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On acyclic edge-coloring of the complete bipartite graphs $K_{2p-1,2p-1}$ for odd prime p

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ABSTRACT

An acyclic edge-coloring of a graph is a proper edge-coloring without bichromatic (2-colored) cycles. The acyclic chromatic index of a graph G, denoted by $a'(G)$, is the least integer k such that G admits an acyclic edge-coloring using k colors. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in a graph G. A complete bipartite graph with n vertices on each side is denoted by $K_{n,n}$. Basavaraju, Chandran and Kummini proved that $a'(K_{n,n}) \geq n+2 = \Delta+2$ when n is odd. Basavaraju and Chandran showed that $a'(K_{p,p}) \leq p+2$ which implies $a'(K_{p,p}) = p+2 = \Delta+2$ when p is an odd prime, and the main tool in their proof is perfect 1-factorization of $K_{p,p}$. In this paper we study the case of $K_{2p-1,2p-1}$ which also possess perfect 1-factorization, where p is odd prime. We show that $K_{2p-1,2p-1}$ admits an acyclic edge-coloring using $2p+1$ colors and so we get $a'(K_{2p-1,2p-1}) = 2p+1 = \Delta+2$ when p is an odd prime.

1. Introduction

Let $G = (V, E)$ be a finite and simple graph. A *proper edge-coloring* of G is an assignment of colors to the edges so that no two adjacent edges have same color. So it is a map $\theta : E \to \mathcal{C}$ with $\theta(e) \neq \theta(f)$ for any adjacent edges $e, f \in E$, where $\mathcal{C}$ is the set of colors. The *chromatic index*, denoted by $\chi'(G)$, is the minimum number of colors needed to properly color the edges of G. A proper edge-coloring of G is *acyclic* if there is no two colored cycle in G. The minimum number of colors required in an acyclic edge-coloring of G is the *acyclic edge chromatic number* (also called *acyclic chromatic index*) and is denoted by $a'(G)$. The notion of acyclic coloring was first introduced by Grünbaum [7] in 1973, and the concept of acyclic edge-coloring was first studied by Fiamčík [6]. Let $\Delta = \Delta(G)$ be the maximum degree of a vertex in G. It is obvious that any proper edge-coloring requires at least Δ colors. Vizing [16] proved that there always exists a proper edge-coloring with $\Delta + 1$ colors. Since any acyclic edge coloring is proper, we must have $a'(G) \geq \chi'(G) \geq \Delta$. On the other hand, in 1978, Fiamčík [6] (also Alon, Sudakov and Zaks [1]) posed the following conjecture:

$$\text{for any graph } G, \ a'(G) \leq \Delta + 2. \tag{1}$$

In [1], it was proved that there exists a constant c such that $a'(G) \leq \Delta + 2$ for any graph with girth is at least $c\Delta \log \Delta$. It was also proved in [1] that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. Later Něsetřil and Wormald [15] improved this bound and showed that $a'(G) \leq \Delta + 1$ for a random regular graph G. In another direction, there have been many results

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giving upper bounds on $a'(G)$ for arbitrary graphs or a class of graphs. Recently, Ndreca et al. obtained $a'(G) \leq 9.62\Delta$ [14] which is currently the best upperbound for an arbitrary graph G. See [17, Section 3.3] for a nice account of recent results.

The above conjecture (1) was shown to be true for some special classes of graphs. Burnstein [5] showed that $a'(G) \leq 5$ when $\Delta = 3$. Hence the conjecture is true when $\Delta \leq 3$. Muthu, Narayanan and Subramanian proved that the conjecture holds true for grid-like graphs [11] and outerplanner graphs [12]. It has been observed that determining $a'(G)$ is a hard problem from both theoretical and algorithmic points of view [17, p. 2119]. In fact, we do not yet know the values of $a'(G)$ for some simple and highly structured graphs like complete graphs and complete bipartite graphs in general. Fortunately, we can get the exact value of $a'(G)$ for some cases of complete bipartite graphs, thanks to the perfect 1-factorization.

Let $K_{n,n}$ be the complete bipartite graph with n vertices on each side. The complete bipartite graph $K_{n,n}$ is said to have a *perfect 1-factorization* if the edges of $K_{n,n}$ can be decomposed into n disjoint perfect matchings such that the union of any two perfect matchings gives a Hamiltonian cycle. It is known that when $n+2 \in \{p, 2p-1, p^2\}$, where p is an odd prime, or $n+2 < 50$ and odd, then $K_{n+2,n+2}$ has a perfect 1-factorization (see [4]). One can easily see that if $K_{n+2,n+2}$ has a perfect 1-factorization then $a'(K_{n,n}) \leq a'(K_{n+1,n+1}) \leq n+2$. And also we have

$$a'(K_{n,n}) \geq n+2 = \Delta + 2 \quad \text{when } n \text{ is odd}$$

due to Basavaraju, Chandran and Kummini [3]. Hence $a'(K_{n,n}) = n+2 = \Delta+2$ when $n+2 \in \{p, 2p-1, p^2\}$. The main idea here is to give different colors to the edges in different 1-factors in $K_{n+2,n+2}$, and removal of two vertices on each side and their associated edges gives the required edge-coloring of $K_{n,n}$. Similarly, by a result of Guldan [8, Corollary 1], we can also get $a'(K_{n+1,n+1}) = n+2 = \Delta+1$ when $n+2 \in \{p, 2p-1, p^2\}$. But a different approach is needed to deal with $K_{n+2,n+2}$ when $n+2 \in \{p, 2p-1, p^2\}$. In 2009, Basavaraju and Chandran [2] proved that $a'(K_{p,p}) = p+2 = \Delta+2$ for any odd prime p. The main tool in their approach is again perfect 1-factorization of $K_{p,p}$. In the remaining two cases, namely, $n+2 \in \{2p-1, p^2\}$ the value of $a'(K_{n+2,n+2})$ is not yet known. In this paper we study the case of $K_{2p-1,2p-1}$ which also possesses a perfect 1-factorization, where p is odd prime. We show that $K_{2p-1,2p-1}$ admits an acyclic edge-coloring using $2p+1$ colors.

2. Our result

We state our main result as follows.

Theorem 1. *$a'(K_{2p-1,2p-1}) = 2p+1 = \Delta+2$, where p is an odd prime.*

We follow the proof technique of [2] to present the proof of Theorem 1. Accordingly we first consider a perfect 1-factorization of $K_{2p-1,2p-1}$. Next we consider another perfect matching which satisfies certain conditions. Then we present an edge-coloring of $K_{2p-1,2p-1}$ using $2p+1$ colors and show that it is acyclic. In general, for odd n if $K_{n,n}$ possesses a perfect 1-factorization, the difficulty is to identify a suitable perfect matching that can help to get an acyclic edge-coloring of $K_{n,n}$ using only $n+2$ colors. The main contribution of this paper is to identify such a suitable perfect matching and provide an acyclic edge-coloring of $K_{2p-1,2p-1}$ using $2p+1$ colors, where p is an odd prime.

Proof of Theorem 1. We label the vertices of $K_{2p-1,2p-1}$ on each side with elements of the set $I = \{1, 2, \ldots, 2p-1\} = \mathbb{Z}_{2p} \backslash \{0\}$, and so a perfect matching (1-factor) can be represented by a permutation of the label set I. Let us now present a perfect 1-factorization of $K_{2p-1,2p-1}$ using permutations of the label set I. Let M_j be the perfect matching corresponding to the permutation π_j for $j \in I$ which we define below. In the definitions of π_j below, $k \in I$ $(=\mathbb{Z}_{2p} \backslash \{0\})$ and the operations are understood to be done modulo $2p$ (that is in $\mathbb{Z}_{2p}$).

For $i = 1, 2, \ldots, p-1$, define

$$\pi_{2i}(k) = \begin{cases} 2i & \text{if } k = 2i \\ i+p & \text{if } k = i \\ i & \text{if } k = i+p \\ 2i-k & \text{otherwise.} \end{cases}$$

For $i = 0, 1, 2, \ldots, p-1$ and $i \neq \frac{p-1}{2}$, define

$$\pi_{2i+1}(k) = \begin{cases} 2i+1 & \text{if } k = 2i+1 \\ k-(2i+1) & \text{if } k \neq 2i+1 \text{ and } k \text{ is odd} \\ k+(2i+1) & \text{if } k \text{ is even.} \end{cases}$$

Also

$$\pi_p(k) = 2p - k = -k.$$

A perfect 1-factorization of $K_{2p-1,2p-1}$ is presented in [13, p. 31] applying Laufer's technique [10] on the formulation of perfect 1-factorization of the complete bipartite graph K_{2p} given by Kobayashi [9]. The formulation presented above is a simple modification of the formulation given in [13, p. 31] to suit our representation. So the decomposition of the edges into

$\{M_j : j \in I\}$ forms a perfect 1-factorization of $K_{2p-1,2p-1}$. That is $M_{j_1} \cup M_{j_2}$ forms a Hamiltonian cycle in $K_{2p-1,2p-1}$ for any $j_1, j_2 \in I$ with $j_1 \neq j_2$.

We now consider another perfect matching M satisfying the following condition: for $j \in I$ there is exactly one edge common to M and M_j. The perfect matching M that we consider is described below.

The multiplicative group $\mathbb{Z}_{2p}^*$ can be represented by the set $\{1, 3, 5, \ldots, p-2, p+2, p+4, \ldots, 2p-1\}$. Let x be a generator of $\mathbb{Z}_{2p}^*$ and let y be its inverse in $\mathbb{Z}_{2p}^*$. Note that $|\mathbb{Z}_{2p}^*| = order(x) = order(y) = p-1$. Let M be the perfect matching corresponding to the permutation π given by

$$\pi(k) = \begin{cases} k+p & \text{if } k \text{ is even} \\ p & \text{if } k = p \\ ky+p & \text{if } k \in \mathbb{Z}_{2p}^*. \end{cases}$$

Claim 1. *We have $|M \cap M_j| = 1$ for $j \in I$.*

Proof. We need to show that for each $j \in I$, there is exactly one $k \in I$ such that $\pi(k) = \pi_j(k)$. By a careful analysis of the different cases we can check that it is indeed true and the value of k corresponding to each $j \in I$ is as given below:

$$\begin{array}{rclcll} \pi(p) & = & p & = & \pi_p(p) & \\ \pi(i) & = & i+p & = & \pi_{2i}(i) & \text{for } i \in \{2, 4, \ldots, p-1\} \\ \pi(i+p) & = & i & = & \pi_{2i}(i+p) & \text{for } i \in \{1, 3, \ldots, p-2\} \\ \pi\left(\dfrac{(2i+1)x}{x-1+p}\right) & = & \dfrac{(2i+1)}{x-1+p}+p & = & \pi_{2i+1}\left(\dfrac{(2i+1)x}{x-1+p}\right) & \text{for } i \in \{0, 1, \ldots, p-1\} \setminus \left\{\dfrac{p-1}{2}\right\}. \quad \square \end{array}$$

An edge-coloring of $K_{2p-1,2p-1}$ using $2p+1$ colors:

Let $M' = M \setminus \{(p, p)\}$ and color the edges of $K_{2p-1,2p-1}$ as follows to get a coloring θ using $2p+1$ colors:

- the edges in $M_j^* = M_j \setminus M'$ are colored with c_j for $j \in I$;
- the edges in $M^* = M' \setminus \{(1, y+p)\}$ are colored with c_{2p};
- the edge $(1, y+p)$ is colored with c_{2p+1}.

Claim 2. *The edge-coloring θ is acyclic.*

Proof. Obviously θ is a proper edge-coloring. Note that $M_{j_1} \cup M_{j_2}$ forms a Hamiltonian cycle for $j_1, j_2 \in I$ with $j_1 \neq j_2$ as $\{M_j : j \in I\}$ is a perfect 1-factorization of $K_{2p-1,2p-1}$. One can easily see that the union $M_{j_1}^* \cup M_{j_2}^*$ of the color classes c_{j_1} and c_{j_2} is a 'proper' subset of $M_{j_1} \cup M_{j_2}$. Therefore there cannot be a cycle involving the edges from the color classes c_{j_1} and c_{j_2} for $j_1, j_2 \in I$ with $j_1 \neq j_2$. Note also that there cannot be any bichromatic cycle involving the color c_{2p+1} since there is only one edge colored with c_{2p+1}. So the remaining part is to prove that for $j \in I$ there is no cycle in the induced subgraph of the union $M^* \cup M_j^*$ of the color classes c_{2p} and c_j. For this purpose we now analyze the cycles in the induced subgraph of

$$M \cup M_j = \begin{cases} M^* \cup M_p^* \cup \{(1, y+p)\} & \text{if } j = p \\ M^* \cup M_j^* \cup \{(1, y+p), (p, p)\} & \text{otherwise} \end{cases}$$

for $j \in I$. Observe that $|M \cup M_j| = 4p-3$. In order to the prove the remaining part we show that:

- there is exactly one cycle C_j of length $4(p-1)$ in the induced subgraph of $M \cup M_j$, and the other edge which is not in the cycle is the edge in $M \cap M_j$;
- the edge (p, p) is in the cycle C_j of $M \cup M_j$ for $j \in I$ with $j \neq p$, and in the case where $j = p$ the edge $(1, y+p)$ is in the cycle C_p of $M \cup M_p$.

First note that the union $M \cup M_j$ (of two perfect matchings) forms a collection of disjoint cycles, and the cycles of $M \cup M_j$ can be seen from the permutation $\pi^{-1} \circ \pi_j$ for $j \in I$. The inverse permutation of π is given by

$$\pi^{-1}(k) = \begin{cases} k+p & \text{if } k \in \mathbb{Z}_{2p}^* \\ p & \text{if } k = p \\ kx+p & \text{if } k \text{ is even.} \end{cases}$$

We now present some useful identities and then present the cycle structure of the permutations $\pi^{-1} \circ \pi_j$ from which we can see the cycles of $M \cup M_j$, by dividing j's into four groups.

Since x is a generator of $\mathbb{Z}_{2p}^*$ we get that $(p-2)$ is the least positive integer such that

$$1 + x + \cdots + x^{p-2} \equiv 0 \pmod{2p}.$$

Since x is a generator of $\mathbb{Z}_{2p}^*$, there exists some $t \in \{1, 2, \ldots, p-2\}$ such that

$$x^t \equiv 2 + p \pmod{2p}.$$

So we get

$$2x^{p-1-t} \equiv 1 + p \pmod{2p}.$$

Then we have

$$x^t \equiv 2 \pmod{p} \quad \text{and} \quad 2x^{p-1-t} \equiv 1 \pmod{p}.$$

We will use the above identities in the discussion below whenever needed. To present the cycle structure of $\pi^{-1} \circ \pi_j$, a mapping $\pi^{-1} \circ \pi_j(v_s) = v_{s+1}$ is described with a line

$$v_s \overset{\pi_j}{\mapsto} u_s \overset{\pi^{-1}}{\longmapsto} v_{s+1}.$$

This corresponds to one edge $(v_s, u_s) \in M_j$ and another edge $(v_{s+1}, u_s) \in M$. The next line starts with v_{s+1} and it is given by

$$v_{s+1} \overset{\pi_j}{\mapsto} u_{s+1} \overset{\pi^{-1}}{\longmapsto} v_{s+2}.$$

So a cycle $(v_0\, v_1\, v_2\, \ldots, v_{\ell-1})$ of length ℓ in $\pi^{-1} \circ \pi_j$ corresponds to a cycle of length 2ℓ in the graph $K_{2p-1,2p-1}$. Note that the operations on the elements of I are done modulo $2p$ in the definition of π_j's and π.

Case 1: $j = p$.

In this case $M \cap M_p = \{(p, p)\}$ and so we get $\pi^{-1} \circ \pi_j(p) = p$, that is p is the only fixed element in $\pi^{-1} \circ \pi_p$. Let us look at the cycle containing 1 in $\pi^{-1} \circ \pi_p$.

$$\begin{array}{rcccl}
1 & \overset{\pi_p}{\mapsto} & -1 & \overset{\pi^{-1}}{\longmapsto} & -1+p \\
-1+p & \overset{\pi_p}{\mapsto} & 1+p & \overset{\pi^{-1}}{\longmapsto} & (1+p)x + p \equiv x \pmod{2p} \\
x & \overset{\pi_p}{\mapsto} & -x & \overset{\pi^{-1}}{\longmapsto} & -x+p \\
-x+p & \overset{\pi_p}{\mapsto} & x+p & \overset{\pi^{-1}}{\longmapsto} & (x+p)x + p \equiv x^2 \pmod{2p} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x^{p-2} & \overset{\pi_p}{\mapsto} & -x^{p-2} & \overset{\pi^{-1}}{\longmapsto} & -x^{p-2}+p \\
-x^{p-2}+p & \overset{\pi_p}{\mapsto} & x^{p-2}+p = y+p & \overset{\pi^{-1}}{\longmapsto} & x^{p-1} \equiv 1 \pmod{2p}.
\end{array}$$

Observe that $\{x^r : 0 \le r \le p-2\} = \mathbb{Z}_{2p}^*$ as x is a generator of $\mathbb{Z}_{2p}^*$, and so $\{x^r, -x^r + p : 0 \le r \le p-2\} = I \backslash \{p\}$. Therefore the cycle is of length $2(p-1)$, and so the corresponding cycle C_p in the graph $K_{2p-1,2p-1}$ is of length $4(p-1)$. Note also that the edge $(1, y+p) \in M$ is in the cycle C_p.

Case 2: $j = 2i$ for $i \in \{2, 4, \ldots, p-1\}$.

In this case $M \cap M_{2i} = \{(i, i+p)\}$ and so we get $\pi^{-1} \circ \pi_{2i}(i) = i$, that is i is the only fixed element in $\pi^{-1} \circ \pi_j$. Let us look at the cycle containing p in $\pi^{-1} \circ \pi_j$.

$$\begin{array}{rcccl}
p & \overset{\pi_{2i}}{\mapsto} & 2i-p & \overset{\pi^{-1}}{\longmapsto} & 2i \\
2i & \overset{\pi_{2i}}{\mapsto} & 2i & \overset{\pi^{-1}}{\longmapsto} & 2ix + p \equiv ix^{t+1} + p \pmod{2p} \\
ix^{t+1}+p & \overset{\pi_{2i}}{\mapsto} & 2i - ix^{t+1} - p & \overset{\pi^{-1}}{\longmapsto} & 2i - ix^{t+1} \\
2i - ix^{t+1} & \overset{\pi_{2i}}{\mapsto} & ix^{t+1} & \overset{\pi^{-1}}{\longmapsto} & ix^{t+2} + p \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
ix^{p-2}+p & \overset{\pi_{2i}}{\mapsto} & 2i - ix^{p-2} + p & \overset{\pi^{-1}}{\longmapsto} & 2i - ix^{p-2} \\
2i - ix^{p-2} & \overset{\pi_{2i}}{\mapsto} & ix^{p-2} & \overset{\pi^{-1}}{\longmapsto} & ix^{p-1} + p \equiv i + p \pmod{2p} \\
i+p & \overset{\pi_{2i}}{\mapsto} & i & \overset{\pi^{-1}}{\longmapsto} & ix + p \\
ix+p & \overset{\pi_{2i}}{\mapsto} & 2i - ix - p & \overset{\pi^{-1}}{\longmapsto} & 2i - ix \\
2i - ix & \overset{\pi_{2i}}{\mapsto} & ix & \overset{\pi^{-1}}{\longmapsto} & ix^2 + p \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
ix^{t-1}+p & \overset{\pi_{2i}}{\mapsto} & 2i - ix^{t-1} + p & \overset{\pi^{-1}}{\longmapsto} & 2i - ix^{t-1} \\
2i - ix^{t-1} & \overset{\pi_{2i}}{\mapsto} & ix^{t-1} & \overset{\pi^{-1}}{\longmapsto} & ix^t + p \\
ix^t + p \equiv 2i + p \pmod{2p} & \overset{\pi_{2i}}{\mapsto} & p & \overset{\pi^{-1}}{\longmapsto} & p.
\end{array}$$

Observe that $S_1 = \{ix^r + p \pmod{2p} : 0 \le r \le p-2\} = \mathbb{Z}_{2p}^*$ as i is even. One can check that $S_2 = \{2i - ix^r \pmod{2p} : 1 \le r \le p-2 \text{ and } r \ne t\}$ are distinct even numbers and also $|S_1 \cup S_2 \cup \{2i, p\}| = 2(p-1)$. Therefore the cycle is of length $2(p-1)$, and so the corresponding cycle C_{2i} in the graph $K_{2p-1,2p-1}$ is of length $4(p-1)$. Note also that the edge $(p, p) \in M$ is in the cycle C_{2i}.

Case 3: $j = 2i$ for $i \in \{1, 3, \ldots, p-3\}$.

In this case $M \cap M_{2i} = \{(i+p, i)\}$ and so we get $\pi^{-1} \circ \pi_j(i+p) = i+p$, that is $i+p$ is the only fixed element in $\pi^{-1} \circ \pi_j$. Let us look at the cycle containing p in $\pi^{-1} \circ \pi_j$.

$$\begin{array}{rcccl}
p & \overset{\pi_{2i}}{\longmapsto} & 2i-p & \overset{\pi^{-1}}{\longmapsto} & 2i \\
2i & \overset{\pi_{2i}}{\longmapsto} & 2i & \overset{\pi^{-1}}{\longmapsto} & 2ix+p \equiv ix^{t+1} \pmod{2p} \\
ix^{t+1} & \overset{\pi_{2i}}{\longmapsto} & 2i-ix^{t+1} & \overset{\pi^{-1}}{\longmapsto} & 2i-ix^{t+1}+p \\
2i-ix^{t+1}+p & \overset{\pi_{2i}}{\longmapsto} & ix^{t+1}+p & \overset{\pi^{-1}}{\longmapsto} & ix^{t+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
ix^{p-2} & \overset{\pi_{2i}}{\longmapsto} & 2i-ix^{p-2} & \overset{\pi^{-1}}{\longmapsto} & 2i-ix^{p-2}+p \\
2i-ix^{p-2}+p & \overset{\pi_{2i}}{\longmapsto} & ix^{p-2}+p & \overset{\pi^{-1}}{\longmapsto} & ix^{p-1} \equiv i \pmod{2p} \\
i & \overset{\pi_{2i}}{\longmapsto} & i+p & \overset{\pi^{-1}}{\longmapsto} & (i+p)x+p \equiv ix \pmod{2p} \\
ix & \overset{\pi_{2i}}{\longmapsto} & 2i-ix & \overset{\pi^{-1}}{\longmapsto} & 2i-ix+p \\
2i-ix+p & \overset{\pi_{2i}}{\longmapsto} & ix+p & \overset{\pi^{-1}}{\longmapsto} & ix^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
ix^{t-1} & \overset{\pi_{2i}}{\longmapsto} & 2i-ix^{t-1} & \overset{\pi^{-1}}{\longmapsto} & 2i-ix^{t-1}+p \\
2i-ix^{t-1}+p & \overset{\pi_{2i}}{\longmapsto} & ix^{t-1}+p & \overset{\pi^{-1}}{\longmapsto} & ix^t \\
ix^t \equiv 2i+p \pmod{2p} & \overset{\pi_{2i}}{\longmapsto} & p & \overset{\pi^{-1}}{\longmapsto} & p.
\end{array}$$

Observe that $S_1 = \{ix^r : 0 \le r \le p-2\} = \mathbb{Z}_{2p}^*$ as i is odd. One can check that $S_2 = \{2i - ix^r + p : 1 \le r \le p-2 \text{ and } r \ne t\}$ are distinct even numbers and also $|S_1 \cup S_2 \cup \{2i, p\}| = 2(p-1)$. Therefore the cycle is of length $2(p-1)$, and so the corresponding cycle C_{2i} in the graph $K_{2p-1,2p-1}$ is of length $4(p-1)$. Note also that the edge $(p, p) \in M$ is in the cycle C_{2i}.

Case 4: $j = 2i+1$ for $i \in \{0, 1, \ldots, p-1\}$ with $i \ne \frac{p-1}{2}$.

In this case $M \cap M_{2i+1} = \{(\frac{jx}{x-1+p}, \frac{j}{x-1+p} + p)\}$ and so we get $\pi^{-1} \circ \pi_j(\frac{jx}{x-1+p}) = \frac{jx}{x-1+p}$, that is $\frac{jx}{x-1+p}$ is the only fixed element in $\pi^{-1} \circ \pi_j$. Let us now look at the cycle containing p in $\pi^{-1} \circ \pi_j$.

$$\begin{array}{rcccl}
p & \overset{\pi_j}{\mapsto} & p-j & \overset{\pi^{-1}}{\longmapsto} & (p-j)x+p \equiv -jx \pmod{2p} \\
-jx & \overset{\pi_j}{\mapsto} & -jx-j & \overset{\pi^{-1}}{\longmapsto} & (-jx-j)x+p = -j(x^2+x)+p \\
-j(x^2+x)+p & \overset{\pi_j}{\mapsto} & -j(x^2+x+1)+p & \overset{\pi^{-1}}{\longmapsto} & -j(x^3+x^2+x) \\
-j(x^3+x^2+x) & \overset{\pi_j}{\mapsto} & -j(x^3+x^2+x+1) & \overset{\pi^{-1}}{\longmapsto} & -j(x^4+x^3+x^2+x)+p \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-j(x^{p-4}+\cdots+x) & \overset{\pi_j}{\mapsto} & -j(x^{p-4}+\cdots+1) & \overset{\pi^{-1}}{\longmapsto} & -j(x^{p-3}+\cdots+x)+p \\
-j(x^{p-3}+\cdots+x)+p & \overset{\pi_j}{\mapsto} & -j(x^{p-3}+\cdots+1)+p & \overset{\pi^{-1}}{\longmapsto} & -j(x^{p-2}+\cdots+x) \equiv j \pmod{2p} \\
j & \overset{\pi_j}{\mapsto} & j & \overset{\pi^{-1}}{\longmapsto} & j+p \\
j+p & \overset{\pi_j}{\mapsto} & 2j+p & \overset{\pi^{-1}}{\longmapsto} & 2j \\
2j & \overset{\pi_j}{\mapsto} & 3j & \overset{\pi^{-1}}{\longmapsto} & 3j+p \\
3j+p & \overset{\pi_j}{\mapsto} & 4j+p & \overset{\pi^{-1}}{\longmapsto} & 4j \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(p-2)j+p & \overset{\pi_j}{\mapsto} & (p-1)j+p & \overset{\pi^{-1}}{\longmapsto} & (p-1)j \\
(p-1)j & \overset{\pi_j}{\mapsto} & pj \equiv p \pmod{2p} & \overset{\pi^{-1}}{\longmapsto} & p.
\end{array}$$

Observe that the elements in $S_1 = \{j + p, 2j, 3j + p, 4j, \ldots, (p-2)j + p\} = \{r(j+p) \bmod 2p : 1 \le r \le p-1\}$ are the distinct even numbers in I. Also the other elements $S_2 = \{-jx, -j(x^2+x) + p, -j(x^3+x^2+x), \ldots, -j(x^{p-3} + \cdots + x) + p, -j(x^{p-2} + \cdots + x) = j, p\}$ are distinct (all are odd numbers). The missing element in this list is $\frac{jx}{x-1+p}$ which is the fixed element in $\pi^{-1} \circ \pi_j$. Therefore the cycle is of length $2(p-1)$, and so the corresponding cycle C_{2i+1} in the graph $K_{2p-1,2p-1}$ is of length $4(p-1)$. Note also that the edge $(p, p) \in M$ is in the cycle C_{2i+1}. Hence the proof. □

Remark 1. For an odd prime p, if G is a graph obtained by removing just one edge from $K_{2p-1,2p-1}$ then $a'(G) = 2p = \Delta + 1$. This is also true even if one deletes any number of edges between 1 and $2p-3$ from $K_{2p-1,2p-1}$. The proof is similar to the proof of [2, Theorem 2].

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