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On minimum average stretch spanning trees in polygonal 2-trees ☆,☆☆

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ABSTRACT

A spanning tree of an unweighted graph is a *minimum average stretch spanning tree* if it minimizes the ratio of sum of the distances in the tree between the end vertices of the graph edges and the number of graph edges. For a polygonal 2-tree on n vertices, we present an algorithm to compute a minimum average stretch spanning tree in $O(n \log n)$ time. This algorithm also finds a minimum fundamental cycle basis in polygonal 2-trees. We show that there is a unique minimum cycle basis in a polygonal 2-tree and it can be computed in linear time.

1. Introduction

Average stretch is a parameter used to measure the quality of a spanning tree in terms of distance preservation, and finding a spanning tree with minimum average stretch is a classical problem in network design. Let $G = (V(G), E(G))$ be an unweighted graph and T be a spanning tree of G. For an edge $(u, v) \in E(G)$, $d_T(u, v)$ denotes the distance between u and v in T. The average stretch of T is defined as

$$\text{AvgStr}(T) = \frac{1}{|E(G)|} \sum_{(u,v)\in E(G)} d_T(u, v) \tag{1}$$

A *minimum average stretch spanning tree* of G is a spanning tree that minimizes the average stretch. Given an unweighted graph G, the minimum average stretch spanning tree (Mast) problem is to find a minimum average stretch spanning tree of G. Due to the unified notation for tree spanners, the Mast problem is equivalent to the problem, Mfcb, of finding a minimum fundamental cycle basis in unweighted graphs [17]. Minimum average stretch spanning trees are used to solve symmetric diagonally dominant linear systems [17]. Further, minimum fundamental cycle bases have various applications including determining the isomorphism of graphs, frequency analysis of computer programs, and generation of minimal perfect hash functions (see [4,11] and the references therein]). Due to these vast applications, finding a minimum average stretch spanning tree is useful in theory and practice. The Mast problem was studied in a graph theoretic game in the context of the k-server problem by Alon et al. [1]. The Mfcb problem was introduced by Hubika and Syslo in 1975 [12]. The Mfcb problem was proved to be NP-hard by Deo et al. [4] and APX-hard by Galbiati et al. [11]. Another closely related problem is the problem of probabilistically embedding a graph into its spanning trees. A graph G is said to be *probabilistically*

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embedded into its spanning trees with *distortion* t, if there is a probability distribution D of spanning trees of G, such that for any two vertices the expected stretch of the spanning trees in D is at most t. The problem of probabilistically embedding a graph into its spanning trees with low distortion has interesting connections with low average stretch spanning trees.

In the literature, spanning trees with low average stretch have received significant attention in special graph classes such as k-outerplanar graphs and series-parallel graphs. In case of planar graphs, Kavitha et al. remarked that the complexity of MFCB is unknown and there is no $O(\log n)$ approximation algorithm [13]. For k-outerplanar graphs, the technique of peeling-an-onion decomposition is employed to obtain a spanning tree whose average stretch is at most c^k, where c is a constant [7]. In case of series-parallel graphs, a spanning tree with average stretch at most $O(\log n)$ can be obtained in polynomial time (see Section 5 in [8]). The bounds on the size of a minimum fundamental cycle basis are studied in graph classes such as planar, outerplanar and grid graphs [13]. The study of probabilistic embeddings of graphs is discussed in [7,8]. To the best of our knowledge, there is no published work to compute a minimum average stretch spanning tree and minimum fundamental cycle basis in any subclass of planar graphs.

We consider polygonal 2-trees in this work, which are also referred to as polygonal-trees. They have a rich structure that make them very natural models for biochemical compounds, and provide an appealing framework for solving associated enumeration problems.

Definition 1. (See [14].) A cycle is a polygonal 2-tree. For a polygonal 2-tree G such that $(u,v) \in E(G)$, adding a path P between u and v in such a way that $E(G) \cap E(P) = \emptyset$, $V(G) \cap V(P) = \{u, v\}$, and $|E(P)| \geq 2$ results in a polygonal 2-tree.

A cycle consisting of k edges is a *k-gonal tree*. For a k-gonal 2-tree G such that $(u,v) \in E(G)$, adding a path P between u and v in such a way that $E(G) \cap E(P) = \emptyset$, $V(G) \cap V(P) = \{u, v\}$, and $|E(P)| = k-1$ results in a *k-gonal 2-tree*. For example, a *2-tree* is a 3-gonal tree. The class of polygonal 2-trees is a subclass of planar graphs and it includes 2-connected outerplanar graphs and k-gonal trees. 2-trees, in other words 3-gonal trees, are extensively studied in the literature. In particular, previous work on various flavors of counting and enumeration problems on 2-trees is compiled in [10]. Formulas for the number of labeled and unlabeled k-gonal trees with r polygons (induced cycles) are computed in [15]. The family of k-gonal trees with same number of vertices is claimed as a chromatic equivalence class by Chao and Li, and the claim has been proved by Wakelin and Woodal [14]. The class of polygonal 2-trees is shown to be a chromatic equivalence class by Xu [14]. Further, various subclasses of generalized polygonal 2-trees have been considered, and it has been shown that they also form a chromatic equivalence class [14,19,20]. The enumeration of outerplanar k-gonal trees is studied by Harary, Palmer and Read to solve a variant of the cell growth problem [6]. Molecular expansion of the species of outerplanar k-gonal trees is shown in [6]. Also outerplanar k-gonal trees are of interest in combinatorial chemistry, as the structure of chemical compounds like catacondensed benzenoid hydrocarbons forms an outerplanar k-gonal tree.

Our results. We state our main theorem.

Theorem 2. *Given a polygonal 2-tree G on n vertices, a minimum average stretch spanning tree of G can be obtained in $O(n \log n)$ time.*

Due to the equivalence of MAST and MFCB (shown in Lemma 5), our result implies the following corollary. For a set $\mathcal{B}$ of cycles in G, the size of $\mathcal{B}$, denoted by size($\mathcal{B}$), is the number of edges in $\mathcal{B}$ counted according to their multiplicity.

Corollary 3. *Given a polygonal 2-tree G on n vertices, a minimum fundamental cycle basis $\mathcal{B}$ of G can be obtained in $O(n \log n + \text{size}(\mathcal{B}))$ time.*

We characterize polygonal 2-trees using a kind of ear decomposition and present the structural properties of polygonal 2-trees that are useful in finding a minimum average stretch spanning tree (in Section 3). We then identify a set of edges in a polygonal 2-tree, called safe edges, whose removal results in a minimum average stretch spanning tree (in Section 4). We present an algorithm with necessary data-structures to identify the safe set of edges efficiently and compute a minimum average stretch spanning tree in sub-quadratic time (in Section 5). We finally characterize polygonal 2-trees using cycle bases, which is of our independent interest (in Section 6).

A graph G can be probabilistically embedded into its spanning trees with distortion t if and only if the multigraph obtained from G by replicating its edges has a spanning tree with average stretch at most t (see [1]). It is easy to observe that, a spanning tree T of G is a minimum average stretch spanning tree for G if and only if T is a minimum average stretch spanning tree for a multigraph of G. As a consequence of our result, we have the following corollary.

Corollary 4. *For a polygonal 2-tree G on n vertices, the minimum possible distortion of probabilistically embedding G into its spanning trees can be obtained in $O(n \log n)$ time.*

2. Graph preliminaries

We consider simple, connected, unweighted and undirected graphs. We use standard graph terminology from [24]. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively in G. We denote $|V(G)|$ by n and $|E(G)|$ by m. The union of graphs G_1 and G_2 is defined as a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ and is denoted by $G_1 \cup G_2$. The intersection of graphs G_1 and G_2 written as $G_1 \cap G_2$ is a graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. The removal of a set X of edges from G is denoted by $G - X$. For a set $X \subset V(G)$, $G[X]$ denotes the induced graph on X. An edge $e \in E(G)$ is a *cut-edge* (bridge) if $G - e$ is disconnected. A graph consisting of at least three vertices is *2-connected* if it cannot be disconnected by removing less than two vertices. A *2-connected component* of G is a maximal 2-connected subgraph of G.

Let T be a spanning tree of G. An edge $e \in E(G) \setminus E(T)$ is a *non-tree* edge of T. For a non-tree edge (u, v) of T, a cycle formed by the edge (u, v) and the unique path between u and v in T is referred to as a *fundamental cycle*. For an edge $(u, v) \in E(G)$, *stretch* of (u, v) is the distance between u and v in T. The *total stretch* of T is defined as the sum of the stretches of all the edges in G. We remark that there are slightly different definitions existing in the literature that refer to the average stretch of a spanning tree. We use the definition in Eq. (1), presented by Emek and Peleg in [8], to refer to the average stretch of a spanning tree. Proposition 14 in [17] states that, T is a minimum total stretch spanning tree of G if and only if the set of fundamental cycles of T is a minimum fundamental cycle basis of G. Then, we can have the following lemma.

Lemma 5. *Let G be an unweighted graph and T be a spanning tree of G. T is a minimum average stretch spanning tree of G if and only if the set of fundamental cycles of T is a minimum fundamental cycle basis of G.*

We use the following convention crucially. A path is a connected graph in which there are two vertices of degree one and the rest of the vertices are of degree two. An edge can be considered as a connected graph consisting of single edge.

Lemma 6. *Let G' be a 2-connected component in an arbitrary graph G and T be a subgraph of G.*

(a) If T is a spanning tree of G, then $T \cap G'$ is a spanning tree of G'.
(b) If T is a path in G, then $T \cap G'$ is a path.

Proof. We first prove the following claim: If T is a tree, then $T \cap G'$ is a tree.

Let $T' = T \cap G'$. Suppose T' is not connected, then there exist two vertices x and y in $V(G')$ such that there is no path between x and y in T'. Since T is a tree, there is a path P between x and y in T. Since T' is not connected, we can observe that $V(P) \setminus V(G')$ contains at least one vertex, say, u. Further, the two edges incident on u in P are not in G'. Now we can obtain a graph $G' \cup P$ which is a 2-connected component in G. This contradicts the maximality of the 2-connected component G'. Therefore, T' is connected. As T' is acyclic, we conclude that T' is a tree.

If T is a spanning tree of G, then the set of vertices in $T \cap G'$ is $V(G')$. Therefore from the above claim, $T \cap G'$ is a spanning tree of G'. Thus (a) holds. Further, (b) also holds from the above claim. □

Special graph classes. A *partial 2-tree* is a subgraph of a 2-tree. A graph is a *series-parallel* graph, if it can be obtained from an edge, by repeatedly duplicating an edge between its end vertices or replacing an edge by a path. An alternative equivalent definition for series-parallel graphs is given in [9].

3. Structure of polygonal 2-trees and computation of induced cycles

In this section, we present crucial structural properties of polygonal 2-trees in Lemma 10. This lemma will be used significantly in proving the correctness of our algorithm. Another major result in this section is Theorem 14, which computes a kind of ear decomposition for polygonal 2-trees. This helps in obtaining an efficient algorithm to solve Mast. The notion of open ear decomposition is well known to characterize 2-connected graphs [24]. An *open ear decomposition* of G is a partition of $E(G)$ into a sequence $(P_0, \ldots, P_k)$ of edge disjoint graphs called *ears* such that,

1. for each $i \geq 0$, P_i is a path,
2. $P_0 \cup P_1$ is a cycle,
3. for each $i \geq 1$, end vertices of P_i are distinct and the internal vertices of P_i are not in $P_0 \cup \ldots \cup P_{i-1}$.

Further, a restricted version of open ear decomposition called *nested ear decomposition* is used to characterize series-parallel graphs [9]. An open ear decomposition $(P_0, \ldots, P_k)$ of G is said to be *nested* if it satisfies the following properties:

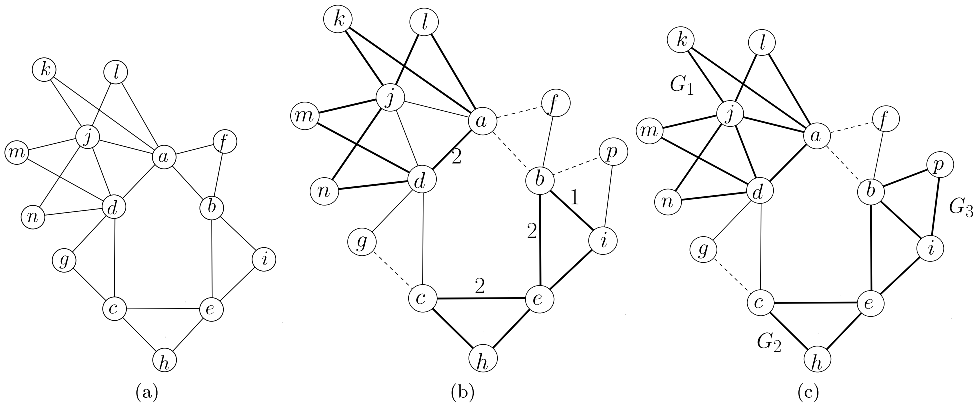


Fig. 1. (a) For the polygonal 2-tree shown, $(P_0, \ldots, P_{10})$ is a nice ear decomposition, where $P_0 = (a, b)$, $P_1 = (a, d, c, e, b)$, $P_2 = (a, f, b)$, $P_3 = (c, g, d)$, $P_4 = (c, h, e)$, $P_5 = (b, i, e)$, $P_6 = (a, j, d)$, $P_7 = (a, k, j)$, $P_8 = (a, l, j)$, $P_9 = (d, m, j)$, and $P_{10} = (d, n, j)$ are paths in G. (b) For the polygonal 2-tree G shown, let $A = \{(a, f), (a, b), (b, p), (c, g)\}$. The edges in bound$(A, G)$ are shown in thick. Support$((a, d))$ = Support$((b, e))$ = Support$((c, e)) = \{(a, b), (a, f)\}$ and Support$((b, i)) = \{(b, p)\}$. cost$((a, d)) = 2$, cost$((c, e)) = 2$, cost$((b, e)) = 2$, cost$((b, i)) = 1$ and for the rest of the edges in bound(A, G), cost is zero. (c) For the polygonal 2-tree G shown, let $A = \{(a, b), (a, f), (c, g)\}$. The 2-connected components G_1, G_2 and G_3 in $G - A$ are polygonal 2-trees. Let $P = (a, d, c, e, b, f)$ be the shortest path between vertices a and f in $G - A$. Then P intersects exactly with one edge in the graphs G_1, G_2 and G_3.

1. For each $i \geq 1$, there exists $j < i$, such that the end vertices of path P_i are in P_j.
2. Let the end vertices of P_i and $P_{i'}$ be in P_j, where $0 \leq j < i, i' \leq k$ and $i \neq i'$. Let $Q_i \subseteq P_j$ be the path between the end vertices of P_i and $Q_{i'} \subseteq P_j$ be the path between the end vertices of $P_{i'}$. Then $E(Q_i) \subseteq E(Q_{i'})$ or $E(Q_{i'}) \subseteq E(Q_i)$ or $E(Q_i) \cap E(Q_{i'}) = \emptyset$.

We define *nice ear decomposition* to characterize polygonal 2-trees and we show how it helps in efficiently compute the induced cycles. A nested ear decomposition $(P_0, \ldots, P_k)$ is said to be *nice* if it has the following property: P_0 is an edge and for each $i \geq 1$, if x_i and y_i are the end vertices of P_i, there is some $j < i$, such that (x_i, y_i) is an edge in P_j. A nice ear decomposition of a polygonal 2-tree is shown in Fig. 1a. Definition 1 naturally gives a nice ear decomposition for polygonal 2-trees. Further, a unique polygonal 2-tree can be constructed easily from a nice ear decomposition. Thus we have the following observation.

Observation 7. *A graph G is a polygonal 2-tree if and only if G has a nice ear decomposition.*

In the following lemmas, we present results from the literature that establish polygonal 2-trees as a subclass of 2-connected partial 2-trees, which we formalize in Lemma 10.

Lemma 8. *(See Theorem 42 in [2].) A graph G is a partial 2-tree if and only if every 2-connected component of G is a series-parallel graph.*

According to Lemma 8, 2-connected series-parallel graphs and 2-connected partial 2-trees are essentially the same.

Lemma 9. *(See Lemma 1, Lemma 7 and Theorem 1 in [9].) A graph G is 2-connected if and only if G has an open ear decomposition in which the first ear is an edge. Further, for a 2-connected series-parallel graph, every open ear decomposition is nested. A graph is series-parallel if and only if it has a nested ear decomposition.*

The above lemma implies that every 2-connected partial 2-tree has a nested ear decomposition *starting* with an edge (first ear is an edge) and vice versa. We strengthen the first part of this result in Lemma 13.

3.1. Necessary and sufficient conditions

From Propositions 1.7.2 and 12.4.2 in [5], partial 2-trees do not contain a K_4-subdivision (as a subgraph). The following lemma presents a few necessary properties of polygonal 2-trees, which are useful in the rest of the paper.

Lemma 10. *Let G be a polygonal 2-tree. Then the following statements are true.*

(a) G is a 2-connected partial 2-tree and G does not contain a K_4-subdivision.
(b) Any two induced cycles in G share at most one edge and at most two vertices.
(c) For $u, v \in V(G)$ such that $(u, v) \notin E(G)$, $G - \{u, v\}$ has at most two components.

Proof. From Lemma 9, a graph is a 2-connected partial 2-tree if and only if it has a nested ear decomposition starting with an edge. From Observation 7, a graph is a polygonal 2-tree if and only if it has a nice ear decomposition. Observe that nice ear decomposition is a restricted version of nested ear decomposition. Therefore, a polygonal 2-tree is a 2-connected partial 2-tree. Recall that partial 2-trees do not contain K_4-subdivision as a subgraph. It follows that polygonal 2-trees do not contain K_4-subdivision as a subgraph.

We now prove that any two induced cycles in G share at most one edge and at most two vertices. Let $D = (P_0, \dots, P_k)$ be a nice ear decomposition of G. The proof is by induction on the number of ears in G. If the number of ears in D is one, then the claim is trivially true. If the number of ears in D is at least two, then we remove the internal vertices of P_k from G and let G' be the resultant graph. Let $D' = (P_0, \dots, P_{k-1})$. As G' is a polygonal 2-tree and D' is a nice ear decomposition of G', inductively G' satisfies (b). Let u and v be the end vertices of P_k. For the induced cycle $C = P_k \cup (u, v)$, $C \cap G'$ is (u, v). Therefore C has at most one edge and two vertices in common with the induced cycles in G'. Because $V(P_k) \cap V(G') = \{u, v\}$ and $(u, v) \in E(G')$, C is the only induced cycle not in G'. Hence, any two induced cycles in G share at most one edge and at most two vertices.

We now prove the last claim of this lemma. The proof is by contradiction. We assume that the removal of vertices u and v from G such that $(u, v) \notin E(G)$ disconnects G into at least three components G_1, G_2 and G_3. Note that $\{u, v\}$ is a minimal vertex separator in G, because G is 2-connected. It follows that, for each $1 \leq i \leq 3$, there is an induced path P_i between u and v in G, such that the internal vertices of P_i are in G_i and $|E(P_i)| \geq 2$. We have induced cycles $C_1 = P_1 \cup P_3$ and $C_2 = P_2 \cup P_3$ that share at least two edges, which contradicts that any two induced cycles in G have at most one common edge. □

We now present a sufficient condition for a graph to be a polygonal 2-tree.

Lemma 11. *If G is a 2-connected partial 2-tree and every two induced cycles in G share at most one edge, then G is a polygonal 2-tree.*

Proof. On the contrary, assume that G is not a polygonal 2-tree. By Lemma 13, since G is a 2-connected partial 2-tree, G has a nested ear decomposition $D = (P_0, \dots, P_k)$ such that P_0 is an edge and for each $i \geq 1$, $|E(P_i)| \geq 2$. Since G is not a polygonal 2-tree, D is not a nice ear decomposition. Therefore, there exists an index $i \in \{1, \dots, k\}$ with the property that, let u and v be the end vertices of P_i, then for every $j < i$, $(u, v) \notin E(P_j)$. For every $j \geq i$, since $|E(P_j)| \geq 2$ and no internal vertex of P_j is in $P_1, \dots, P_{j-1}$, $(u, v) \notin E(P_j)$. Thereby $(u, v) \notin E(G)$. As $P_0 \cup \ldots \cup P_{i-1}$ is 2-connected, there exist two internally vertex disjoint paths P'_1 and P'_2 between u and v. Since $(u, v) \notin E(G)$, P'_1, P'_2 and P_i are internally vertex disjoint paths and each of these paths have at least one internal vertex. Due to Lemma 12, for $1 \leq i \neq j \leq 3$, there is no path between any internal vertex in P_i and any internal vertex in P_j that excludes the vertices u and v. Now we have two induced cycles $P'_1 \cup P_i$ and $P'_2 \cup P_i$ that share at least two edges. This contradicts the premise of the lemma. Therefore, G is a polygonal 2-tree. □

3.2. Computation of induced cycles in polygonal 2-trees

Our algorithm will perform several computations on the induced cycles of a polygonal 2-tree. It is therefore important to obtain the set of induced cycles in a polygonal 2-tree in linear time. We prove this in Theorem 14. The proof is based on the following two lemmas and a linear-time algorithm for obtaining an open ear decomposition [21].

Lemma 12. *Let G be a partial 2-tree and let P_1, P_2 and P_3 be three internally vertex disjoint paths between vertices u and v in G such that $(u, v) \notin E(G)$. Then $G - \{u, v\}$ has at least three components.*

Proof. Assume that $G - \{u, v\}$ has at most two components. Then without loss of generality, there is a path P between $x \in V(P_1)$ and $y \in V(P_2)$, such that $V(P) \cap V(P_3) = \emptyset$, $V(P) \cap V(P_1) = \{x\}$, and $V(P) \cap V(P_2) = \{y\}$. Then there is a K_4-subdivision on the vertices x, y, u and v in G. It contradicts that a partial 2-tree does not contain a K_4-subdivision. Thus $G - \{u, v\}$ has at least three components. □

Lemma 13. *Let G be a 2-connected partial 2-tree. Then there exists a nested ear decomposition $(P_0, \dots, P_k)$ of G, such that P_0 is an edge and for each $i \geq 1$, $|E(P_i)| \geq 2$.*

Proof. From Lemma 9, G has a nested ear decomposition $D = (P_0, \dots, P_k)$ such that P_0 is an edge. Suppose D does not satisfy the given constraint, then we update D as follows, so that the resultant nested ear decomposition satisfies the given constraint. Let P_i be the first path in the sequence D, such that $|E(P_i)| = 1$, where $i \geq 1$. Let P_j be the first path in the sequence D, such that the end vertices of P_i are in P_j, where $j < i$. Let x and y be the end vertices of P_i. We obtain new paths $P_{i'}$ and $P_{j'}$ from P_i and P_j as follows: $P_{i'}$ is the path between x and y in P_j and $P_{j'}$ is $P_j \cup P_i - X$, where X is the set of internal vertices in $P_{i'}$. We replace P_j with $P_{j'}$, delete P_i and add $P_{i'}$ immediately after $P_{j'}$. By performing the update steps mentioned above for at most $k - 2$ times, we obtain a nested ear decomposition that satisfies the desired constraint. □

In the lemma below, we show that a nested ear decomposition as in Lemma 13 is a nice ear decomposition for polygonal 2-trees and it can be computed in linear time.

Theorem 14. *Let G be a polygonal 2-tree on n vertices. Let D be a nested ear decomposition of G as in Lemma 13 and $\mathcal{B}$ be the set of induced cycles in G. Then D is a nice ear decomposition. Further, D and $\mathcal{B}$ can be computed in linear time and* $\text{size}(\mathcal{B})$ *is $O(n)$.*

Proof. On the contrary, assume that D is not a nice ear decomposition. By Lemma 13, G has a nested ear decomposition $D = (P_0, \ldots, P_k)$ such that P_0 is an edge and for each $i \geq 1$, $|E(P_i)| \geq 2$. Since D is not a nice ear decomposition, there exists an index $i \in \{2, \ldots, k\}$ with the property that, letting u and v be the end vertices of P_i, and for every $j < i$, $(u, v) \notin E(P_j)$. For every $j \geq i$, since $|E(P_j)| \geq 2$ and no internal vertex of P_j is in $P_1, \ldots, P_{j-1}$, $(u, v) \notin E(P_j)$. Thereby $(u, v) \notin E(G)$. As $P_0 \cup \ldots \cup P_{i-1}$ is 2-connected, there exist two internally vertex disjoint paths P'_1 and P'_2 between u and v. Since $(u, v) \notin E(G)$, P'_1, P'_2 and P_i are internally vertex disjoint paths. Due to Lemma 12, $G - \{u, v\}$ has at least three components, which is a contradiction to Lemma 10(c).

We now prove that a nice ear decomposition of G can be obtained in $O(n)$ time. First obtain an open ear decomposition D' starting with an edge by using linear-time algorithm in [21]. We then apply Lemma 13 on D'. As we can use a linked list representation for the paths in D', during the application of Lemma 13, we spend only a constant amount of time at each ear. Thus applying Lemma 13 takes linear time, because the number of ears in D' is at most n. From the first part of this lemma, the resultant ear decomposition is a nice ear decomposition. Also note that $|E(G)| \leq 2n - 3$. Thus a nice ear decomposition $(P_0, \ldots, P_k)$ is computed in $O(n)$ time.

From the nice ear decomposition $D = (P_0, \ldots, P_k)$ of G, we now present a linear-time procedure to obtain the set of induced cycles in G. Since P_0 is an edge, $C_1 = P_0 \cup P_1$ is an induced cycle in G. For every $i \geq 2$, by letting x_i and y_i be the end vertices of P_i, we obtain an induced cycle $C_i = P_i \cup (x_i, y_i)$ in G. Observe that $C_1, \ldots, C_k$ are the only induced cycles in G. This can be proved easily by applying induction on the number of ears in D. The number of ears in D is at most n. Thus the set of induced cycles in G can be obtained in $O(n)$. The ears $P_0, \ldots, P_k$ are a partition of $E(G)$. Therefore, $|E(C_0)| + \ldots + |E(C_k)| \leq |E(G)| + n$. Thus $\text{size}(\mathcal{B})$ is $O(n)$. □

4. Structure of paths, trees and MASTs in polygonal 2-trees

For the rest of the paper, G denotes a polygonal 2-tree. In this section we design an iterative procedure to delete a subset of edges from a polygonal 2-tree, so that the graph on the remaining edges is a minimum average stretch spanning tree. This result is shown in Theorem 21.

Important definitions: We introduce some necessary definitions on polygonal 2-trees. Let G' be a subgraph of G. Two induced cycles in G' are *adjacent* if they share an edge. An edge in G' is *internal* if it is part of at least two induced cycles; otherwise it is *external*. An induced cycle in G' is *external* if it has an external edge; otherwise it is *internal*. A fundamental cycle of a spanning tree, created by a non-tree edge is said to be *external* if the associated non-tree edge is external. For a cycle C in G', the *enclosure* of C is defined as $G'[V(C)]$ and is denoted by $Enc(C)$. A set $A \subseteq E(G)$ consisting of k (≥ 0) edges is said to be an *iterative* set for G if the edges in A can be ordered as $e_1, \ldots, e_k$ such that e_1 is external and not a bridge in G, and for each $2 \leq i \leq k$, e_i is external and not a bridge in $G - \{e_1, \ldots, e_{i-1}\}$. Let A be an iterative set of edges in G. For every edge $(u, v) \in A$, both u and v are not present in the same 2-connected component in $G - A$. We define $\text{bound}(A, G)$ to be the set of external edges in $G - A$ that are not bridges. For an edge $e \in \text{bound}(A, G)$, G_e denotes the 2-connected component in $G - A$ that has e. The following definition is illustrated in Fig. 1b.

Definition 15. Let A be an iterative set of edges in G and $e \in \text{bound}(A, G)$. The support of e is defined as $\{(u, v) \in A \mid$ there is a path P joining u and v in $G - A$ such that $P \cap G_e = e\}$ and is denoted by $\text{Support}(e)$. The $\text{cost}(e)$ is defined as $|\text{Support}(e)|$.

4.1. Structural properties of paths

In the following lemmas we present a result on the structure of paths connecting the end vertices of edges in an iterative set A. This is useful for setting up an iterative approach for computing a minimum average stretch spanning tree. We apply the necessary properties of polygonal 2-trees (cf. Lemma 10) and sufficient condition for a graph to be a polygonal 2-tree (cf. Lemma 11) in the proofs of the following lemmas.

Lemma 16. *Let A be an iterative set of edges for G and $(u, v) \in A$, P be a path joining u and v in $G - A$, G' be a 2-connected component in $G - A$ that has at least two vertices from P, and let $P' = P \cap G'$ be a path with end vertices x and y. Then the following are true:*

(a) $(x, y) \in E(G')$.
(b) If P is a shortest path, then P' is an edge.
(c) Every 2-connected component in $G - A$ is a polygonal 2-tree.

Proof. To show that $(x, y) \in E(G')$, assume to the contrary that $(x, y) \notin E(G')$. Since G' is 2-connected, there exist two internally vertex disjoint paths P_1 and P_2 between x and y in G'. Since A is an iterative set of edges for G and $(u, v) \in A$, $|\{u, v\} \cap V(G')| \leq 1$. It follows that $P' \subset P$. Then from the cycle $P \cup (u, v)$, we choose a path P_3 joining x and y, in such a way that P_3 is edge disjoint from P'. Consequently, none of the internal vertices in P_3 are from G'. Therefore, P_1, P_2 and P_3 are internally vertex disjoint paths joining x and y that have at least one internal vertex. By Lemma 12, $G - \{x, y\}$ has at least three components. Then the contrapositive of Lemma 10(c) implies that G is not a polygonal 2-tree. This contradicts that G is a polygonal 2-tree. Thus $(x, y) \in E(G')$.

If P is a shortest path and P' is not an edge, then we can replace P' in P by (x, y) and obtain a path shorter than P. Therefore, P' is an edge.

We now prove the third claim of this lemma. Let H be a 2-connected component in $G - A$. From Lemma 10(a), G is a partial 2-tree. Thereby H is a 2-connected partial 2-tree. Since A is an iterative set, the edges in A can be ordered as $e_1, \ldots, e_k$, such that e_1 is external and not a bridge in G and for each $2 \leq i \leq k$, e_i is external and not a bridge in $G - \{e_1, \ldots, e_{i-1}\}$. We delete the edges in A from G one by one, in the order $e_1, \ldots, e_k$. Observe that each time, when an edge e_i is deleted, exactly one induced cycle is destroyed and no new induced cycles are created. Also we know that any two induced cycles in G share at most one edge. Consequently, any two induced cycles in H share at most one edge. Therefore, Lemma 11 implies that H is a polygonal 2-tree. □

Lemma 16 is illustrated in Fig. 1c.

Lemma 17. *Let A be an iterative set of edges for G. Then $(u, v) \in$ Support(e) if and only if there is a shortest path P joining u and v in $G - A$ and P has e.*

We use the following lemma to prove Lemma 17.

Lemma 18. *Let P be a path with end vertices u and v in G. Let $G_1, \ldots, G_r$ be the 2-connected components in G from which P has at least two vertices. For each $1 \leq i \leq r$, let P_i be a shortest path joining the end vertices of $G_i \cap P$. Let P' be the path obtained from P by replacing every $G_i \cap P$ with P_i. Then P' is a shortest path joining u and v in G.*

Proof. Assume that there exists a path P'' joining u and v in G such that $|E(P'')| < |E(P')|$. For each i, let x_i and y_i be the end vertices of P_i. The set of edges in P' that are bridges in G are definitely in P''. Therefore, there exists an $1 \leq i \leq r$, such that the subpath between x_i and y_i in P'' is shorter than P_i. This contradicts that P_i is a shortest path joining x_i and y_i. □

The above lemma holds for arbitrary graphs.

Proof of Lemma 17. ($\Rightarrow$) Let $(u, v) \in$ Support(e). By the definition of Support(e), there is a path P' joining u and v in $G - A$ such that $G_e \cap P'$ is e. Let $G_1, \ldots, G_r$ be the 2-connected components in $G - A$ from which P' has at least two vertices. For each $1 \leq i \leq r$, by Lemma 6(b), $P_i = G_i \cap P'$ is a path; let x_i and y_i be the end vertices of P_i; due to Lemma 16(a), $(x_i, y_i) \in E(G_i)$. Let P be the path obtained from P' after replacing every P_i by (x_i, y_i). Since $G_e \cap P'$ is e, P has e. From Lemma 18, P is a shortest path joining u and v in $G - A$ and P has e.

($\Leftarrow$) Let P be a shortest path joining u and v in $G - A$ such that P has e. Let G_e be a 2-connected component containing e in $G - A$. Since P has e, G_e has at least two vertices from P. From Lemma 16(b), $G_e \cap P$ is an edge. Further, $G_e \cap P$ is e. Thus $(u, v) \in$ Support(e). □

4.2. Structural properties of spanning trees

Lemma 19. *Let T be a spanning tree of G and e be an external edge in G such that $e \in E(T)$. For the spanning tree T, let $C_{\min}$ be the smallest fundamental cycle containing e and let $C_{\max}$ be a largest fundamental cycle containing e. Let e' and e'' be the non-tree edges associated with $C_{\min}$ and $C_{\max}$, respectively. Then, (a) e'' is an external edge (b) $Enc(C_{\min}) \subseteq Enc(C_{\max})$.*

We use the following lemma to prove Lemma 19.

Lemma 20. *Let T be an arbitrary spanning tree of G. Let C be a fundamental cycle of T formed by a non-tree edge (x, y) in G. Let C_1 be an induced cycle containing (x, y) in $Enc(C)$ and C_2 be another induced cycle containing (x, y) in G. Then*

(a) $V(C) \cap V(C_2) = \{x, y\}$.
(b) For vertices $u \in V(C) \setminus \{x, y\}$ and $v \in V(C_2) \setminus \{x, y\}$, any path joining u and v in G goes through x or y.

Proof. If $C = C_1$, then we are done, because every two induced cycles share at most two vertices. We now consider the case that $C \neq C_1$ and thus there is an edge (x', y') in C_1 that is not in C. Assume that $V(C) \cap V(C_2)$ has a vertex that is different

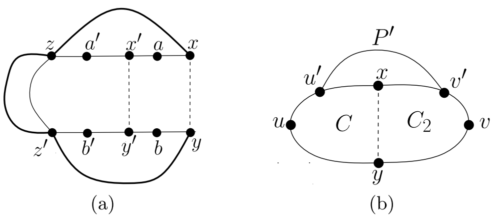


Fig. 2. (a) In the graph shown, the following are the six internally vertex disjoint paths on the vertices $\{x, y, x', z\}$: (x, y), (x', a, x), (x', y', b, y), (z, a', x'), $(z, \ldots, x)$ using thick edges, $(z, \ldots, z', \ldots, y)$ using thick edges. (b) A K_4-subdivision on vertices $\{x, y, u', v'\}$.

from x and y. In the path consisting of at least two edges from x to y in C_2, let z and z' be the first and last vertices from C, respectively. From Lemma 10(b), we know that any two induced cycles in a polygonal 2-tree share at most two vertices. Thus $V(C_1) \cap V(C_2) = \{x, y\}$ and $z, z' \notin V(C_1)$. Since (x', y') is not in C, clearly $(x', y') \neq (x, y)$. Further, without loss of generality, assume that $x' \neq x$. From Lemma 10(b), $x' \notin V(C_2)$ and thus $x' \neq z$. The graph $C \cup C_2$ is shown in Fig. 2a, where the edges in C and C_2 other than (x, y) are shown by solid edges and bold edges, respectively. There is a K_4-subdivision in $C \cup C_2$ on the vertices $\{x, y, z, x'\}$, because of the following six paths that are internally vertex disjoint: the edge (x, y); the path joining x' and x in C_1 without going through y; the path joining x' and y in C_1 without going through x; the path between z and x in C_2 without going through y; the path between z and y (via z') in C_2 without going through x; the path between z and x' in C without going through y'. This contradicts that G does not contain a K_4-subdivision. Thus $V(C) \cap V(C_2) = \{x, y\}$.

We now prove the second part of the lemma. Let P be a path that joins vertices u and v, such that $x, y \notin V(P)$. In the sequence of vertices in P from u to v, let u' be the last vertex in C and v' be the first subsequent vertex in C_2. Let $P' \subseteq P$ be the path joining the vertices u' and v'. From the first part of this lemma, x and y are the only vertices common in C and C_2. Thereby u' is different from v'. It follows that the edges in P' are disjoint from the edges in $C \cup C_2$. Now, we consider the graph $H = C \cup C_2 \cup P'$. The subgraph H of G, shown in Fig. 2b, is a K_4-subdivision on the vertices x, y, u', v', because for every two vertices in $\{x, y, u', v'\}$, there is an internally vertex disjoint path. We have a contradiction, as G does not contain a K_4-subdivision. Hence the lemma. □

Proof of Lemma 19. Assume that e'' is an internal edge in G. Then e'' is contained in at least two induced cycles C_1 and C_2 in G. Without loss of generality assume that C_1 is in $Enc(C_{max})$ and let $P = C_2 - e''$ be a path. From Lemma 20(a), $V(C_2) \cap V(C_{max}) = \{x, y\}$. Thus the path between x and y in T and the path P are internally vertex disjoint. As a consequence, there is an edge (u, v) in P but not in T; otherwise the tree T has a cycle. By Lemma 20(b), the fundamental cycle formed by the non-tree edge (u, v) is of larger length than C_{max} and also has e. Because C_{max} is a maximum length fundamental cycle containing e, this is a contradiction. Therefore, e'' is an external edge in G.

We now prove the second part of the lemma. Let P_1 be the maximal path in $C_{min} \cap C_{max}$, such that e is in P_1. Let a and b be the end vertices of P_1. Let P_2 be the path between a and b in C_{min} that is internally vertex disjoint from P_1. Let P_3 be the path between a and b in C_{max} that is internally vertex disjoint from P_1. Let C' and C'' be the induced cycles containing e in $Enc(C_{max})$ and $Enc(C_{min})$, respectively. If P_2 is an edge, then $Enc(C_{min}) \subseteq Enc(C_{max})$ and we are done. If P_1 is an edge, then e is being shared by two induced cycles C' and C''. This contradicts that e is an external edge. Consider the case when P_1 as well as P_2 consist of at least two edges. As $|C_{max}| \geq |C_{min}|$, P_3 consist of at least two edges. Since $G - \{a, b\}$ has at least three components and $(a, b) \notin E(G)$, by Lemma 10, G is not a polygonal 2-tree. Hence the lemma. □

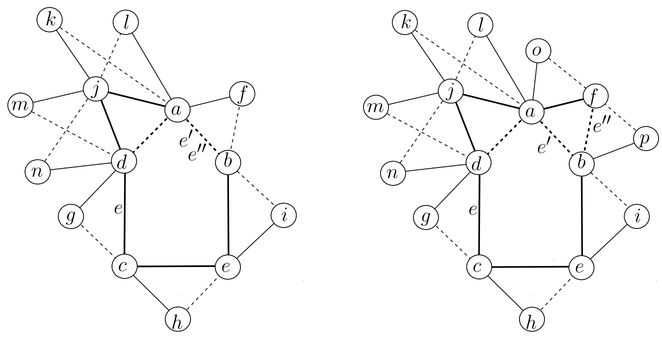
4.3. Structural properties of MASTs

A set A of edges in G is referred to as a *safe* set for G, if A is an iterative set of edges for G and a minimum average stretch spanning tree of G is in $G - A$.

Theorem 21. *Let A be a safe set of edges for G such that* $\text{bound}(A, G) \neq \emptyset$. *Let e be an edge in* $\text{bound}(A, G)$ *for which* $\text{cost}(e)$ *is minimum. Then $A \cup \{e\}$ is a safe set for G.*

Proof. For a safe set A, let T^* be a minimum average stretch spanning tree of G; that is, $T^* \subset G - A$ as $\text{bound}(A, G) \neq \emptyset$. If $e \notin E(T^*)$, then we are done. Assume that $e \in E(T^*)$. Clearly, $A \cup \{e\}$ is an iterative set for G. To show that $A \cup \{e\}$ is a safe set for G, we use the technique of cut-and-paste to obtain a spanning tree T' (by deleting the edge e from T^* and adding an appropriately chosen edge e') and show that $\text{AvgStr}(T') \leq \text{AvgStr}(T^*)$.

Let G_e be the 2-connected component in $G - A$ containing e and $G_1, \ldots, G_k$ be the 2-connected components in $G - A$. For clarity, $G_e \in \{G_1, \ldots, G_k\}$. From Lemma 16(c), G_e is a polygonal 2-tree. For $1 \leq i \leq k$, by Lemma 6(a), $T_i = T^* \cap G_i$ is a spanning tree of G_i. For the spanning tree T^*, let C_{min} be the smallest fundamental cycle containing e in G_e and let C_{max} be a largest fundamental cycle containing e in G_e. Let $e', e'' \in E(G_e)$ be the non-tree edges associated with C_{min} and C_{max}, respectively. From Lemma 19, e'' is an external edge in G_e and $Enc(C_{min}) \subseteq Enc(C_{max})$. Let $e' = (x_{min}, y_{min})$,



(a) $X = \{(g, c)\}, Y = \{(f, b)\}$ (b) $X = \{(g, c)\}, Y = \{(f, b), (f, p)\}$

Fig. 3. Dashed and solid edges shown in thick are the edges of G_e. Dashed edges are the non-tree edges of T^* and solid edges are the edges of T^*.

$e'' = (x_{\max}, y_{\max})$. For a non-tree edge (u, v) in T^*, we use P_{uv} to denote the path between u and v in T^* and C_{uv} to denote the fundamental cycle of T^* formed by (u, v). Let $X = \{(u, v) \in E(G) \setminus E(T^*) \mid e \in E(P_{uv}), e' \notin Enc(C_{uv})\}$, $Y = \{(u, v) \in E(G) \setminus E(T^*) \mid e \in E(P_{uv}), e' \in Enc(C_{uv}), (u, v) \neq e'\}$, $Z = \{(u, v) \in E(G) \setminus E(T^*) \mid e \notin E(P_{uv})\}$. The set of non-tree edges in T^* is $X \uplus Y \uplus \{e'\} \uplus Z$. Let $T' = T^* + e' - e$. The set of non-tree edges in T' is $X \uplus Y \uplus Z \uplus \{e\}$. To prove the theorem, we prove the following claims.

Claim 1. *$X \subseteq A$.*

Claim 2. Support$(e) \subseteq X$.

Claim 3. Support$(e'') \subseteq Y$.

Claim 4. $X \subseteq$ Support(e).

Claim 5. *For every $(u, v) \in Z$, the path between u and v in T^* is in T'.*

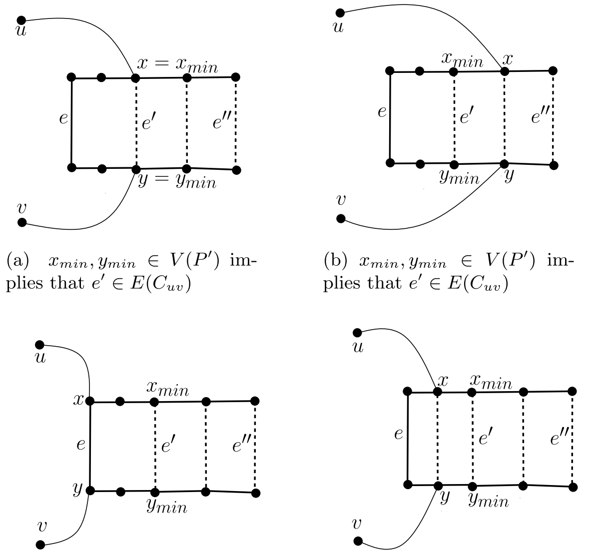
Assuming that the above five claims are true, we complete the proof of the theorem. We know that cost$(e) \leq$ cost(e''). As e and e'' are in G_e, from the definition of Support, we further know that Support$(e) \cap$ Support$(e'') = \emptyset$. Therefore, from Claims 2, 3 and 4, it follows that $|X| \leq |Y|$. Since $e', e \in E(C_{\min})$, $e \in E(T^*)$ and $e' \notin E(T^*)$, the stretch of e' in T^* is equal to the stretch of e in T'. From Claim 5, stretch does not change for the edges in Z. For all the edges in X, stretch increases by $|C_{\min}| - 2$. Further, for all the edges in Y, stretch decreases by $|C_{\min}| - 2$. If $|X| < |Y|$, as shown in Fig. 3b, then AvgStr$(T') <$ AvgStr(T^*); it contradicts that T^* is a minimum average stretch spanning tree. Thereby $|X| = |Y|$, shown in Fig. 3a. This implies that AvgStr$(T') =$ AvgStr(T^*). Since T^* is a minimum average stretch spanning tree, T' is also a minimum average stretch spanning tree. Clearly, T' is in $G - (A \cup \{e\})$. Hence $A \cup \{e\}$ is a safe set for G.

We now prove the five claims.

Proof of Claim 1. On the contrary, assume that $(u, v) \in X$ and $(u, v) \notin A$. To arrive at a contradiction, we show that e is an internal edge. Since $(u, v) \in X$, there is a fundamental cycle C_{uv} of T^* formed by the non-tree edge (u, v) containing e. As $(u, v) \notin A$, clearly (u, v) is in $G - A$. Further, P_{uv} is in $G - A$, because $T^* \subset G - A$. So we know that C_{uv} is in $G - A$. If C_{uv} is not in G_e, then $G_e \cup C_{uv}$ becomes a 2-connected subgraph in $G - A$, because G_e is in $G - A$, C_{uv} is in $G - A$, and e is both in G_e and C_{uv}. But, we know that G_e is a maximal 2-connected subgraph (2-connected component), thereby C_{uv} is in G_e. Clearly, C_{uv} and $C_{\min}$ are not edge disjoint cycles. If $Enc(C_{\min}) \subseteq Enc(C_{uv})$, then either $(u, v) \in Y$ or $(u, v) = e'$, which contradicts the fact that $(u, v) \in X$. Also, $Enc(C_{uv})$ is not contained in $Enc(C_{\min})$, because $C_{\min}$ is a minimum length fundamental cycle containing e. Therefore, both $C_{\min}$ and C_{uv} are not contained in each other. Thus, e is an internal edge in $G - A$. This is a contradiction, as we know that e is external. □

Proof of Claim 2. Let $(u, v) \in$ Support(e). In order to prove that $(u, v) \in X$, we show the following: (a) $(u, v) \notin E(T^*)$, (b) P_{uv} has e and (c) e' is not in $Enc(C_{uv})$.

By the definition of Support(e), $(u, v) \in A$. As $T^* \subset G - A$, it follows that $(u, v) \notin E(T^*)$. By Lemma 17, there is a shortest path P joining u and v in $G - A$ and P has e. Let $G'_1, \ldots, G'_r$ be the 2-connected components in $G - A$ containing at least two vertices from P. Due to Lemma 16(b), for each $1 \leq i \leq r$, $P \cap G'_i$ is an edge, say (x_i, y_i). Thus $P \cap G_e$ is e. Further, P contains at most one vertex from e', because $e' \in E(G_e)$. The set of edges in P that are cut-edges in $G - A$ are present in T^*. Due to Lemma 6(a), replacing every edge (x_i, y_i) in P by the path between x_i and y_i in T^*, P_{uv} is obtained. Since $P \cap G_e$ is e and e is in T^*, it implies that P_{uv} has e. Thus P_{uv} has e, and e' is not in $Enc(C_{uv})$. □



(a) $x_{min}, y_{min} \in V(P')$ implies that $e' \in E(C_{uv})$

(b) $x_{min}, y_{min} \in V(P')$ implies that $e' \in E(C_{uv})$

(c) $P' \cap G_e = e$

(d) P' has at most one vertex from $\{x_{min}, y_{min}\}$

Fig. 4. Cases in Claim 4: Dashed edges and solid edges shown in thick are the edges of G_e. Solid edges shown in thick are the edges of $G_e \cap T^*$. The path $G_e \cap P_{uv}$ is denoted by P'.

Proof of Claim 3. Let $(u, v) \in \text{Support}(e'')$. In order to prove that $(u, v) \in Y$, we show the following: (a) $(u, v) \notin E(T^*)$, (b) P_{uv} has e and (c) e' is in $Enc(C_{uv})$.

Because $(u, v) \in A$ and $T^* \subset G - A$, we have $(u, v) \notin E(T^*)$. As $e, e'' \in E(G_e)$, due to Lemma 17, there is a shortest path P joining u and v in $G - A$ and P has e''. Let $G'_1, \ldots, G'_r$ be the 2-connected components in $G - A$ such that for each $1 \le i \le r$, $P \cap G'_i$ is an edge, say (x_i, y_i), due to Lemma 16(b). By Lemma 6(a), we replace every edge (x_i, y_i) in P by the path between x_i and y_i in T^* and obtain the tree path P_{uv}. Note that $P \cap G_e$ is e'', $e'' = (x_{\max}, y_{\max})$, and e'' in P got replaced with the path between $x_{\max}$ and $y_{\max}$ in T^*. Also, we know that the path between $x_{\max}$ and $y_{\max}$ in T^* has e. Further by Lemma 19(b), e' is in $Enc(C_{\max})$. These observations imply that P_{uv} has e and $Enc(C_{uv})$ contains e'. □

Proof of Claim 4. Let $(u, v) \in X$. By Claim 1, clearly $(u, v) \in A$. Lemma 6(b) implies that $P_{uv} \cap G_e$ is a path. Let $P' = P_{uv} \cap G_e$ be a path and let x and y be the end vertices of P'. If P' is an edge, shown in Fig. 4c, then the claim holds. On the contrary assume that P' has at least two edges. By Lemma 16(a), $(x, y) \in E(G)$. Further, $(x, y) \notin E(T^*)$ as it would then form a cycle in the tree. If $x_{\min}, y_{\min} \in V(P')$, shown in Fig. 4a and Fig. 4b, then (u, v) must be in Y. As we know that $(u, v) \in X$, the path P' is strictly contained in the path joining the vertices $x_{\min}$ and $y_{\min}$ in T^*. Then the fundamental cycle of T formed by (x, y) is of lesser length than the length of $C_{\min}$, shown in Fig. 4d; a contradiction because $C_{\min}$ is a minimum length fundamental cycle in G_e containing e. Therefore, $P_{uv} \cap G_e$ is e. Thus $(u, v) \in \text{Support}(e)$. □

Proof of Claim 5. Let $(u, v) \in Z$. By the definition of Z, clearly $e \notin E(P_{uv})$. It implies that $e' \notin Enc(C_{uv})$ as the path between the end vertices of e' in T^* has e. Therefore P_{uv} has at most one end vertex from e and e'. Since the symmetric difference of $E(T^*)$ and $E(T')$ is $\{e, e'\}$, the path P_{uv} in T^* remains the same in T'. Hence the theorem. □

We now show the termination condition for applying Theorem 21.

Lemma 22. *Let A be a safe set of edges for G such that* $\text{bound}(A, G) = \emptyset$. *Then $G - A$ is a minimum average stretch spanning tree of G.*

Proof. Since A is a safe set for G, a minimum average stretch spanning tree is contained in $G - A$. Since $\text{bound}(A, G)$ is $\emptyset$, $G - A$ is acyclic. Therefore, $G - A$ is a minimum average stretch spanning tree of G. □

5. Computation of an MAST in polygonal 2-trees

A quick overview of our approach to solve Mast is presented in Algorithm 1 below. The detailed implementation is presented in Algorithm 2.

In order to obtain a minimum average stretch spanning tree efficiently, we need to efficiently find an edge in $\text{bound}(A, G)$ with minimum cost in every iteration, where A is a safe set for G. In this section, we present necessary data-structures,

Algorithm 1: An algorithm to find an MAST of a polygonal 2-tree G.

```
A ← ∅;
for each edge e ∈ E(G) do c[e] ← 0;
while G − A has a cycle do
    Choose an edge e from G − A, such that e belongs to exactly one induced cycle in G − A and c[e] is minimum ;
    Let C be the induced cycle containing e in G − A ;
    for each ê ∈ E(C) \ {e} do c[ê] ← c[ê] + c[e] + 1;
    A ← A ∪ {e} ;
Return G − A;
```

Algorithm 2: An algorithm to find an MAST of a polygonal 2-tree G.

```
1  Perform the steps described in Initialization ;
2  while Q ≠ ∅ do
3      e ← Q.extract-min() ;
4      A ← A ∪ {e} ;
5      C ← Cycles[e] \ pCycles[e] ;
6      for each edge ê ∈ E(C) \ {e} do
7          c[ê] ← c[ê] + c[e] + 1 ;
8          pCycles[ê] ← pCycles[ê] ∪ C ;
9          unpCount[ê] ← unpCount[ê] − 1 ;
10         if unpCount[ê] = 1 then Q.insert(ê, c[ê]) ;
11         if unpCount[ê] = 0 then Q.delete(ê) ;
12 Return G − A;
```

so that a minimum average stretch spanning tree in polygonal 2-trees on n vertices can be computed in $O(n \log n)$ time. A pseudo-code for achieving this is given in Algorithm 2. For each edge $e \in \text{bound}(A, G)$, we show how to compute $cost(e)$ efficiently in Lemma 24.

Notation. Let Q be a min-heap that supports the following operations: Q.insert(x) inserts an arbitrary element x into Q, Q.extract-min() extracts the minimum element from Q, Q.decrease-key(x, k) decreases the key value of x to k in Q, Q.delete(x) deletes an arbitrary element x from Q. Q.delete(x) can be implemented by calling Q.decrease-key($x, -\infty$) followed by Q.extract-min() [3]. For a set A of safe edges for G, an induced cycle in G is said to be *processed* if it is not in $G - A$; otherwise it is said to be *unprocessed*. For an edge $e \in E(G)$, we use the sets $Cycles[e]$ and $pCycles[e]$ to store the set of induced cycles and processed induced cycles, respectively containing e; $unpCount[e]$ is used to store the number of unprocessed induced cycles containing e. For an edge $e \in E(G)$, we use $c[e]$ to store some intermediate values while computing $\text{cost}(e)$; whenever e becomes an edge in $\text{bound}(A, G)$, we make sure that $c[e]$ is $\text{cost}(e)$.

Initialization. Given a polygonal 2-tree G, we first compute the set of induced cycles in G. For each induced cycle C in G and for each edge $e \in E(C)$, we insert the cycle C in the set $Cycles[e]$. For each $e \in E(G)$, we perform $unpCount[e] \leftarrow |Cycles[e]|$, $pCycles[e] \leftarrow \emptyset$, $c[e] \leftarrow 0$. We further initialize the set A of safe edges with $\emptyset$. Later, we construct a min-heap Q with the edges e in $\text{bound}(A, G)$ i.e., external edges that are not bridges in G, based on $c[e]$.

Now we shall look at Algorithm 2.

Algorithm 2 maintains the following loop invariants:

L1. The min heap Q only consists of, the set of edges in $\text{bound}(A, G)$.
L2. For an edge $e \in E(G)$, $pCycles[e]$ is the set of processed induced cycles containing e and $unpCount[e]$ is equal to the number of unprocessed induced cycles containing e.
L3. For every edge $e \in \text{bound}(A, G)$, $Cycles[e] \setminus pCycles[e]$ is the unique external induced cycle in $G - A$ containing e.
L4. A is a safe set for G (cf. Theorem 21).
L5. For every edge $e \in \text{bound}(A, G)$, $c[e] = \text{cost}(e)$ (cf. Lemma 24).

Proofs of Loop Invariants L1–L3

Proof. An edge gets inserted into Q only when it is in a unique induced cycle of $G - A$. Further, all the bridges in $G - A$ are getting deleted in line 11. Thus L1 holds. In lines 8 and 9, processed induced cycles and the count of unprocessed cycles are updated. Thus L2 holds. For each edge $e \in \text{bound}(A, G)$, $unpCount[e]$ is one. Therefore, $Cycles[e] \setminus pCycles[e]$ gives the unique induced cycle in $G - A$ containing e, thereby L3 holds. □

The proof of loop invariant L5 is deferred to next subsection. We finish the running time analysis of Algorithm 2 below. The algorithm terminates when Q becomes $\emptyset$, that is, $\text{bound}(A, G) = \emptyset$. Then by Lemma 22, $G - A$ is a minimum average stretch spanning tree of G.

Lemma 23. *For a polygonal 2-tree G on n vertices, Algorithm 2 takes $O(n \log n)$ time.*

Proof. The set of induced cycles in G can be obtained in linear time (cf. Theorem 14), thereby line 1 takes linear time. As the size of induced cycles in G is $O(n)$ (Theorem 14), line 5 and lines 7–9 contribute $O(n)$ towards the run time of the algorithm. Also every edge in G gets inserted into the heap Q and gets deleted from Q only once and $|E(G)| \leq 2n - 3$. It takes $O(\log n)$ time for the operations `insert()`,`delete()` and `extract-min()` [3]. Thus Algorithm 2 takes $O(n \log n)$ time. $\square$

5.1. Cost updating procedure

During the execution of our algorithm, for each edge e in $G - A$, such that e is external and not a bridge, we need to compute $\text{cost}(e)$ efficiently. This is done in Algorithm 2 in line 7. We prove the correctness of this step in Corollary 26 using Lemma 24.

Algorithm 2 runs for $m - n + 1$ iterations. For $0 \leq j \leq m - n + 1$, let $A_j \subset E(G)$ denote the set of safe edges in G at the end of jth iteration. Let e be an edge extracted from the heap Q in jth iteration and C be the unique induced cycle containing e in $G - A_{j-1}$. That is, C is a cycle in $G - A_{j-1}$ and C is not a cycle in $G - A_j$ as e is added to A in iteration j. Then we say that C is processed in iteration j and e is the *destructive* edge for C. For each external edge in G, we know that $\text{Support}(e)$ is $\emptyset$, which implies that $\text{cost}(e)$ is 0.

Lemma 24. *Let $e \in \text{bound}(A_j, G)$ such that e is an internal edge in G, where $0 \leq j < m - n + 1$. Let C be the unique external induced cycle in $G - A_j$ containing e and $C_1, \dots, C_k$ be the other induced cycles in G containing e. For $1 \leq i \leq k$, let e_i be the destructive edge of C_i. Then* $\text{Support}(e) = \text{Support}(e_1) \cup \dots \cup \text{Support}(e_k) \cup \{e_1, \dots, e_k\}$.

We use the following lemma to prove Lemma 24. For a path P and for vertices $x, y \in V(P)$, $P(x, y)$ denotes a subpath in P with end vertices x and y. For an edge (x, y) in G, if (x, y) is external, then there is a unique shortest path between x and y in $G - (x, y)$.

Lemma 25. *Let A be a safe set for G and P be a path with end vertices u and v in $G - A$. Let (a, b) be an edge in P such that $(a, b) \in \text{bound}(A, G)$. Let $G' = G - (A \cup \{(a, b)\})$ and P' be the path obtained from P by replacing (a, b) with the shortest path between a and b in G'. Then P is a shortest path in $G - A$ if and only if P' is a shortest path in G'.*

Proof. ($\Rightarrow$) Assume that P' is not a shortest path joining u and v in G'. Then there exist a path P'' joining u and v in G' such that $|E(P'')| < |E(P')|$. Consider the case when P'' has both a and b. Because there is a unique shortest path joining a and b in G', $P'(a, b)$ is $P''(a, b)$. Then, without loss of generality the path $P''(u, a)$ consisting of lesser number of edges than $P'(u, a)$. As $P(u, a)$ is $P'(u, a)$, replacing the path $P(u, a)$ in P with $P''(u, a)$ leads to a path shorter than P in $G - A$, which contradicts that P is a shortest path. Consider the other case when P'' has at most on vertex from $\{a, b\}$. In the path P from u to v, without loss of generality, assume that a appears before b. In the sequence of vertices in P from u to a, let a' be the last vertex in P''. Similarly in the sequence of vertices in P from b to v, let b' be the first vertex in P''. Let G'' be the graph obtained by performing union on $P(u, v) \cup \{(u, v)\}$ and the polygonal 2-tree containing (a, b) in $G - A$. Since the intersection of $P(u, v) \cup \{(u, v)\}$ and $G - A$ is (a, b), G'' is a polygonal 2-tree. Now we have three internally vertex disjoint paths between a' and b' in G'': $P''(a', b')$, $P(a', b')$, and the path other than $P(a', b')$ in $P(u, v) \cup \{(u, v)\}$. Since $|\{a, b\} \cap \{a', b'\}| \leq 1$ and P is a shortest path in $G - A$, $(a', b') \notin E(G'')$. By Lemma 12, $G'' - \{a', b'\}$ has at least three components. Applying Lemma 10(c), G'' is not a polygonal 2-tree, which contradicts that G'' is a polygonal 2-tree.

($\Leftarrow$) Assume that P is not a shortest path joining u and v in $G - A$. Then there exists a path P'' joining u and v in $G - A$ such that $|E(P'')| < |E(P)|$. Consider the case where $a, b \in V(P'')$. P'' is a disjoint union of $P''(u, a)$, $P''(a, b)$ and $P''(b, v)$. Similarly P is a disjoint union of $P(u, a)$, $P(a, b)$ and $P(b, v)$. Since $P(a, b) = P''(a, b) = e$, without loss of generality the path $P''(u, a)$ consists of lesser number of edges than $P(u, a)$. Replacing the path $P'(u, a)$ in P' with $P''(u, a)$ leads to a path shorter than P' in G', which contradicts that P' is a shortest path. Consider the other case where at most one vertex from $\{a, b\}$ is in P''. Then P'' is in G', and also we have $|E(P'')| < |E(P)| < |E(P')|$. Consequently, P'' is shorter than P' in G'. This contradicts that P' is a shortest path joining u and v in G'. $\square$

Proof of Lemma 24. For $1 \leq i \leq k$, let $f(i) + 1$ be the iteration number in which C_i is processed in Algorithm 2.

($\Leftarrow$) Let $(u, v) \in \text{Support}(e_i)$ for some $1 \leq i \leq k$. Then by Lemma 17, there is a shortest path P joining u and v in $G - A_{f(i)}$ and P has e_i. From the premise, e_i gets added to A in the iteration $f(i) + 1$. Thereby $e_i \in A_{f(i)+1}$, which implies that e_i is not in $G - A_{f(i)+1}$. Consider the path $P' = (P - e_i) \cup (C_i - e_i)$ in $G - A_{f(i)+1}$. As e_i is exterior in $G - A_{f(i)+1}$, $C_i - e_i$ is a shortest path between the end vertices of e_i in $G - A_{f(i)+1}$. Also, $C_i - e_i$ has e. Thus P' has e. By forward

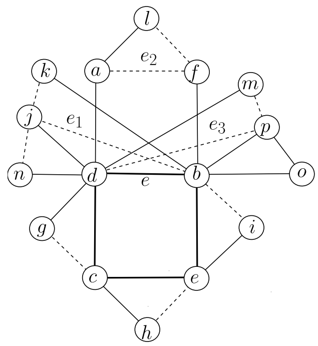


Fig. 5. For the polygonal 2-tree G shown, dashed edges are the edges in A, solid edges are the edges of $G - A$ and solid edges shown in thick are the edges of G_e. Support$(e_1) = \{(j, k)\}$, Support$(e_2) = \{(l, f)\}$, Support$(e_3) = \{(m, p)\}$ and Support$(e) = \{(j, k), (l, f), (m, p), e_1, e_2, e_3\}$.

direction of Lemma 25, P' is a shortest path joining u and v in $G - A_{f(i)+1}$. Note that e is in $G - A_j$. By forward direction of Lemma 25, it follows that there is a shortest path P_j joining u and v in $G - A_j$ and P_j has e. Thus $(u, v) \in$ Support(e). Observe that $e_i \in Enc(P_j \cup (u, v))$. Further, the path between the end vertices of e_i in P_j is a shortest path containing e in $G - A_j$. Therefore, we have $e_i \in$ Support(e).

($\Rightarrow$) Let $(u, v) \in$ Support(e). Then by Lemma 17, there is a shortest path P' joining u and v in $G - A_j$ and P' has e. Now $Enc(P' \cup (u, v))$ contains e_i for some $1 \leq i \leq k$. Consider the case $(u, v) \neq e_i$. Let x_i and y_i be the end vertices of e_i. We replace the path between x_i and y_i in P' by the edge e_i and let P be the resultant path consisting of e_i. From the reverse direction of Lemma 25, P is a shortest path joining u and v in $G - A_{f(i)}$, and P has e_i. Thereby $(u, v) \in$ Support(e_i). As a result, $(u, v) \in$ Support$(e_i) \cup \{e_i\}$ for some $1 \leq i \leq k$. □

Lemma 24 is illustrated in Fig. 5.

Corollary 26. *Let $e, e_1, \ldots, e_k$ be the edges as mentioned in Lemma 24. Then* cost$(e) =$ cost$(e_1) + \ldots +$ cost(e_k).

This concludes the presentation of our main result, namely Theorem 2.

6. On minimum cycle bases in polygonal 2-trees

This section is of our independent interest, which presents results on minimum cycle basis in polygonal 2-trees. In particular, we show that there is a unique minimum cycle basis in polygonal 2-trees, which can be computed in linear time. Also, we present an alternative characterization for polygonal 2-trees using cycle basis. This is shown in Theorem 31.

A graph is *Eulerian* if the degree of every vertex is even. Let G be an arbitrary unweighted graph and $H_1, \ldots, H_k$ be subgraphs of G. Then the graph $H_1 \oplus \ldots \oplus H_k$ consists only of the edges that appear odd number of times in $H_1, \ldots, H_k$. A minimal set $\mathcal{B}$ of Eulerian subgraphs of G is a *cycle basis* of G, if every cycle in G can be expressed as exclusive-or ($\oplus$) sum of a subset of graphs in $\mathcal{B}$. A *minimum cycle basis* (MCB) of G is a cycle basis that minimizes the sum of the lengths of the cycles in the cycle basis. It is well known that every Eulerian graph in any minimum cycle basis is a cycle. The cardinality of a cycle basis is $m - n + 1$ [5].

Planar graphs and Halin graphs are characterized based on their cycle bases. A cycle basis is said to be *planar basis* if every edge in the graph appears in at most two cycles in the cycle basis. A graph is planar if and only if it has a planar basis [18]. A 3-connected planar graph is Halin if and only if it has a planar basis and every cycle in the planar basis has an external edge [23]. Halin graphs that are not necklaces have a unique minimum cycle basis [22]. Also, outerplanar graphs have a unique minimum cycle basis [16].

Lemma 27. *(See Proposition 1.9.1 in [5].) The induced cycles in an arbitrary graph G generate its entire cycle space.*

Lemma 28. *The number of induced cycles in a polygonal 2-tree G is $m - n + 1$.*

Proof. We apply induction on the number of internal edges in G. Let $E_{in}(G)$ denote the set of internal edges in G. If $|E_{in}(G)| = 0$, then G has one induced cycle and $m - n + 1$ is one. For the induction step, let $|E_{in}(G)| > 0$. We decompose G into polygonal 2-trees $G_1, \ldots, G_k$ such that $G_1 \cup \ldots \cup G_k = G$ and $G_1 \cap \ldots \cap G_k$ is an edge in G, where $k \geq 2$. Let $m_i = |E(G_i)|$ and $n_i = |V(G_i)|$. For every $1 \leq i \leq k$, $|E_{in}(G_i)| < |E_{in}(G)|$ as one internal edge of G has become external in G_i. By induction hypothesis, for every $1 \leq i \leq k$, the number of induced cycles in G_i is $m_i - n_i + 1$. Observe that the set of induced cycles in G is equal to the disjoint union of the set of induced cycles in $G_1, \ldots, G_k$. Further, we know that $m = m_1 + \ldots + m_k - k + 1$ and $n = n_1 + \ldots + n_k - 2k + 2$. Consequently, we can see that the number of induced cycles in G is $m - n + 1$. □

Lemma 29. *For an arbitrary 2-connected partial 2-tree G, if the set of induced cycles in G is a cycle basis, then G is a polygonal 2-tree.*

Proof. Assume that G is not a polygonal 2-tree. Then by Lemma 10, there exist two induced cycles C_1 and C_2 in G such that $|E(C_1) \cap E(C_2)| \geq 2$. Let $C_3 = C_1 \oplus C_2$. Since C_1 and C_2 are induced cycles, clearly $C_1 \cap C_2$ is a path and C_3 is a cycle. Let P be the maximal common path in C_1 and C_2. Let P_1 and P_2 be the maximal private paths in C_1 and C_2, respectively.

Consider the case when C_3 is an induced cycle. The set $\{C_1, C_2, C_3\}$ is not a part of a cycle basis. It contradicts that the set of induced cycles in G is a cycle basis.

Consider the other case when C_3 is not an induced cycle. Let $C'_1, \ldots, C'_k$ be the set of induced cycles in $Enc(C_3)$. Since C_1 and C_2 are induced cycles and C_3 is not an induced cycle, there exists a chord e in C_3 such that, one end vertex of e is in P_1 and the other end vertex of e is in P_2. Note that at least one induced cycle in $Enc(C)$ has e, whereas C_1 and C_2 do not have e. It follows that $\{C_1, C_2\}$ is different from $\{C'_1, \ldots, C'_k\}$. We can express C_3 as $C_1 \oplus C_2$ as well as $C'_1 \oplus \ldots \oplus C'_k$. Therefore, $\{C_1, C_2\} \cup \{C'_1, \ldots, C'_k\}$ is not part of a cycle basis. It is a contradiction, because we know that the set of induced cycles in G is a cycle basis. Therefore, our assumption is incorrect and hence G is a polygonal 2-tree. □

Theorem 30. *For a polygonal 2-tree G, the set of induced cycles is a unique minimum cycle basis.*

Proof. Recall that the cardinality of a cycle basis is $m - n + 1$. Therefore, from Lemma 27 and Lemma 28, it follows that the set of induced cycles in G is a cycle basis. Assume that $\mathcal{B}$ is a minimum cycle basis of G such that $\mathcal{B}$ contains at least one non-induced cycle. Let C be a smallest non-induced cycle in $\mathcal{B}$ and $C_1, \ldots, C_k$ be the set of induced cycles in $Enc(C)$. Observe that $C_1 \oplus \ldots \oplus C_k$ is C. Clearly, there exists $1 \leq i \leq k$ such that $C_i \notin \mathcal{B}$ as $\mathcal{B}$ is a cycle basis. We replace C with C_i and obtain a cycle basis such that its size is strictly less than the size of $\mathcal{B}$ as $|E(C)| > |E(C_i)|$. We got a contradiction, because $\mathcal{B}$ is a minimum cycle basis. Therefore, the set of induced cycles in G is a unique minimum cycle basis of G. □

The following theorem follows from Lemma 29 and Theorem 30.

Theorem 31. *A graph G is a polygonal 2-tree if and only if G is a 2-connected partial 2-tree and the set of induced cycles in G is a cycle basis.*

As the set of induced cycles in polygonal 2-trees is a minimum cycle basis, Theorem 14 computes a minimum cycle basis in polygonal 2-trees in linear time.

Concluding remarks. For a polygonal 2-tree on n vertices, we have designed an $O(n \log n)$-time algorithm for the problem, Mast, of finding a minimum average stretch spanning tree. By using this algorithm, we have obtained a minimum fundamental cycle basis $\mathcal{B}$ of a polygonal 2-tree on n vertices in $O(n \log n) + \text{size}(\mathcal{B})$ time. We have also shown that polygonal 2-trees have a unique minimum cycle basis and it can be computed in linear time. The problem of finding a minimum routing cost spanning tree is closely related to Mast. A *minimum routing cost spanning tree* is a spanning tree of a graph that minimizes the sum-total distance between every two vertices in the spanning tree. The complexity of finding a minimum routing cost spanning tree in polygonal 2-trees (also in planar graphs) is open, whereas it is NP-hard in weighted undirected graphs.

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