatial dimension. This extension can be given a physical interpretation in terms of a Lorentz transformation.

It is interesting to ask whether similar analogies exist in the case of multi-level diffusion, in which the velocity of the diffusing particle is an N-state Markov process with $N > 2$. In this connection, it is worth pointing out that the master equation for the total positional probability density $P(x, t)$ then satisfies a partial differential equation of order N in time, provided the transition matrix of the velocity process is such that a single equation can be obtained for P by an elimination procedure [33]. It is also of interest to examine the extension of the analogy between diffusive processes and relativistic wave equations to more than one spatial dimension [34, 35], a non-trivial problem in which open questions remain.

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References

[1] Einstein A 1905 *Ann. Phys. (Lpz)* **17** 549
[2] von Smoluchowski M 1906 *Ann. Phys. (Lpz)* **21** 756
[3] Taylor G I 1921 *Proc. London Math. Soc.* **20** 196
[4] Fürth R 1920 *Z. Phys.* **2** 244
[5] Goldstein S 1950 *Q. J. Mech. Appl. Math.* **4** 129
[6] Kac M 1974 *Rocky Mountain J. Math.* **4** 497
[7] Weiss G H and Rubin R 1983 *Adv. Chem. Phys.* **52** 363
[8] Balakrishnan V and Chaturvedi S 1988 *Physica* A **148** 581
[9] Balakrishnan V, Lakshmibala S and Van den Broeck C 1988 *Physica* A **153** 57
[10] Van den Broeck C and Mazo R M 1984 *J. Chem. Phys.* **81** 3624
[11] John T M and Murthy K P N 1986 *J. Stat. Phys.* **45** 753
[12] Sancho J M 1984 *J. Math. Phys.* **25** 354
[13] Balakrishnan V and Van den Broeck C 2001 *Phys. Rev.* E **65** 012101
[14] Nicolis G, Balakrishnan V and Nicolis C 2002 *Phys. Rev.* E **65** 051109
[15] Nelson E 1985 *Quantum Fluctuations* (Princeton, NJ: Princeton University Press)
[16] Gaveau B, Jacobson T, Kac M and Schulman L S 1984 *Phys. Rev. Lett.* **53** 419
[17] Serva M 1988 *Ann. Inst. H Poincaré* **49** 415
[18] McKeon D G C and Ord G N 1992 *Phys. Rev. Lett.* **69** 3
[19] Ord G N and Deakin A S 1996 *Phys. Rev.* A **54** 3772
[20] Ord G N 1997 *Present Status of the Quantum Theory of Light* ed S Jeffers *et al* (Dordrecht: Kluwer Academic)
[21] McKeon D G C and Ord G N 2004 *Can. J. Phys.* **62** 19
[22] Ibison M 1999 *Chaos Solitons Fractals* **10** 1
[23] Maharana K 2000 *Phys. Rev.* E **62** 1683
[24] Balakrishnan V and Lakshmibala S 1995 *Selected Topics in Mathematical Physics* ed R Sridhar, K Srinivasa Rao and V Lakshminarayanan (New Delhi: Allied Publishers) p 269
[25] Godoy S V and Fujita S 1982 *J. Phys. Soc. Japan* **56** 52
[26] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)
[27] Shapiro V E and Loginov V M 1978 *Physica* A **91** 563
[28] van Kampen N G 1976 *Phys. Rep.* **24** 171
[29] Balakrishnan V 1993 *Pramana—J. Phys.* **40** 259
[30] Van den Broeck C 1990 *Physica* A **168** 677
[31] Balakrishnan V, Van den Broeck C and Hänggi P 1988 *Phys. Rev.* A **38** 4213

[32] Masoliver J, Lindenberg K and Weiss G H 1989 *Physica* A **157** 891
[33] Balakrishnan V 2003 *Proc. SPIE* **5114** *Noise in Complex Systems and Stochastic Dynamics* ed L Schimansky-Geier, D A Abbott, A Neiman and C Van den Broeck (Bellingham, WA: SPIE) p 40
[34] Ord G N and McKeon D G C 1993 *Ann. Phys.* **272** 244
[35] Raetz G 2000 *Quaternions and General Relativity* ch 17, sec 1 (http://home.pcisys.net/∼bestwork./)