



On the complexity of making a distinguished vertex minimum or maximum degree by vertex deletion



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ABSTRACT

In this paper, we investigate the approximability of two node deletion problems. Given a vertex weighted graph $G = (V, E)$ and a specified, or “distinguished” vertex $p \in V$, $\text{MDD}(\min)$ is the problem of finding a minimum weight vertex set $S \subseteq V \setminus \{p\}$ such that p becomes the minimum degree vertex in $G[V \setminus S]$; and $\text{MDD}(\max)$ is the problem of finding a minimum weight vertex set $S \subseteq V \setminus \{p\}$ such that p becomes the maximum degree vertex in $G[V \setminus S]$. These are known NP-complete problems and they have been studied from the parameterized complexity point of view in [1]. Here, we prove that for any $\epsilon > 0$, both the problems cannot be approximated within a factor $(1 - \epsilon) \log n$, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$. We also show that for any $\epsilon > 0$, $\text{MDD}(\min)$ cannot be approximated within a factor $(1 - \epsilon) \log n$ on bipartite graphs, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$, and that for any $\epsilon > 0$, $\text{MDD}(\max)$ cannot be approximated within a factor $(1/2 - \epsilon) \log n$ on bipartite graphs, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$. We give an $O(\log n)$ factor approximation algorithm for $\text{MDD}(\max)$ on general graphs, provided the degree of p is $O(\log n)$. We then show that if the degree of p is $n - O(\log n)$, a similar result holds for $\text{MDD}(\min)$. We prove that $\text{MDD}(\max)$ is APX-complete on 3-regular unweighted graphs and provide an approximation algorithm with ratio 1.583 when G is a 3-regular unweighted graph. In addition, we show that $\text{MDD}(\min)$ can be solved in polynomial time when G is a regular graph of constant degree.

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1. Introduction

The problems of making a distinguished vertex minimum or maximum degree by vertex deletion in undirected graphs are very natural, albeit unexplored problems in graph theory, and see a wide array of applications. We formally state these two problems as follows.

- $\text{MDD}(\min)$: Given a graph $G = (V, E)$ with a distinguished vertex $p \in V$, find a vertex set $S \subseteq V \setminus \{p\}$ of minimum size such that the vertex p is the unique vertex of minimum degree in $G[V \setminus S]$.

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- $MDD(\max)$: Given a graph $G = (V, E)$ with a distinguished vertex p , find a vertex set $S \subseteq V \setminus \{p\}$ of minimum size such that the vertex $p \in V$ is the unique vertex of maximum degree in $G[V \setminus S]$.

Variants of these problems include the *weighted case*, in which we are interested in finding a vertex set S of minimum weight instead of minimum cardinality, when each vertex in G has a weight associated with it.

These problems have been previously studied in [1,2] with reference to directed graphs and electoral networks. The most natural motivation lies in competitive social networks, which are undirected, and in which the degree of a node is widely seen as a measure of its popularity, influence or importance. An agent may wish to decrease the influence of a competing agent (minimize the degree of a distinguished vertex) or increase his own influence (maximizing degree of a distinguished vertex) at minimum cost, by shielding the minimum number of other agents from the network.

Another application lies in terrorist networks studied extensively in [3,4], in which the connectivity of a particular node in the network may be decreased by targeting the minimum number of other nodes. The $MDD(\min)$ problem finds a direct application in this scenario, as well as in similar scenarios involving cartel networks.

The third major application could lie in biology – in protein networks. There have been a multitude of papers published [5,12,14] which try to correlate the parameter of a particular node in the network – such as degree, centrality, etc. – with the importance of the corresponding protein. While degree is seen as a reasonably good indicator of connectivity and influence, it may be interesting to look at how many other proteins would have to disappear from the network in order to make a particular protein influential. This is a direct application of $MDD(\max)$, and the minimum number of other proteins which need to be deleted could provide a measure of essentiality of the protein corresponding to the distinguished vertex. The research in this area has been mainly empirical so far, and this could provide another metric to judge the importance of a particular protein given its interaction network.

Both $MDD(\min)$ and $MDD(\max)$ are known to be NP-complete [1]. Previous work on these two problems involved approaches using parameterized complexity [1], but a classical complexity approach has not yet been taken as per our knowledge. In this paper, we take a classical complexity theory approach towards the problems and make the following contributions:

- We show that $MDD(\min)$ on a graph G is equivalent to $MDD(\max)$ on the graph G^c .
- We prove that both $MDD(\min)$ and $MDD(\max)$ are hard to approximate within a factor smaller than $\log n$, where n represents the number of vertices in the input graph.
- On bipartite graphs, we prove that $MDD(\min)$ and $MDD(\max)$ are hard to approximate within a factor smaller than $O(\log n)$.
- We propose an $O(\log n)$ factor approximation algorithm for $MDD(\max)$ when the input graph G satisfies a certain property. As a consequence of this, we show that if $d(p) = O(\log n)$ in the input graph G , $MDD(\max)$ is approximable within a factor of $O(\log n)$.
- We show that $MDD(\min)$ is solvable in polynomial time on k -regular graphs, as long as $k = O(1)$.
- For 3-regular unweighted graphs, we propose an approximation algorithm for $MDD(\max)$ with approximation ratio 1.583. On 3-regular bipartite graphs, we prove that $MDD(\max)$ is APX-complete.

2. Preliminaries

All the discussion in this paper concerns undirected graphs. The word *graph* is used to mean undirected graph without any ambiguity.

2.1. Notation

In a graph $G = (V, E)$, the sets $N_G(v) = \{u \in V(G) : (u, v) \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *neighbourhood* and the *closed neighbourhood* of a vertex v in G , respectively. The *degree* of a vertex v in G is $|N_G(v)|$ (or the number of neighbours of v in graph G) and is denoted by $d_G(v)$. Note that even if $v \notin V(H)$, $d_H(v)$ could be non-zero, if $v \in V(G)$ and H is a subgraph of G . We shall use $N(v)$, $N[v]$ and $d(v)$ instead of $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively, when there is no ambiguity regarding the graph under consideration. In a similar vein, for a set of vertices S , we define $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$.

A graph $G = (V, E)$ is called *k-regular* if $d_G(v) = k$, $\forall v \in V$. For $S \subseteq V$, $G[S]$ denotes the subgraph induced by S on G . The *complement of a graph* $G = (V, E)$ is the graph $G^c = (V, E^c)$, where $(u, v) \in E^c$ if and only if $(u, v) \notin E$, $\forall u, v \in V$, $u \neq v$. Unless otherwise mentioned, n denotes the number of vertices in the input graph.

In a graph $G = (V, E)$, $S \subseteq V$ is called a *dominating set* in G if $N[S] = V$. Given a graph $G = (V, E)$, an instance of $MDD(\max)$, we say that $S \subseteq V \setminus \{p\}$ is a *solution to $MDD(\max)$* for G , if the vertex p is the maximum degree vertex in $G[V \setminus S]$. S is called a *minimal solution to $MDD(\max)$* for G if, for each $u \in S$, $S \setminus \{u\}$ is not a solution to $MDD(\max)$ for G . A *minimum solution to $MDD(\max)$* on graph G is a solution S to $MDD(\max)$ of minimum weight/cardinality. Similarly, a solution (and minimal solution, minimum solution) to $MDD(\min)$ for G is defined.

2.2. Known results

We now state the definitions of a few known NP-complete optimization problems such as the *minimum dominating set* problem, *f-dependent set* problem and *minimum set cover* problem, and state approximability and inapproximability bounds for them.

Definition 1 (*MinDom*). Given a graph $G = (V, E)$, the minimum dominating set problem *MinDom* is to find a dominating set S of minimum cardinality.

Given a universe $\mathcal{U} = \{x_1, x_2, \dots, x_r\}$ and a collection of subsets $\mathcal{F} = \{F_1, F_2, \dots, F_t\}$ where $F_i \subseteq \mathcal{U}$, a set $T \subseteq \mathcal{F}$ is called a *set cover* for \mathcal{U} if $\cup_{F \in T} F = \mathcal{U}$. Size of a set cover T is defined as the number of sets in it.

Definition 2 (*MinSetCover*). Given an instance $(\mathcal{U}, \mathcal{F})$, the minimum set cover problem *MinSetCover* is to find a set cover T of minimum size.

Both *MinDom* and *MinSetCover* are known to be equivalent with respect to approximation preserving reductions [9] and both cannot be approximated within a factor better than $\log n$.

Proposition 3. (See [6].) For any $\epsilon > 0$, *MinDom* and *MinSetCover* cannot be approximated within a factor $(1 - \epsilon) \log n$, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$. Note that for *MinSetCover*, $n = |\mathcal{U}| + |\mathcal{F}| = r + t$.

Another inapproximability result for *MinDom* is also known, and we will use it in some of our proofs.

Proposition 4. (See [11].) *MinDom* is APX-complete for cubic (3-regular) as well as bicubic (3-regular bipartite) graphs.

Definition 5 (*f-dependent set deletion*). Given a vertex weighted graph $G = (V, E)$ and a function $f : V \rightarrow \mathbb{N}$, the *f-dependent set deletion* problem is to find a set $S \subseteq V$ of minimum weight such that degree of each vertex v in $G[V \setminus S]$ is at most $f(v)$.

Proposition 6. (See [7,10].) The *f-dependent set problem* can be approximated within a factor of $2 + \log \alpha$, where $\alpha = \max\{f(v) \mid v \in V\}$. If $\alpha \in \{0, 1\}$, then *f-dependent set problem* is approximated within a factor of 2.

The *f-dependent set problem* is a generalization of *MinDom* and has a similar inapproximability result which is as follows.

Proposition 7. (See [10].) Unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$, for any $\epsilon > 0$, *f-dependent set problem* cannot be approximated within a factor of $(1 - \epsilon) \log \alpha$, where $\alpha = \max\{f(v) \mid v \in V\}$ and $f(v) \geq 3$ for all $v \in V$.

2.3. Equivalence of *MDD (min)* and *MDD (max)*

We now show a result that we will use repeatedly in this paper.

Theorem 8. *MDD (max)* in a graph G is equivalent to *MDD (min)* in graph G^c , and vice versa.

Proof. Given an instance $G = (V, E)$ of *MDD (max)*, we construct the graph G^c as an instance of *MDD (min)*. An optimal solution to *MDD (max)* for G for *MDD (max)* must be an optimal solution to *MDD (min)* for G^c , since the two operations – deletion and complementation – are commutative as far as our problem is concerned. From this observation, the theorem statement follows. \square

From [Theorem 8](#), it also follows that both *MDD (min)* and *MDD (max)* are equivalent with respect to approximation preserving reductions.

3. Hardness results

In this section, we show that both *MDD (min)* and *MDD (max)* are hard to approximate within a factor smaller than $O(\log n)$. We prove these results by establishing approximation preserving reductions from *MinDom* and using the inapproximability result from [Proposition 3](#).

Theorem 9. For any $\epsilon > 0$, *MDD (min)* cannot be approximated within a factor $(1 - \epsilon) \log n$, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$.

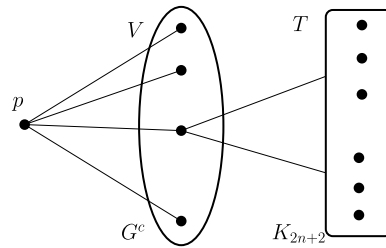


Fig. 1. Construction of H .

Proof. Given an instance $G = (V, E)$ of MinDom , we construct an instance $H = (V', E')$ of $\text{MDD}(\min)$ in polynomial time, as follows (Fig. 1). Here, we assume that n is the number of vertices in G . First, we construct the complement G^c of G . Then, we create a new vertex p and join it to all the vertices in V by introducing n edges $(p, v) \forall v \in V$. Next, we add a complete graph K_{2n+2} over a set T of $2n+2$ new vertices. For each vertex $v \in V$, if the degree of v is x in G , i.e. $d_G(v) = x$, we add x edges from v to any x vertices of T , to form graph H . Notice that now, $\forall v \in V(G)$, $d_H(v) = n$, due to the complementation of G in the construction of H . It is easy to observe that H has $3n+3$ vertices as $V' = V \cup \{p\} \cup T$.

We now claim that:

Claim 10. $S \subseteq V$ is a dominating set in G if and only if p is the vertex of minimum degree in $H[V' \setminus S]$ (i.e. S is a solution to $\text{MDD}(\min)$ for H).

Proof. Let $S \subseteq V$ be a dominating set in G . Then for all $v \in V \setminus S$, v is adjacent to some vertex in S . Therefore, the degree of a vertex v in $H[V' \setminus S]$ is at least $n - |S| + 1$. At the same time, the degree of p in $H[V' \setminus S]$ is $n - |S|$. The degree of each vertex in $V' \setminus (V \cup \{p\})$ at least $2n+1$, by construction. Since degree of p in $H[V' \setminus S]$ is $n - |S|$, it follows that p is the minimum degree vertex in $V' \setminus S$, and therefore, S is a solution to $\text{MDD}(\min)$ on H .

Conversely, let $S \subseteq V' \setminus \{p\}$ be a vertex deletion set in H which makes p the vertex of minimum degree in $H[V' \setminus S]$. Since $|T| = 2n+2$ and all vertices in T have large degree, the optimal vertex deletion set in H cannot have size larger than $|V|$. Therefore, an optimal vertex deletion set in H is a subset of V . Based on this observation, we shall assume that any vertex deletion set S in H is a subset of V . Since S is a vertex deletion set, $d_{H[V' \setminus S]}(p) = n - |S|$ and $d_{H[V' \setminus S]}(u) \geq n - |S| + 1$, for all $u \in V' \setminus (S \cup \{p\})$. Let $v \in V \setminus S$ be any vertex. Since $d_{H[V' \setminus S]} \geq n - |S| + 1$, there exists at least one vertex $u \in S$ such that v and u are not adjacent in H . This implies that S is a dominating set in G . \square

From Claim 10, it follows that the reduction explained is a cost preserving reduction. Since $|V'| = 3(n+1)$, which is linear in n , and using Proposition 3, it can be observed that for sufficiently large n and for any $\epsilon' > 0$, $\text{MDD}(\min)$ cannot be approximated within a factor of $(1 - \epsilon') \log |V'|$, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$. Theorem 9 is therefore proved. \square

Using Theorem 8, it follows as a corollary that:

Corollary 11. For any $\epsilon > 0$, $\text{MDD}(\max)$ cannot be approximated within a factor $(1 - \epsilon) \log n$, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$.

We now prove that a similar hardness result holds for $\text{MDD}(\min)$ even when the input G is restricted to bipartite graphs. We do this by establishing a cost preserving reduction from MinSetCover to $\text{MDD}(\min)$ on bipartite graphs, similar to that of Theorem 9.

Theorem 12. For any $\epsilon > 0$, $\text{MDD}(\min)$ on bipartite graphs cannot be approximated within a factor $(1 - \epsilon) \log n$, unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$.

Proof. We prove this theorem by establishing a cost preserving reduction from MinSetCover to $\text{MDD}(\min)$. Let $(\mathcal{U}, \mathcal{F})$ be an instance of MinSetCover with $\mathcal{U} = \{x_1, x_2, \dots, x_r\}$, $\mathcal{F} = \{F_1, F_2, \dots, F_t\}$ (refer Definition 2). Here we assume that $|\mathcal{U}| \ll |\mathcal{F}|$. We construct a graph $G = (V, E)$ corresponding to \mathcal{U} and \mathcal{F} as follows. We introduce a vertex for every element in $\mathcal{U} \cup \mathcal{F}$. Let $U = \{a_1, a_2, \dots, a_r\}$ be the set of vertices corresponding to elements in \mathcal{U} , where vertex a_i corresponds to element $x_i \in \mathcal{U}$ and $F = \{b_1, b_2, \dots, b_t\}$ be the set of vertices corresponding to the elements in \mathcal{F} , where vertex b_i corresponds to subset F_i . The vertex set of G , $V = U \cup F \cup C \cup D \cup \{p\}$, where C and D have t vertices each. Therefore $|V| = 3t + r + 1$. The edge set E is defined as follows. We make a complete bipartite graph $K_{t,t}$ on $C \cup D$ with vertex bipartition as C and D . We introduce an edge (p, b_i) , for every $b_i \in F$. We add an edge $(a_i, b_j) \in E$ if and only if $x_i \notin F_j$. At this stage, if the degree of a vertex $b_i \in F$ is strictly less than t , then we add sufficient edges from b_i to vertices in D in order to increase $d_G(b_i)$ to t . Note that $d_G(b_j) \geq t \forall b_j \in F$. Similarly, we add edges from each vertex $a_i \in U$ to vertices in C such that $d_G(a_i) \geq t \forall a_i \in U$. Clearly, G is a bipartite graph. For a sketch of G see Fig. 2.

We now prove the following claim.

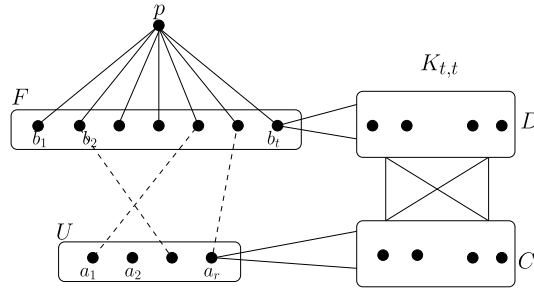


Fig. 2. $MDD(\min)$ for bipartite graphs: construction of the graph G from an instance $(\mathcal{U}, \mathcal{F})$ of set cover.

Claim 13. $T = \{F_{i_1}, F_{i_2}, \dots, F_{i_\ell}\}$ is a set cover for \mathcal{U} if and only if $S = \{b_{i_1}, b_{i_2}, \dots, b_{i_\ell}\}$ is a solution to $MDD(\min)$ for G .

Proof. Note that $d_{G[V \setminus S]}(p) = t - \ell$ and $d_{G[V \setminus S]}(b_i) \geq t, \forall b_i \in F \setminus S$. If T is a set cover for \mathcal{U} , then for each $k \in \{1, \dots, r\}$, $\exists j \in \{1, 2, \dots, \ell\}$ such that $x_k \in F_{i_j}$. This implies that for every $a_k \in U, \exists j \in \{1, 2, \dots, \ell\}$ such that $(a_k, b_{i_j}) \notin E$ and therefore, $d_{G[V \setminus S]}(a_k) \geq t - \ell + 1, \forall a_k \in U$. Also, $d_{G[V \setminus S]}(v) \geq t$ for every $v \in C \cup D$. Thus, p is the unique minimum degree vertex in $G[V \setminus S]$, and so S is a solution to $MDD(\min)$ for G .

Next, we show that for a given minimal solution $S \subseteq V$ to $MDD(\min)$ for G , we can construct a set cover T for \mathcal{U} with $|T| \leq |S|$. Let $S \subseteq V$ be a minimal solution to $MDD(\min)$ for G . Then $S \cap F \neq \emptyset$, since we need to necessarily reduce the degree of p . This implies that $d_{G[V \setminus S]}(p) \leq t - 1$. It is intuitive that $S \cap (C \cup D) = \emptyset$. Suppose $S \cap U \neq \emptyset$. Let $a_k \in S \cap U$ be any arbitrary vertex. Then, for the corresponding element $x_k \in \mathcal{U}$ there exists a set $F_j \in \mathcal{F}$ such that $x_k \in F_j$. Based on this property, we construct a new set S' of vertices by replacing each vertex $a_k \in S \cap U$ by a vertex b_j (where $x_k \in F_j$). Therefore, it follows that $|S'| \leq |S|$ and $S' \subseteq F$. Now we show that the set $T = \{F_i | b_i \in S'\}$ is a set cover for $(\mathcal{U}, \mathcal{F})$. If $a_k \in S \cap U$, then by construction of S' , there exists an $F_j \in T$ such that $x_k \in F_j$. For $a_k \in U \setminus S$, we have $d_{G[V \setminus S]}(a_k) > t - |S \cap F|$. This implies that there exists at least one $b_j \in S \cap F$ such that $(a_k, b_j) \notin E$. From the construction of G , we have that $x_k \in F_j$. Note that $S \cap F \subseteq S'$ and therefore $F_j \in T$. Hence, T is a set cover for \mathcal{U} . \square

The reduction in Claim 13 is cost preserving. Since $|V| = O(n)$, where $n = r + t$, using Proposition 3, it can be proved that for any $\epsilon > 0$, $MDD(\min)$ on bipartite graphs cannot be approximated within a factor of $(1 - \epsilon) \log |V|$, unless $NP \subseteq Dtime(n^{\log \log n})$. Theorem 12 is therefore proved. \square

Note that the complement of a bipartite graph is not necessarily bipartite, and so Theorem 8 cannot be used to extend Theorem 12 to $MDD(\max)$ on bipartite graphs. We use a different reduction to show the hardness of $MDD(\max)$ on bipartite graphs.

Theorem 14. For any $\epsilon > 0$, $MDD(\max)$ on bipartite graphs cannot be approximated within a factor $(\frac{1}{2} - \epsilon) \log n$, unless $NP \subseteq Dtime(n^{\log \log n})$.

Proof. We prove this theorem by establishing a cost preserving reduction from $MinSetCover$. Let $\mathcal{U} = \{x_1, x_2, \dots, x_r\}$, $\mathcal{F} = \{F_1, F_2, \dots, F_\ell\}$ and $|\mathcal{U}| \ll |\mathcal{F}|$. We construct a bipartite graph G as follows. First, we construct the natural bipartite graph representation of $(\mathcal{U}, \mathcal{F})$. For this we introduce two sets of vertices as $U = \{a_1, a_2, \dots, a_r\}$ and $F = \{b_1, b_2, \dots, b_\ell\}$, corresponding to elements in \mathcal{U} and \mathcal{F} , respectively. Here, (a_i, b_j) is an edge if and only if $x_i \in F_j$. In the next step, we introduce a new vertex p and edges (p, a_i) , for $1 \leq i \leq r$. We shall denote the resulting graph as $G' = (U \cup F \cup \{p\}, E')$. In the final step of the construction of G , we introduce a few degree one vertices to G' so that $d_G(v) = t$, for each vertex $v \in U \cup \{p\}$. We do this as follows. For each $v \in U \cup \{p\}$, we introduce a new set of vertices I_v of size $t - d_{G'}(v)$ to the graph G' and make v adjacent to all the vertices in I_v . Let $I = \cup_{v \in U \cup \{p\}} I_v$. We call the graph finally obtained as $G = (V, E)$ where $V = U \cup F \cup I \cup \{p\}$ and E is the set of edges as defined above. We have that $|V| \leq (|\mathcal{U}| + 2)(|\mathcal{F}| + 1) \leq n^2$, where $n = r + t$. We also observe that G is a bipartite graph, $d(v) = t$ for all $v \in U \cup \{p\}$ and $d(v) < t$ for every other vertex. For a sketch of this construction, refer to Fig. 3.

We now make the following claim.

Claim 15. $T = \{F_{i_1}, F_{i_2}, \dots, F_{i_\ell}\}$ is a set cover for \mathcal{U} if and only if $S = \{b_{i_1}, b_{i_2}, \dots, b_{i_\ell}\}$ is a solution to $MDD(\max)$ on G .

Proof. For each $x_i \in \mathcal{U}$ there is an $F_{i_k} \in T$ such that $x_i \in F_{i_k}$ and the corresponding vertex $b_{i_k} \in S$. This implies that $d_{G[V \setminus S]}(a_i) \leq t - 1, \forall a_i \in U$ and $d(p) = t$ (since none of the neighbours of p is deleted). Hence p is the vertex of maximum degree in $G[V \setminus S]$ (S is a solution to $MDD(\max)$ on G).

For the converse, let S be a minimal solution to $MDD(\max)$ on G . Without loss of generality, we can assume that $S \subseteq F$. Suppose $S \not\subseteq \mathcal{F}$ and $v \in S \setminus F$ be any vertex. If $v = a_i \in U$ for some i , then we choose a set $F_j \in \mathcal{F}$ with $x_i \in F_j$ and replace

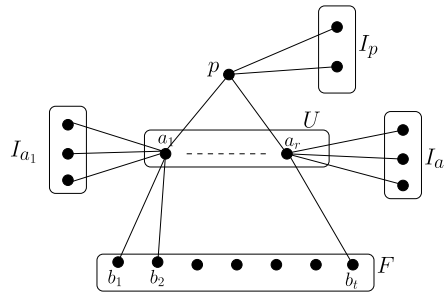


Fig. 3. MDD (max) for bipartite graphs: construction of the graph G from an instance $(\mathcal{U}, \mathcal{F})$ of set cover.

v by b_j in S . If $v \in I_p$ then we simply remove v from S , if $v \in I_{a_k}$, for some $a_k \in U$, then we replace v by b_j in S , where F_j contains the element x_k . It is important to observe that this process of normalizing S does not increase its size. The sub-collection $T = \{F_i : b_i \in S\}$ corresponding to the vertices of S gives a set cover for \mathcal{U} . \square

From Claim 15, it is easy to observe that for any solution S to MDD (max) for G , we can find (in polynomial time) a set cover T for \mathcal{U} with $|T| \leq |S|$. Also, if S_{opt} and T_{opt} are any optimal solutions for MDD (max) and MinSetCover, respectively, then $|S_{opt}| = |T_{opt}|$. Suppose, for some $\epsilon > 0$, there exists a polynomial time algorithm approximating MDD (max) within a factor of $(\frac{1}{2} - \epsilon) \log N$, on bipartite graphs with N vertices. Let S be such an approximate solution to MDD (max) for the bipartite graph G as constructed from an instance $(\mathcal{U}, \mathcal{F})$ of MinSetCover. Therefore, $|S| \leq |S_{opt}|(\frac{1}{2} - \epsilon) \log |V|$. By the above discussion we have, $|T| \leq |S| \leq |T_{opt}|(1 - 2\epsilon) \log n = |T_{opt}|(1 - \epsilon') \log n$, for some $\epsilon' > 0$. This contradicts Proposition 3. Therefore, Theorem 14 is proved. \square

We now consider the complexity of MDD (min) on regular graphs.

Theorem 16. MDD (min) on k -regular graphs is solvable in polynomial time as long as $k = O(1)$.

Proof. Let $G = (V, E)$ be a k -regular graph. We claim that the size of an optimal solution to MDD (min) for the instance G is at most $2k - 1$. We prove this by exhibiting a feasible solution to MDD (min) on G of size at most $2k - 1$. Let $A = \{v \in V \setminus N[p] : N(v) = N(p)\}$ and let $S = N(p) \cup A$. We have that $|N(p)| = k$ and $|A| \leq (k - 1)$ since $d(v) = k$ for all $v \in V$. Therefore, $|S| \leq k + (k - 1) = 2k - 1$. Now, consider some vertex $v \in V \setminus (S \cup \{p\})$. Then $N(v) \neq N(p)$ since $v \notin A$. Also note that $N(v) \cap A = \emptyset$. This implies that v has at least one neighbour in $V \setminus (S \cup \{p\})$. Therefore, $d_{G[V \setminus S]}(v) \geq 1 \forall v \in V \setminus (S \cup \{p\})$. Also, $d_{G[V \setminus S]}(p) = 0$. Since $d_{G[V \setminus S]}(v) \geq 1$ for every $v \in V \setminus S$, p is the minimum degree vertex in $G[V \setminus S]$ and hence S is a feasible solution to MDD (min) on G . Therefore, the size of an optimal solution to G is at most $2k - 1$. Let \mathcal{A} be the collection of all subsets of $V \setminus \{p\}$ of size at most $2k - 1$. Then any optimal solution belongs to this collection \mathcal{A} . We have that $|\mathcal{A}| = \sum_{i=1}^{2k-1} \binom{n}{i}$ and if this is polynomial in n , then an optimal solution to G can be found in polynomial time by explicit enumeration of all possibilities. $\sum_{i=1}^{2k-1} \binom{n}{i} \approx 2^{nH_2(\frac{2k-1}{n})}$ which is a polynomial in n as long as $k = O(1)$, (where $H_2(x) = -(x \log_2 x + (1 - x) \log_2 (1 - x))$, $\forall x \in [0, 1]$). Therefore, in this case, the optimal solution can be found in polynomial time. \square

From Theorem 8 and Theorem 16, it can be observed that MDD (max) is polynomial time solvable on k -regular graphs provided $k = n - O(1)$. However, we shall prove that MDD (max) on k -regular graphs is APX-complete when $k = O(1)$.

Theorem 17. MDD (max) is APX-complete on cubic graphs.

Proof. We exhibit a simple L -reduction [11] from MinDom on cubic graphs to MDD (max) on cubic graphs. Consider a cubic graph $G = (V, E)$ and an instance of MinDom. Let G_1 be the graph on 6 vertices $\{p, a, b, c, d, e\}$, as given in Fig. 4, and let $V' = V \cup \{p, a, b, c, d, e\}$. We construct an instance $(G' = (V', E'), p)$ of MDD (max), where $G' = G_1 \cup G$. Clearly, G' is a cubic graph. It is easy to see that the optimal solution to MDD (max) for the instance (G_1, p) is the set $\{d, e\}$. This implies that any minimal solution to MDD (max) for G' contains both d and e , and none of $\{a, b, c\}$. Now, to find a solution for G' we only need to bound the degree of every vertex in G by 2.

If S is a dominating set for G , then $d_{G[V \setminus S]}(v) \leq 2$ for every $v \in V \setminus S$. Therefore, $S' = S \cup \{d, e\}$ is a solution to MDD (max) for G' with $|S'| = |S| + 2$. Conversely, let S' be a minimal solution to MDD (max) for G' . Then $S' \cap \{a, b, c\} = \emptyset$ and $\{d, e\} \subseteq S'$. This implies that $d_{G'[V \setminus S']}(p) = 3$ and $d_{G'[V \setminus S']}(v) \leq 2$ for every $v \in V' \setminus S'$ and hence also $d_{G[V \setminus S]}(v) \leq 2$ for every $v \in V \setminus S'$. Thus, $S' \setminus \{d, e\}$ is a dominating set for G and $|S'| = |S| + 2$.

If S_{opt} is a minimum dominating set for G , then $S_{opt} \cup \{d, e\}$ is a minimum solution to MDD (max) for G' . Conversely, if S'_{opt} is a minimum solution to MDD (max) for G' , then $S'_{opt} \setminus \{d, e\}$ is a minimum dominating set for G . Choosing $\alpha = 2$,

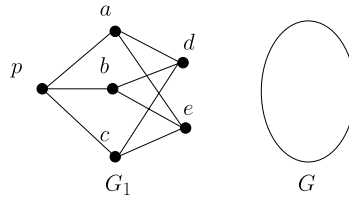


Fig. 4. Construction of $G' = G \cup G_1$.

we have that $S'_{opt} \leq \alpha S_{opt}$. Let S' be a minimal solution to $MDD(\max)$ for G' and let S be the corresponding solution to $MinDom$ for G . Then for $\beta = 1$, we have that $|S| - |S_{opt}| \leq \beta(|S'| - |S'_{opt}|)$. This gives an L -reduction from $MinDom$ on cubic graphs to $MDD(\max)$ on cubic graphs. From Proposition 4, we see that $MDD(\max)$ for cubic graphs is APX-hard. In the next section, we provide a constant factor approximation algorithm to $MDD(\max)$ on cubic graphs, thereby showing that it is APX-complete. \square

We also arrive at the following theorem for bicubic (3-regular bipartite) graphs, by a construction similar to that of Theorem 17. Note that graph G_1 in that construction is bipartite, and so for a bipartite graph G , $G' = G_1 \cup G$ would be bipartite.

Theorem 18. $MDD(\max)$ is APX-complete for bicubic graphs.

Proof. The reduction is similar to that of Theorem 17. The constant approximation ratio comes from an algorithm we present in the next section for cubic graphs. \square

4. Approximation algorithms

In this section, we show that the vertex weighted version of $MDD(\max)$ is approximable within a factor of $O(\log n)$, on graphs for which the neighbourhood of vertex p satisfies a particular property. Using Theorem 8, we will extend these algorithms to $MDD(\min)$. Here we shall assume that $d(p) = t$ in the input instance $G = (V, E)$ of $MDD(\max)$. We define $Y = \{v \mid v \in V \text{ and } d(v) \geq t\}$ and $D = N[Y]$. We will first provide approximation algorithms for special cases of the problem in Lemma 19 and Lemma 20, when $Y \cap N[p] = \emptyset$, and then move on to a generalization that captures the aforementioned property even when $Y \cap N[p] \neq \emptyset$.

Lemma 19. If the input instance G for $MDD(\max)$ satisfies the condition $D \cap N[p] = \emptyset$ then it can be approximated within a factor of $2 + \log t$.

Proof. Consider the f -dependent set problem with input as $G[V \setminus N[p]]$ and $f(v) = t - d_{N(p)}(v) - 1$, for all $v \in V \setminus N[p]$. Let S be an approximate solution to the f -dependent set problem, for this instance, generated by the Okun–Barak Algorithm [10]. We shall show that S is a $(2 + \log t)$ -factor approximate solution of $MDD(\max)$, for the instance G . From the definition of f on $V \setminus N[p]$, it follows that vertex p is the unique vertex of maximum degree in $G[V \setminus S]$. Therefore, S is a vertex deletion for $MDD(\max)$ for the instance G . Next, we prove that any minimum solution S_o to $MDD(\max)$ for the instance G , $S_o \cap N(p) = \emptyset$. Suppose, $A = S_o \cap N(p) \neq \emptyset$. Let $S'_o = S_o \setminus A$. Then S'_o is also a vertex deletion set. In the process of deleting the vertices of A from S_o , we increase the degree of vertex p by $|A|$ and vertices in $N(A) \cap [V \setminus (S'_o)]$ by at most $|A|$. Since degree of each vertex in A is at most $t - 1$, it follows that p has maximum degree in $G[V \setminus S'_o]$. \square

Lemma 20. If the input instance G for $MDD(\max)$ satisfies the conditions $Y \cap N[p] = \emptyset$ and $D \cap N[p] \neq \emptyset$ then it can be approximated within a factor of $2 + \log t$.

Proof. Similar to the proof of Lemma 19. Note here that $\forall v : v \in D \cap N[p]$, v will never be part of the solution to $MDD(\max)$. \square

We are now interested in a more general (but not the most general) case, when $Y \cap N[p] \neq \emptyset$. Let $G = (V \cup \{p\}, E)$ be an instance of $MDD(\max)$ with $Y \cap N(p) \neq \emptyset$. For such an instance we construct a set $L \subseteq N(p)$ as given in Algorithm 1.

Theorem 21. Let G be an instance of $MDD(\max)$ with $|L| = O(\log n)$. Then $MDD(\max)$ can be approximated within a factor of $O(\log n)$.

Proof. From the definition of L it follows that for every vertex $v \in N(p) \setminus L$, $d_{G[V \setminus L]}(v) < d(p) - |L|$. Let S be any solution to $MDD(\max)$ for G . Then $d_{G[V \setminus S]}(p) > d_{G[V \setminus S]}(u)$, for all $u \in V \setminus S$.

Algorithm 1: Construction of the set L .

Input: A graph $G = (V, E)$ and $p \in V$ with $Y \cap N(p) \neq \emptyset$;
Output: $L \subseteq N(p)$;
 $L = \emptyset$;
while \exists a vertex $u \in (N(p) \setminus L)$ with $|N(u) \setminus L| \geq |N(p) \setminus L|$ **do**
 $L = L \cup \{u\}$;
end
return(L);

Next we show that any minimal vertex deletion set S in G does not contain any vertex from $N(p) \setminus L$. Suppose $S \cap (N(p) \setminus L) = A \neq \emptyset$. Let $|A| = \alpha$. Now consider the set $S' = S \setminus A$. We show that S' is a vertex deletion set. Since, $d_{G[V \setminus S]}(p) > d_{G[V \setminus S]}(u)$, for all $u \in V \setminus S$, $d_{G[V \setminus S]}(p) = d_{G[V \setminus S]}(p) + \alpha > d_{G[V \setminus S]}(u) = d_{G[V \setminus S]}(u) + \alpha$, for all $u \in V \setminus S$. As $d_{G[V \setminus L]}(v) < d(p) - |L|$, for all $v \in A$, we have $d_{G[V \setminus S]}(p) > d_{G[V \setminus S]}(v)$, for all $v \in A$.

From above arguments it follows that any optimal vertex deletion set S_o in G does not contain any vertex from $N(p) \setminus L$.

Next, we present a polynomial time algorithm that computes an $O(\log n)$ -factor solution to $MDD(\max)$ for the input instance G with $|L| = O(\log n)$.

Since $|L| = O(\log n)$, Algorithm 2 runs in polynomial time. Let $K_o = S_o \cap L$. Let S_{K_o} be the f -dependent set computed in Algorithm 2 for the set K_o . It is not hard to observe that $w(S_o \setminus K_o) = w(S_{o,f,K_o})$, where S_{o,f,K_o} is an optimal f dependent set for the instance considered in the algorithm associated with set K_o . Since the algorithm is choosing the least weight vertex deletion set, we have

$$\frac{w(S)}{w(S_o)} \leq \frac{w(K_o) + w(S_{K_o})}{w(S_o)} = \frac{w(K_o) + w(S_{K_o})}{w(K_o) + w(S_o \setminus K_o)} \leq \frac{w(S_{K_o})}{w(S_o \setminus K_o)} = \frac{w(S_{K_o})}{w(S_{o,f,K_o})} \leq O(\log n). \quad \square$$

Theorem 22. For any $\epsilon > 0$, $MDD(\max)$ cannot be approximated within a factor $(1 - \epsilon) \log n$, unless $NP \subseteq Dtime(n^{\log \log n})$, even on graphs with $L = \emptyset$.

Proof. Follows from Theorem 9 and Theorem 8. Note that in the reduction in the proof of Theorem 9, the size of L is zero. \square

Algorithm 2: Computation of $O(\log n)$ factor solution S for $MDD(\max)$.

Input: A graph $G = (V, E)$, $p \in V$ with $Y \cap N(p) \neq \emptyset$ and $|L| = O(\log n)$, $w : V \rightarrow \mathbb{Z}^+$;
Output: A vertex deletion set S for $MDD(\max)$ on G ;
 $S = \emptyset$;
 $wt = \infty$;
for each subset K of L do
 Compute an f -dependent set S' using Okun-Barak's algorithm [10] with input as $G[V \setminus K]$, $w'(v) = \begin{cases} \infty & \text{for } v \in N(p) \setminus K \\ w(v) & \text{for } v \in V \setminus N(p), \end{cases}$ and
 $f(v) = d(p) - |K| - 1$ for $v \in V \setminus K$;
 $S' = S' \cup K$;
 if $w(S') < wt$ **then**
 $S = S'$ and $wt = w(S')$;
 end
end
return(S);

From Theorem 21 and Theorem 22, we see that Algorithm 2 approximates the problem when $L = O(\log n)$, which is also a $\log n$ hard problem, to the best possible extent unless $NP \subseteq Dtime(n^{\log \log n})$.

From Theorem 21 and since $L \subseteq N(p)$, it follows that if $d(p) = O(\log n)$ then the same algorithm gives an $O(\log n)$ approximate solution. As a corollary to Theorem 21 we have the following result using Theorem 8.

Corollary 23. $MDD(\min)$ can be approximated within a factor of $O(\log n)$ provided $d(p) \geq n - O(\log n)$.

We now consider algorithms for $MDD(\max)$ on regular graphs. We arrive at the following lemma:

Lemma 24. Let $G = (V, E)$ be a k -regular graph with $|V| = n$ and S be any solution to $MDD(\max)$ for G . Let $(S, V \setminus S)$ be the set of edges across the sets S and $V - S$ and $f = |N(p) \setminus S|$. Then $|S| \geq \frac{(k-f+1)n-1}{2k-f+1} \geq \frac{n-1}{k+1}$.

Proof. By using estimations on $|(S, V \setminus S)|$, we see that

$$k|S| \geq |(S, V \setminus S)| \geq (k - f) + (k - f + 1)(n - |S| - 1). \tag{1}$$

Note that the leftmost term represents the maximum number of edges that can arise out of S , and that the rightmost term is a lower bound on the number of edges arising out of $V \setminus S$. From (1), the proof of the lemma follows. \square

From Lemma 24, we have

Theorem 25. *MDD(max) can be approximated within a factor of $(k + 1)$ on k -regular graphs.*

However, it is possible to improve this approximation bound for $MDD(max)$ on cubic graphs. For this, we use the algorithms for $MinDom$ and $MinDissoVD$ (Minimum Dissociation Vertex Deletion) given by Halldórsson [8] and Tu and Yang [13], respectively. Dissociation number of a given graph is the size of a maximum induced sub-graph of G whose maximum degree is 1. $MinDissoVD$ is the vertex deletion problem corresponding to dissociation number – the minimum number (or weight) of vertices to be deleted such that the remaining graph has maximum degree 1.

Proposition 26. (See [8].) *MinDom on unweighted cubic graphs can be approximated within a factor of 1.583.*

Proposition 27. (See [13].) *MinDissoVD on unweighted cubic graphs can be approximated within a factor of 1.57.*

Theorem 28. *MDD(max) for unweighted cubic graphs can be approximated within a factor of 1.583.*

Proof. Let S be a minimal solution to $MDD(max)$ for G . It is easy to observe that if $d_{G[V \setminus S]}(p) = 0$ then $S = V \setminus \{p\}$. Also, it is easy to observe that for any feasible solution S to G , $d_{G[V \setminus S]}(p) \neq 1$. There are only two other choices left for $d_{G[V \setminus S]}(p)$ which are 2 and 3. We shall try to find a solution in each of the cases and choose the smallest of these three kinds of solutions.

First we compute a solution S to $MDD(max)$ for G such that $d_{G[V \setminus S]}(p) = 3$. In this case, it is important to observe that $1 \leq |N(x) \cap (V \setminus N[p])| \leq 2$, for all $x \in N(p)$.

We now construct the graph G' from G as follows. First, take a copy G' of G . Remove $N[p]$ from G' . For each $x \in N(p)$ with exactly two neighbours a and b in $V \setminus N[p]$, we introduce two new vertices x^1 and x^2 and four new edges $(x^1, a), (x^1, b), (x^2, a), (x^2, b)$ into G' . For each vertex $x \in N(p)$ with exactly one neighbour a in $V \setminus N[p]$, we introduce exactly one new vertex x^1 and a new edge (x^1, a) to G' . We shall refer to this resulting graph as $G' = (V', E')$ and denote X as the set of vertices that are added to the vertex set $V \setminus N[p]$. Let X_v be the set of vertices which are introduced with respect to the vertex $v \in N(p)$, so that $V' = (V \setminus N[p]) \cup X = (V \setminus N[p]) \cup (\cup_{v \in N(p)} X_v)$. By construction, $1 \leq |X_v| \leq 2 \forall v \in N(p)$.

Let D' be a dominating set in G' . If $D' \cap X = \emptyset$, then it can be observed that D' is a solution to $MDD(max)$ for G with $d_{G[V \setminus D']}(p) = 3$. If $D' \cap X \neq \emptyset$, then we construct a set D with $D \cap X = \emptyset$ and $|D| \leq |D'|$ as follows. For any $v \in N(p)$ if $D' \cap X_v \neq \emptyset$, then replace $X_v \cap D'$ by $|X_v \cap D'|$ vertices from $N_{G'}(X_v)$, choosing vertices which were not already in D' . We shall denote the resulting new vertex set as D . Using the fact that one vertex from $N_{G'}(X_v)$ is enough to dominate the vertices of X_v , we can conclude that this new vertex set D is a dominating set for G' . We claim that D is also a solution to $MDD(max)$ for G . Now, since D is a dominating set for G' , then every vertex in $V(G') \setminus D$ has at least one neighbour in D . This implies that every vertex in $(V \setminus \{p\}) \setminus D$ has at least one neighbour in D . This means that $d_{G[V \setminus D]}(v) \leq 2$ for every $v \in V \setminus (D \cup \{p\})$, while $d_{G[V \setminus D]}(p) = 3$. Hence D is a solution to $MDD(max)$ for G . Conversely, if S is a solution to $MDD(max)$ for G with $d_{G[V \setminus S]}(p) = 3$, then S is a dominating set for G' .

Suppose there exists a solution S to $MDD(max)$ for G with $d_{G[V \setminus S]}(p) = 2$. Then the two neighbours of p in $G[V \setminus S]$ (say y and z) are not adjacent. It is also necessary that $N(\{y, z\}) \setminus \{p\} \subseteq S$. Let $x \in N(p) \cap S$ and let $X^* = N(p) \setminus \{x\} = \{y, z\}$. Now consider the graph $G^* = G[V^*]$ with $V^* = V \setminus (N[X^*] \cup \{x\})$. If T is a solution to $MinDissoVD$ for G^* , then (it is easy to prove that) $T \cup \{x\} \cup (N(X^*) \setminus \{p\})$ is a solution to $MDD(max)$ for G . Conversely, if S is a solution to $MDD(max)$ for G with $d_{G[V \setminus S]}(p) = 2$ and $x \in S \cap N(p)$, then $S \setminus (\{x\} \cup N(X^*))$ is a solution to $MinDissoVD$ for G^* .

Using this idea, we give an algorithm for $MDD(max)$ on cubic graphs as in Algorithm 3. Let $S_1 = V \setminus \{p\}$, $S_2 = D_{opt}$ and $S_3 = T_{opt}$, where D_{opt} and T_{opt} are optimal solutions to $MinDom$ for G' and $MinDissoVD$ for G^* , respectively. Then the set S_{opt} defined as a smallest of S_1, S_2 and S_3 gives an optimal solution to $MDD(max)$ for G . Conversely, if S_{opt} is an optimal solution to $MDD(max)$ for G , then either $S_{opt} = S_1$, or S_{opt} is an optimal solution to $MinDom$ for G' or an optimal solution to $MinDissoVD$ for G^* .

Now, let S be the solution returned by Algorithm 3. If $S_{opt} = D_{opt}$, then

$$\frac{|S|}{|S_{opt}|} = \frac{|S|}{|D_{opt}|} \leq \frac{|D|}{|D_{opt}|},$$

where D is the approximate solution to $MinDom$ for G' . Then by Proposition 26, S is an approximate solution within a factor of 1.583.

Algorithm 3: Computation of 1.583 factor solution to MDD (max) on cubic graphs.

```

Input: A 3-regular graph  $G = (V, E)$  and  $p \in V$ ;
Output: A solution  $S$  to MDD (max);
 $S = V \setminus \{p\}$ ;
if there is no  $x \in N(p)$  such that  $N[x] = N[p]$  then
  | Compute a dominating set  $D$  for the graph  $G'$  as in the proof of Theorem 28;
  | if  $|D| < |S|$  then  $S = D$ ;
end
for each  $x \in N(p)$  do
  | Let  $N(p) - \{x\} = \{y, z\}$ ;
  | if  $(y, z) \notin E$  then
  | | Compute a solution  $T$  to MinDissoVD for the input graph  $G[V \setminus (\{x\} \cup N\{y, z\})]$ ;
  | | end
  | |  $S' = T \cup \{x\} \cup N(y, z)$ ;
  | | if  $|S'| < |S|$  then  $S = S'$ ;
  | end
end
return( $S$ );

```

Suppose $S_{opt} \cap N(p) = \{x\}$ and $S_{opt} = T_{opt} \cup \{x\} \cup N(X^*)$. Let T be an approximate solution to MinDissoVD for G^* . Let $\alpha = |\{x\} \cup N(X^*)|$. Then we have

$$\frac{|S|}{|S_{opt}|} = \frac{|S|}{|T_{opt}| + \alpha} \leq \frac{|T| + \alpha}{|T_{opt}| + \alpha} \leq \frac{|T|}{|T_{opt}|}.$$

Therefore, by [Proposition 27](#), S is an approximate solution within a factor of 1.57. Hence, the approximate solution returned by [Algorithm 3](#) is within a factor of 1.583. \square

5. Conclusion

We have shown that both MDD (min) and MDD (max), even when restricted to bipartite graphs, cannot be approximated within a factor $O(\log n)$ unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$. An approximation within a factor of $O(\log n)$ is seen if $d(p) \leq O(\log n)$ or $d(p) \geq n - O(\log n)$ for MDD (max) and MDD (min), respectively. Better approximation results for MDD (min) and MDD (max) on bipartite graphs remain unknown and we conjecture that on general graphs, it is hard to approximate both problems within a factor $O(2^{\log^{1-\epsilon} n})$, for any $\epsilon > 0$.

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