On Acyclic Edge-Coloring of Complete Bipartite Graphs

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Abstract

An acyclic edge-coloring of a graph is a proper edge-coloring without bichromatic (2-colored) cycles. The acyclic chromatic index of a graph G, denoted by $a'(G)$, is the least integer k such that G admits an acyclic edge-coloring using k colors. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in a graph G. A complete bipartite graph with n vertices on each side is denoted by $K_{n,n}$. Basavaraju, Chandran and Kummini proved that $a'(K_{n,n}) \geq n + 2 = \Delta + 2$ when n is odd. Basavaraju and Chandran provided an acyclic edge-coloring of $K_{p,p}$ using $p + 2$ colors and thus establishing $a'(K_{p,p}) = p + 2 = \Delta + 2$ when p is an odd prime. The main tool in their approach is perfect 1-factorization of $K_{p,p}$. Recently, following their approach, Venkateswarlu and Sarkar have shown that $K_{2p-1,2p-1}$ admits an acyclic edge-coloring using $2p+1$ colors which implies that $a'(K_{2p-1,2p-1}) = 2p+1 = \Delta+2$, where p is an odd prime. In this paper, we generalize this approach and present a general framework to possibly get an acyclic edge-coloring of $K_{n,n}$ which possess a perfect 1-factorization using $n + 2 = \Delta + 2$ colors. In this general framework, we show that K_{p^2,p^2} admits an acyclic edge-coloring using $p^2 + 2$ colors and thus establishing $a'(K_{p^2,p^2}) = p^2 + 2 = \Delta + 2$ when $p \geq 5$ is an odd prime.

Keywords: Acyclic edge-coloring, Acyclic chromatic index, Perfect 1-factorization, Complete bipartite graphs

1 Introduction

Let $G = (V, E)$ be a finite and simple graph. A *proper edge-coloring* of G is an assignment of colors to the edges so that no two adjacent edges have same color. So it is a map $\theta : E \to \mathcal{C}$ with $\theta(e) \neq \theta(f)$ for any adjacent edges $e, f \in E$, where $\mathcal{C}$ is the set of colors. The *chromatic index*, denoted by $\chi'(G)$, is the least integer k such that G admits a proper edge-coloring using k colors. A proper coloring of G is *acyclic* if there is no two colored cycle in G. The *acyclic edge chromatic number* (also called *acyclic chromatic index*), denoted by $a'(G)$, is the least integer k such that G admits an acyclic edge-coloring using k colors. The notion of acyclic coloring was first introduced by Grünbaum [11] in 1973, and the concept of acyclic edge-coloring was first studied by Fiamčík [10]. Let $\Delta = \Delta(G)$ be the maximum degree of a vertex in G. It is obvious that any proper edge-coloring requires at least Δ colors. Vizing [22] proved that there always exists a proper edge-coloring with $\Delta + 1$ colors. Since any acyclic edge coloring is proper, we must have $a'(G) \geq \chi'(G) \geq \Delta$. On the other hand, in 1978, Fiamčík [10] (also independently by Alon, Sudakov and Zaks [2]) posed the following conjecture.

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Conjecture 1.1 *for any graph G, $a'(G) \leq \Delta + 2$.*

In [2] it was proved that there exists a constant c such that $a'(G) \leq \Delta + 2$ for any graph with girth is at least $c\Delta \log \Delta$. It was also proved in [2] that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. Later Něsetřil and Wormald [18] improved this bound and showed that $a'(G) \leq \Delta + 1$ for a random regular graph G. In another direction, there have been many results giving upper bounds on $a'(G)$ for an arbitrary graph G. For example, Alon, McDiarmid and Reed [1] proved that $a'(G) \leq 64\Delta$. Molloy and Reed [13] improved this bound and showed that $a'(G) \leq 16\Delta$. Recently, Ndreca *et. al.* obtained $a'(G) \leq 9.62\Delta$ [19] which is currently the best upper bound for an arbitrary graph G. Muthu, Narayanan and Subramanian [14, 17] obtained better bounds: $a'(G) \leq 4.52\Delta$ for graphs with girth at least 220; $a'(G) \leq 6\Delta$ for graphs with girth at least 9. The acyclic edge-coloring of planar graphs has been deeply studied in recent years. See [23, Section 3.3] for a nice account of recent results.

The Conjecture 1.1 was shown to be true for some special classes of graphs. Burnstein [8] showed that $a'(G) \leq 5$ when $\Delta = 3$. Hence the conjecture is true when $\Delta \leq 3$. Muthu, Narayanan and Subramanian proved that the conjecture holds true for grid-like graphs [15] and outerplanner graphs [16]. It has been observed that determining $a'(G)$ is a hard problem from both theoretical and algorithmic points of view [23, p. 2119]. In fact, we do not yet know the values of $a'(G)$ for some simple and highly structured graphs like complete graphs and complete bipartite graphs in general. Fortunately, we can get the exact value of $a'(G)$ for some cases of complete bipartite graphs, thanks to the perfect 1-factorization.

Let $K_{n,n}$ be the complete bipartite graph with n vertices on each side. The complete bipartite graph $K_{n,n}$ is said to have a perfect 1-factorization if the edges of $K_{n,n}$ can be decomposed into n disjoint perfect matchings such that the union of any two perfect matchings gives a Hamiltonian cycle and it is of length $2n$ (see Section 2 for more details). It is known that when $n + 2 \in \{p, 2p - 1, p^2\}$, where p is an odd prime, or $n + 2 < 50$ and odd, then $K_{n+2,n+2}$ has a perfect 1-factorization (see [7]). One can easily see that if $K_{n+2,n+2}$ has a perfect 1-factorization then $a'(K_{n,n}) \leq a'(K_{n+1,n+1}) \leq n+2$. And also we have the following result due to Basavaraju, Chandran and Kummini [4].

Theorem 1.1 $a'(K_{n,n}) \geq n + 2 = \Delta + 2$, *when n is odd.*

Hence $a'(K_{n,n}) = n + 2 = \Delta + 2$ when $n + 2 \in \{p, 2p - 1, p^2\}$. By a result of Guldan [12, Corollary 1], we can also get $a'(K_{n+1,n+1}) = n + 2 = \Delta + 1$ when $n + 2 \in \{p, 2p - 1, p^2\}$.

The main idea here is to give different colors to the edges in different 1-factors in $K_{n+2,n+2}$, and removal of (one) two vertices on each side and their associated edges gives the required edge-coloring of $(K_{n+1,n+1})$ $K_{n,n}$ with $n + 2$ colors. But a different approach is needed to deal with $K_{n+2,n+2}$ when $n + 2 \in \{p, 2p - 1, p^2\}$. In 2009, Basavaraju and Chandran [5] proved that $a'(K_{p,p}) = p + 2 = \Delta + 2$ for any odd prime p. We can view their approach as follows: suitably pick one edge from each 1-factor and partition these edges into two groups and each group can possibly be assigned a different color to get the required result. Following this approach, Venkateswarlu and Sarkar have recently shown that $a'(K_{2p-1,2p-1}) = 2p+1 = \Delta+2$ for any odd prime p [21]. In this paper we view this approach in a more general setting and propose a general framework for the proof. The only remaining infinite class of complete bipartite graphs that are known to have perfect 1-factorization is K_{p^2,p^2}, where p is odd prime. In this general framework we provide an acyclic edge-coloring of K_{p^2,p^2} using $p^2 + 2$ colors when $p \geq 5$. Therefore we state our main result as follows.

Theorem 1.2 $a'(K_{p^2,p^2}) = p^2 + 2 = \Delta + 2$, *where $p \geq 5$ is an odd prime.*

Therefore the acyclic chromatic index is equal to $\Delta + 2$ for all the three known infinite classes of complete bipartite graphs having a perfect 1-factorization, and the Conjecture 1.1 holds true for such graphs.

In the next section we discuss some preliminaries and in Section 3 we present a general framework to possibly get an acyclic edge-coloring of $K_{n,n}$ which possess a perfect 1-factorization using $n + 2$ colors. Then we present the proof of Theorem 1.2 in this framework in Section 4.

2 Preliminaries

Let $n \ (\geq 2)$ be an integer. We treat elements of the ring $\mathbb{Z}_n$ as integers in the range $\{0, 1, \ldots, n - 1\}$. We denote the complete bipartite graph $K_{n,n}$ as $G = (V \cup V', E)$ with $|V| = |V'| = n$ and $E = \{(v \mapsto v') : v \in V \text{ and } v' \in V'\}$. We use $\mapsto$ to define edges though our graph $K_{n,n}$ is undirected. This is only for ease of presentation in associating a perfect matching in $K_{n,n}$ with a permutation of the label set $I (= \{0, 1, \ldots, n - 1\})$, which we discuss below. Accordingly, the use of arrows in the Figures 1 and 2 below is to explicitly emphasize the correspondence between a perfect matching and its associated permutation map. We use the terms 'composition' and 'product' of permutations interchangeably. Note also that a permutation can be decomposed as a product of disjoint cycles uniquely (upto reorder of the cycles and cyclic rotation of the elements within a cycle) and it is called disjoint cycle decomposition. We use $\sqcup$ (instead of the usual union notation $\cup$) to signify union of 'disjoint' sets.

2.1 Perfect matching and Perfect 1-factorization

A *matching* in a graph is a set of edges without common vertices, and a *perfect matching* is a matching which matches all vertices of the graph. In the case of complete bipartite graph $K_{n,n}$, a perfect matching $M \subset E$ is a set of n edges satisfying:

- for each vertex $v' \in V'$ there exists a vertex $v \in V$ such that $(v \mapsto v') \in M$.

- if $(v_1 \mapsto v')$ and $(v_2 \mapsto v')$ are in M then $v_1 = v_2$.

So by labelling the vertices in both V and V' with elements of $I = \{0, 1, \ldots, n - 1\}$ (or an appropriate label set I of size n), we can interpret a perfect matching M in $K_{n,n}$ as a permutation of the label set I, say π_M. For convenience, let us illustrate this through an example. Let $n = 5$ and consider the graph $K_{5,5}$ with the same labels from 0 to 4 for the vertices on the top (V) and the bottom (V'), as depicted in the figure below. Let $M = \{(0 \mapsto 1), (1 \mapsto 2), (2 \mapsto 0), (3 \mapsto 4), (4 \mapsto 3)\}$ then $\pi_M = (012)(34)$.
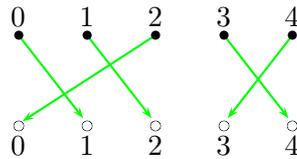
Figure 1: $\pi_M = (012)(34)$

Note that the union of any two perfect matchings of $K_{n,n}$ forms a collection of disjoint cycles. These cycles can also be seen from the disjoint cycle decomposition of the composition of their corresponding permutations. Let M be as mentioned above and $M' = \{(0 \mapsto 1), (1 \mapsto 0), (2 \mapsto 2), (3 \mapsto 3), (4 \mapsto 4)\}$. Then $\pi_{M'} = (01)(2)(3)(4)$.
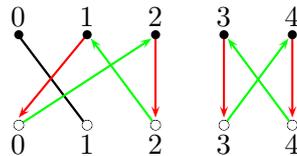
Figure 2: Induced subgraph of $M \cup M'$ (in $K_{5,5}$)

We have $\pi_M^{-1} \circ \pi_{M'} = (0)(12)(34)$ and the arrows are placed accordingly in the figure above. The fixed element (0) corresponds to the common edge $(0 \mapsto 1) \in M \cap M'$ represented by the normal line in Figure 2, and the two cycles $C_1 = (12)$ and $C_2 = (34)$ correspond to the cycles $C_1^g = \{(1 \mapsto 0), (2 \mapsto 0), (2 \mapsto 2), (1 \mapsto 2)\}$ and $C_2^g = \{(3 \mapsto 3), (4 \mapsto 3), (4 \mapsto 4), (3 \mapsto 4)\}$ respectively in $K_{5,5}$. By a careful observation of the example, we can see the following general result.

Lemma 2.1 *Let M and M' be two perfect matchings of $K_{n,n}$ and let $C = (i_0\, i_1 \cdots i_{\ell-1})$, $\ell \geq 2$, be a cycle of length ℓ in the disjoint cycle decomposition of $\pi_M^{-1} \circ \pi_{M'}$, i.e., $\pi_{M'}(i_j) = \pi_M(i_{j+1})$, where the subscripts are taken modulo ℓ. Then the corresponding cycle C^g in $K_{n,n}$ is of length 2ℓ and the participating edges are given by $\{(i_j \mapsto \pi_M(i_j)) : 0 \leq j \leq \ell - 1\} \subset M$ and $\{(i_j \mapsto \pi_{M'}(i_j)) : 0 \leq j \leq \ell - 1\} \subset M'$ which appear alternatively in C^g as depicted in Figure 3: the edges of M' and M are represented by lines (without arrows) with colors red and green respectively.*
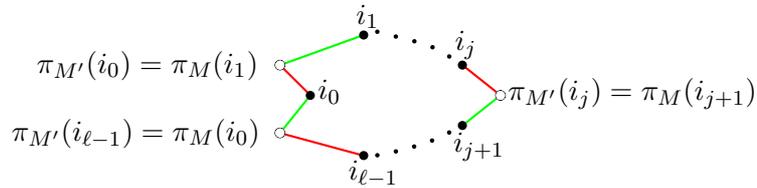
Figure 3: Cycle C^g in the induced subgraph of $M \cup M'$ (in $K_{n,n}$)

A perfect matching is also called a 1-*factor*, and a partitioning of the edges of a graph into 1-factors is a 1-*factorization*. A 1-factorization is *perfect* if the union of any two of its 1-factors (perfect matchings) is a Hamiltonian cycle. As pointed out in the introduction, there are three infinite classes of complete bipartite graphs known to have perfect 1-factorization, namely, $n \in \{p, 2p-1, p^2\}$ with p an odd prime. Let us illustrate it by considering the complete bipartite graph $K_{5,5}$. As discussed above the 1-factors can be described by permutations of $I = \{0, 1, 2, 3, 4\}$. Consider the 1-factors M_i given by the permutations $\pi_i(j) = (i + j) \bmod 5$ for $i, j \in \mathbb{Z}_5$. In Figure 4 the edges of M_1 are the green colored lines and the edges of M_2 are the red colored lines and they correspond to the permutations $\pi_1 = (01234)$ and $\pi_2 = (02413)$ respectively. The induced subgraph formed by the edges $M_1 \sqcup M_2$ is depicted below (without arrows), and the corresponding permutation is equal to $\sigma = \pi_1^{-1} \circ \pi_2 = (01234)$.
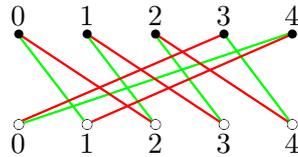
Figure 4: Induced subgraph of $M_1 \sqcup M_2$ (in $K_{5,5}$)

Now we can see from the above lemma that $M_0, M_1, \ldots, M_{n-1}$ is a perfect 1-factorization of the complete bipartite graph $K_{n,n}$ if and only if the permutation $(\pi_i^{-1} \circ \pi_j)$ is a full cycle (of length n) for any $0 \leq i, j \leq (n-1)$ with $i \neq j$, where π_i is the permutation corresponding to M_i. Next we present a general framework to possibly get an acyclic edge-coloring of $K_{n,n}$ which possess a perfect 1-factorization using $n + 2$ colors, where n is odd.

3 A general framework

Let $n\, (\geq 3)$ be an odd integer. Suppose that the complete bipartite graph $K_{n,n}$ has a perfect 1-factorization. Then there exist n disjoint perfect matchings covering all the n^2 edges of $K_{n,n}$, say $M_0, \ldots, M_{n-1}$, such that the union of any two perfect matchings $M_i \sqcup M_j$ gives a Hamiltonian

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cycle (which is of length $2n$). Let $M_i^* \subsetneq M_i$ be a proper subset of M_i and consider the following partial coloring:

$$\text{assign color } c_i \text{ to the edges in } M_i^*. \tag{1}$$

The remaining edges to be colored is given by

$$M = \bigsqcup_{i=0}^{n-1} (M_i \setminus M_i^*),$$

and these edges will be assigned some other colors different from the colors in $\{c_0, \ldots, c_{n-1}\}$.

Lemma 3.1 *There can not be a cycle in the induced subgraph formed by the edges from the union $M_i^* \sqcup M_j^*$ of two color classes c_i and c_j for any $0 \le i, j \le (n-1)$ with $i \ne j$.*

Ideally, to get an acyclic edge-coloring of $K_{n,n}$, our aim should be to use only two more colors for coloring the (uncolored) edges in M, and thus attaining the lower bound on $a'(K_{n,n})$. In other words, partition M into $M^{(1)}$ and $M^{(2)}$, if possible, in such a way that the induced subgraph of $M^{(\rho)} \sqcup M_i^*$ does not contain a cycle for any $0 \le i \le n-1$ and $\rho \in \{1, 2\}$. If such a partition of M exists, then the edges of $M^{(1)}$ and $M^{(2)}$ can be assigned distinct colors, and one can easily see that the proposed edge-coloring is proper and acyclic. These observations and our intuition suggest that to minimize the number of edges (size of M) that are yet to be assigned colors. This can be done by taking M_i^* to be a (proper) maximal subset of M_i, i.e., take $M_i^* = M_i \setminus \{e_i\}$ for some $e_i \in M_i$. Thus the size of the set M will be (near)[1] minimal. Additionally, we have to make sure that a suitable partition of M exists satisfying the other requirements mentioned above. Below we present a strategy to choose M (to be a perfect matching), so that the other requirements on M can possibly be worked out using permutations of the label set I.

As discussed in the previous section, we can interpret each perfect matching M_i in $K_{n,n}$ as a permutation of the label set I and let us denote it by π_i. Now, if possible, select one edge $e_i : (s_i \mapsto t_i)$ from each M_i, $0 \le i \le (n-1)$, such that the set of edges $M = \{e_0, e_1, \ldots, e_{n-1}\}$ gives a perfect matching in $K_{n,n}$. Let π be its corresponding permutation of I. Then we have $M \cap M_i = \{e_i\}$ for $1 \le i \le (n-1)$.

Remark 3.1 *A perfect 1-factorization of complete bipartite graph $K_{n,n}$ is equivalent to a Hamiltonian latin square of order n. Our choice of perfect matching M with $M \cap M_i = \{e_i\}$ for $1 \le i \le (n-1)$ is equivalent to a transversal (of length n). These concepts are well studied in literature (see [9, p. 135–151]). It was conjectured by Ryser that every latin square of odd order has a transversal [6]. Such a transversal is also said to be rainbow matching (see [3] for details). In fact, it is enough to take a transversal of length $n-1$ as pointed out in the footnote below. It was conjectured by Brualdi and Stein independently that every latin square of odd order n has a transversal of length $n-1$ [6, 20]. As far as we know it is not known that in general a Hamiltonian latin square of order n has a transversal (of length n or $n-1$). In fact, we need a transversal which satisfies an additional property as discussed below.*

Now set $M_i^* = M_i \setminus \{e_i\}$ and assign colors as in (1). As discussed our aim is to use two additional colors for coloring the remaining edges (given by M) and obtain an acyclic edge-coloring of $K_{n,n}$. For this purpose, we need to partition M into $M^{(1)}$ and $M^{(2)}$, in such a way that the induced subgraph of $M^{(\rho)} \cup M_i^*$ does not contain a cycle for any $0 \le i \le n-1$ and $\rho \in \{1, 2\}$. In other words, we must have edges from both $M^{(1)}$ and $M^{(2)}$ in any cycle in the induced subgraph of $M \sqcup M_i^* = M \cup M_i$. Note that by Lemma 2.1 the cycles of the induced subgraph of $M \cup M_i$ can be obtained from the cycles of the permutation $\pi^{-1} \circ \pi_i$ in its disjoint

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[1]One can take $M_i^* = M_i$ for exactly one $i \in I$ and other M_i^*'s as mentioned, and observe that Lemma 3.1 holds true in such a case as well.

cycle decomposition. So in order to see such a partition of M exists or not, we analyse cycle structure of the permutations $(\pi^{-1} \circ \pi_i)$ for $0 \leq i \leq (n-1)$. We now see that such a partition of M, if exists, can be obtained from a partition of the label set I, and it is due to the one-to-one correspondence between the label set and a perfect matching.

Suppose $(\pi^{-1} \circ \pi_i) = C_{i0}C_{i1}\cdots C_{ik_i}$ as a product of disjoint cycles. Observe that there is exactly one common edge $M \cap M_i = \{e_i : (s_i \mapsto t_i)\}$, and so $s_i \in I$ is the only fixed element in the permutation $\pi^{-1} \circ \pi_i$, i.e., $(\pi^{-1} \circ \pi_i)(s_i) = s_i$ and $(\pi^{-1} \circ \pi_i)(s) \neq s$ for any $s\,(\neq s_i) \in I$. Let us take $C_{i0} = (s_i)$ and let ℓ_{ij} be the length of the cycle C_{ij}, $1 \leq j \leq k_i$. Then we must have $\ell_{ij} \geq 2$ for $0 \leq i \leq n-1$ and $1 \leq j \leq k_i$. Note that for a cycle C_{ij} in the disjoint cycle decomposition of these permutations, its corresponding cycle C_{ij}^g in $K_{n,n}$, is of length $2\ell_{ij}$; half of the edges are from M and the other half are from M_i^* (see Lemma 2.1). We now try to partition I into $I^{(1)}$ and $I^{(2)}$ by analysing all the cycles C_{ij}, $0 \leq i \leq n-1$ and $1 \leq j \leq k_i$, such that at least one element from both $I^{(1)}$ and $I^{(2)}$ appear in the representations of all those cycles C_{ij}. If such a partition of I exists, then the corresponding partition of M is given by $M^{(1)} = \{(m \mapsto \pi(m)) : m \in I^{(1)}\}$ and $M^{(2)} = \{(m \mapsto \pi(m)) : m \in I^{(2)}\}$, and by Lemma 2.1 we can see that the partition of M into $M^{(1)}$ and $M^{(2)}$ gives the required result. In general, if $K_{n,n}$ possesses a perfect 1-factorization, the difficulty is to identify a suitable perfect matching that can help to get an acyclic edge-coloring of $K_{n,n}$ using only $n+2$ colors. Let us illustrate the technique by considering the case $K_{p,p}$, where p is an odd prime.

3.1 The case of $K_{p,p}$ for an odd prime p

This case was studied in [5] and we present here a slight variant of it. Take M_i to be the perfect matching corresponding to the permutation $\pi_i : a \mapsto a + i \pmod{p}$ for $0 \leq i \leq p-1$. We can see that the decomposition $\{M_i, 0 \leq i \leq p-1\}$ gives a perfect 1-factorization of $K_{p,p}$. Now consider M to be the perfect matching given by the permutation $\pi : a \mapsto ax \pmod{p}$, where x is a generator of $\mathbb{Z}_p^*$. Let y be the multiplicative inverse of x in $\mathbb{Z}_p^*$. Note that $order(x) = order(y) = p-1$ in $\mathbb{Z}_p^*$. We can easily check that $M \cap M_i = \{e_i : (\frac{i}{x-1} \mapsto \frac{ix}{x-1})\}$. We also have the following.

i) $\pi^{-1} \circ \pi_0 = \pi^{-1} = C_{00}C_{01}$, where $C_{i0} = (0)$ represents the common edge $e_0 : (0 \mapsto 0)$ and $C_{01} = (1\, y\, y^2 \cdots y^{p-2})$ is a cycle of length $(p-1)$ containing 1.

ii) $\pi^{-1} \circ \pi_i = C_{i0}C_{i1}$ for $1 \leq i \leq p-1$, where $C_{i0} = (\frac{i}{x-1})$ represents the common edge $e_i : (\frac{i}{x-1} \mapsto \frac{ix}{x-1})$ and $C_{i1} = (0\ iy\ i(y^2+y)\cdots i(y^{p-3}+\cdots+y)\ -i)$ is a cycle of length $(p-1)$ containing 0.

Therefore we can get the required result with a partition of I into $I^{(2)} = \{0, 1\}$ and $I^{(1)} = I \backslash I^{(2)}$. Then the corresponding partition of M is given by $M^{(1)} = M \backslash \{(0 \mapsto 0), (1 \mapsto x)\}$ and $M^{(2)} = \{(0 \mapsto 0), (1 \mapsto x)\}$. Observe that the cycle C_{i1}^g of $K_{n,n}$ corresponding to C_{i1}, $1 \leq i \leq (p-1)$, contains exactly one edge $(0 \mapsto 0)$ which belong to $M^{(2)}$ and C_{01}^g contains the edge $(1 \mapsto x) \in M^{(2)}$. The other $(p-2)$ edges of the cycle C_{i1}^g, $0 \leq i \leq p-1$, belong to $M^{(1)}$. Now the final assignment of the colors is as follows:

– the edges in M_i^* are colored with c_i for $i \in I$;

– the edges in $M^{(1)}$ are colored with c_p;

– the edges in $M^{(2)}$ are colored with c_{p+1}.

From the above discussion and by Lemma 3.1, it is clear that the proposed edge-coloring (with $p+2$ colors) of $K_{p,p}$ is proper and acyclic.

Note that the proposal in [5] is $M_0^* = M_0$ and $M^{(2)} = \{(1 \mapsto x)\}$ and one can easily see that the result is still valid with such a choice as well.

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4 The case of K_{p^2,p^2} for an odd prime $p \geq 5$

In this section we provide an acyclic edge-coloring of K_{p^2,p^2} with $p^2 + 2$ colors, where p is an odd prime ≥ 5. We follow the general framework described in the previous section. Accordingly we now summarize the set-up in this case. We use elements of $\mathcal{I} = \{(a,b) : a,b \in \mathbb{Z}_p\}$ for labelling the vertices of K_{p^2,p^2} on both sides. Let x be a generator of $\mathbb{Z}_p^*$ and let y be its inverse. Observe that $order(x) = order(y) = p - 1$ in $\mathbb{Z}_p^*$. We consider the following.

- Let $M_{(a,b)}$ be the perfect matching corresponding to the permutation $\pi_{(a,b)}$ of the label set $\mathcal{I}$ defined by

$$\pi_{(a,b)}\big((c,d)\big) = \begin{cases} (a, a+b+d) & \text{if } c = 0 \text{ and } a+b+d \neq 0 \\ (a+xb, 0) & \text{if } c = 0 \text{ and } a+b+d = 0 \\ (a+c+xb, 0) & \text{if } c \neq 0 \text{ and } b+d = 0 \\ (a+c, b+d) & \text{if } c \neq 0 \text{ and } b+d \neq 0 \end{cases}$$

  Then from [7] (with $\alpha = 1$ and $\beta = x$), we can see that the perfect matchings $\{M_{(a,b)}, (a,b) \in \mathcal{I}\}$ form a perfect 1-factorization of K_{p^2,p^2}.

- We choose the perfect matching M corresponding to the permutation defined by

$$\pi\big((c,d)\big) = (yc, xd);$$

- We choose the following partition of the label set $\mathcal{I}$:

$$\mathcal{I}^{(2)} = \{(0,1), (1,0), (z,z), (z,zx) \text{ for } z \in \mathbb{Z}_p^*\} \text{ and } \mathcal{I}^{(1)} = \mathcal{I} \setminus \mathcal{I}^{(2)}$$

  Then the corresponding partition of $M = M^{(1)} \sqcup M^{(2)}$ is given by

$$M^{(1)} = \{(c,d) \mapsto (yc, xd) : (c,d) \in \mathcal{I}^{(1)}\},$$
$$M^{(2)} = \{(c,d) \mapsto (yc, xd) : (c,d) \in \mathcal{I}^{(2)}\}.$$

  We have $|\mathcal{I}^{(2)}| = |M^{(2)}| = 2p$ and $|\mathcal{I}^{(1)}| = |M^{(1)}| = p^2 - 2p$.

Let $M^*_{(a,b)} = M_{(a,b)} \setminus M$ for $(a,b) \in \mathcal{I}$. The edge-coloring of K_{p^2,p^2} that we consider is as follows:

- the edges in $M^*_{(a,b)}$ are colored with c_{ap+b} for $(a,b) \in \mathcal{I}$;

- the edges in $M^{(1)}$ are colored with c_{p^2} ;

- the edges in $M^{(2)}$ are colored with c_{p^2+1}.

According to the general framework discussed in the previous section, the two requirements that need to be satisfied to establish the above edge-coloring of K_{p^2,p^2} is proper and acyclic are as follows:

- for $(a,b) \in \mathcal{I}$, there is exactly one fixed element in the permutation $\pi^{-1} \circ \pi_{(a,b)}$. That is there is exactly one edge common to both M and $M_{(a,b)}$;

- for $(a,b) \in \mathcal{I}$, elements from both $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ must appear in the representation of the cycles of length ≥ 2 in the disjoint cycle decomposition of the permutation $\pi^{-1} \circ \pi_{(a,b)}$.

Let us now prove that the above two requirements are satisfied in our set-up. For brevity of expression, we sometimes use the following notation.

$$x' = \frac{1}{x-1} = \frac{y}{1-y} \text{ and } x_i = x + x^2 + \cdots + x^i \text{ for } i = 1, 2, \ldots, p-2,$$

$$y' = \frac{1}{y-1} = \frac{x}{1-x} \text{ and } y_i = y + y^2 + \cdots + y^i \text{ for } i = 1, 2, \ldots, p-2.$$

Proposition 4.1 *For $(a, b) \in \mathcal{I}$, we have $|M \cap M_{(a,b)}| = 1$.*

Proof : An edge $(c, d) \mapsto (c', d')$ is common to both M and $M_{(a,b)}$ if and only if $(c', d') = \pi_{(a,b)}\big((c, d)\big) = \pi\big((c, d)\big)$. Therefore by checking the four cases

$$
\begin{array}{ll}
(a, a + b + d) = (0, xd) & \text{if } c = 0 \text{ and } a + b + d \neq 0 \\
(a + xb, 0) = (0, xd) & \text{if } c = 0 \text{ and } a + b + d = 0 \\
(a + c + xb, 0) = (yc, xd) & \text{if } c \neq 0 \text{ and } b + d = 0 \\
(a + c, b + d) = (yc, xd) & \text{if } c \neq 0 \text{ and } b + d \neq 0
\end{array}
$$

for $(a, b) \in \mathcal{I}$, we get

$$M \cap M_{(a,b)} = \{(ay', bx') \mapsto (-ax', -by')\}$$

and hence the proof. $\square$

The above proposition shows that the first requirement is satisfied in our set-up. In what follows we prove that the other requirement is also satisfied. For this purpose, we now analyse cycle structure of the permutations $\pi^{-1} \circ \pi_{(a,b)}$ for $(a, b) \in \mathcal{I}$. The inverse permutation of π is given by $\pi^{-1}\big((c, d)\big) = (xc, yd)$. So we get

$$
\pi^{-1} \circ \pi_{(a,b)}\big((c, d)\big) = \left\{
\begin{array}{ll}
(xa, y(a + b + d)) & \text{if } c = 0 \text{ and } a + b + d \neq 0 \\
(xa + x^2 b, 0) & \text{if } c = 0 \text{ and } a + b + d = 0 \\
(x(a + c) + x^2 b, 0) & \text{if } c \neq 0 \text{ and } b + d = 0 \\
(x(a + c), y(b + d)) & \text{if } c \neq 0 \text{ and } b + d \neq 0
\end{array}
\right.
$$

The above permutation can be decomposed into the following three permutations.

$$\pi^{(0)}_{(a,b)}\big((c, d)\big) = (x(a + c), y(b + d))$$

$$\pi^{(1)}_{(a,b)}\big((c, d)\big) = \left\{
\begin{array}{ll}
(c, ya + d) & \text{if } c = xa \\
(c, d) & \text{otherwise}
\end{array}
\right.$$

$$\pi^{(2)}_{(a,b)}\big((c, d)\big) = \left\{
\begin{array}{ll}
(c + x^2 b, d) & \text{if } d = 0 \\
(c, d) & \text{otherwise}
\end{array}
\right.$$

Proposition 4.2 *For $(a, b) \in \mathcal{I}$, we have*

$$\pi^{-1} \circ \pi_{(a,b)} = \pi^{(2)}_{(a,b)} \circ \pi^{(1)}_{(a,b)} \circ \pi^{(0)}_{(a,b)}$$

Proof : Note that

$$\pi^{(1)}_{(a,b)}\Big(\pi^{(0)}_{(a,b)}\big((c, d)\big)\Big) = \left\{
\begin{array}{ll}
(x(a + c), ya + y(b + d)) & \text{if } x(a + c) = xa \ (\Leftrightarrow c = 0) \\
(x(a + c), y(b + d)) & \text{otherwise}
\end{array}
\right.$$

and $\pi^{(2)}_{(a,b)}$ splits each of above two cases into two subcases depending on whether the second component is zero or not. Thus we get

$$\pi^{(2)}_{(a,b)}\big(\pi^{(1)}_{(a,b)} \circ \pi^{(0)}_{(a,b)}((c,d))\big) = \begin{cases} (xa, ya + y(b+d)) & \text{if } c = 0 \text{ and } a + b + d \neq 0 \\ (xa + x^2 b, 0) & \text{if } c = 0 \text{ and } a + b + d = 0 \\ (x(a+c) + x^2 b, 0) & \text{if } c \neq 0 \text{ and } b + d = 0 \\ (x(a+c), y(b+d)) & \text{if } c \neq 0 \text{ and } b + d \neq 0 \end{cases}$$

and hence the proof. $\qquad\square$

With the above decomposition and the following result, analysis of the cycle structures can be simplified which we will see later.

Proposition 4.3 *Let* $\sigma\big((c,d)\big) = (yc, yd)$, *then we have*

$$\pi^{-1} \circ \pi_{(xa,xb)} = \sigma^{-1} \circ \big(\pi^{-1} \circ \pi_{(a,b)}\big) \circ \sigma.$$

Proof : We have

$$\pi^{-1} \circ \pi_{(a,b)}\big(\sigma((c,d))\big) = \begin{cases} (xa, y(a + b + yd)) & \text{if } c = 0 \text{ and } a + b + yd \neq 0 \\ (xa + x^2 b, 0) & \text{if } c = 0 \text{ and } a + b + yd = 0 \\ (x(a + yc) + x^2 b, 0) & \text{if } c \neq 0 \text{ and } b + yd = 0 \\ (x(a + yc), y(b + yd)) & \text{if } c \neq 0 \text{ and } b + yd \neq 0 \end{cases}$$

and so we get

$$\sigma^{-1}\big(\pi^{-1} \circ \pi_{(a,b)} \circ \sigma((c,d))\big) = \begin{cases} (x^2 a, y(xa + xb + d) & \text{if } c = 0 \text{ and } xa + xb + d \neq 0 \\ (x^2 a + x^3 b, 0) & \text{if } c = 0 \text{ and } xa + xb + d = 0 \\ (x^2 a + c + x^3 b, 0) & \text{if } c \neq 0 \text{ and } xb + d = 0 \\ (x^2 a + c, y(xb + d)) & \text{if } c \neq 0 \text{ and } xb + d \neq 0 \end{cases}$$

Now one can check that this is equal to $\pi^{-1} \circ \pi_{(xa,xb)}$. $\qquad\square$

Thus for $(a,b) \in \mathcal{I}$, the permutations $\pi^{-1} \circ \pi_{(x^i a, x^i b)}$, $i \in \{0, 1, \ldots, p-2\}$, are all conjugates of each other, and so they all have same cycle structure. In fact we get the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(x^i a, x^i b)}$ by the symbol transformation σ^i, *i.e.*, replacing the symbols (c,d) by $(y^i c, y^i d)$ in the disjoint cycle decomposition of $\pi_{(a,b)}$. Therefore it is enough to study the cycle structure of $\pi^{-1} \circ \pi_{(a',b')}$ for $(a',b') \in \mathcal{I}' = \{(0,0), (0,1), (1,0), (1,b)$ for $b \in \mathbb{Z}_p^*\}$.

We now analyse the cycle structure of $\pi^{-1} \circ \pi_{(a,b)}$ for $(a,b) \in \mathcal{I}$ by dividing into four cases: $(0,0), (*,0), (0,*), (*,*)$, where $*$ represents nonzero elements of $\mathbb{Z}_p$. We discuss these cases one by one and we show that elements from both $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ appear in the representation of the cycles (of length ≥ 2) in the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(a,b)}$ for elements (a,b) in each of these four cases.

In Section 3 we have mentioned cycles with single element explicitly to emphasize that there is exactly one fixed element. As we have already proved it in Proposition 4.1, for convenience we follow the convention and in the discussion below we do not explicitly mention cycles with single element in the disjoint cycle decomposition of permutations. Accordingly we count only the cycles of length ≥ 2 in the disjoint cycle decomposition. For simplicity, we also use some common notation in presenting the disjoint cycle decomposition in each of these cases.

Case: $a = 0$ and $b = 0$

The permutation $\pi_{(0,0)}$ is the identity map, and so $\pi^{-1} \circ \pi_{(0,0)} = \pi^{-1}$. We can see from the definition that the disjoint cycle decomposition of π^{-1} can be given by

$$\pi^{-1} = C_0 C_1 \cdots C_{p-1} C_p,$$

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where the $p+1$ cycles are given by

$$C_j = \big((j,1)\,(jx,y)\,(jx^2,y^2)\cdots(jx^{p-2},y^{p-2})\big) \quad \text{for } j \in \mathbb{Z}_p$$
$$C_p = \big((1,0)\,(x,0)\,(x^2,0)\cdots(x^{p-2},0)\big)$$

and it is evident that they are of length $p-1$. The missing element $(0,0)$ is fixed by π^{-1}.

Note that the cycle C_0 contains exactly one element $(0,1) \in \mathcal{I}^{(2)}$ and other elements belong to $\mathcal{I}^{(1)}$, and also the cycle C_p contains exactly one element $(1,0) \in \mathcal{I}^{(2)}$ and other elements belong to $\mathcal{I}^{(1)}$.

For $j \in \mathbb{Z}_p^*$, the elements of the cycle C_j are of the form (jx^i, y^i) for $i = 0, 1, \ldots, p-2$. Since y is also a generator of $\mathbb{Z}_p^*$ we have either $j = y^{2s}$ or $j = y^{2s+1}$ for some $s \in \{0, 1, \ldots, \frac{p-1}{2}-1\}$. Note that $x^{\frac{p-1}{2}} = y^{\frac{p-1}{2}} = -1$.

- If $j = y^{2s}$ (is a square), then we have $jx^s = y^s$ and $jx^{s+\frac{p-1}{2}} = y^{s+\frac{p-1}{2}}$, and observe that (jx^s, y^s) and $(jx^{s+\frac{p-1}{2}}, y^{s+\frac{p-1}{2}})$ are the only elements of C_j which belong to $\mathcal{I}^{(2)}$ and they are of the form (z, z). The remaining $(p-3)$ elements of C_j belong to $\mathcal{I}^{(1)}$.

- If $j = y^{2s+1}$ (is a non-square), then we have $jx^s = y \cdot y^s$ and $jx^{s+\frac{p-1}{2}} = y \cdot y^{s+\frac{p-1}{2}}$, and observe that (jx^s, y^s) and $(jx^{s+\frac{p-1}{2}}, y^{s+\frac{p-1}{2}})$ are the only elements of C_j which belong to $\mathcal{I}^{(2)}$ and they are of the form (z, zx). The remaining $(p-3)$ elements of C_j belong to $\mathcal{I}^{(1)}$.

Case: $a = 0$ and $b \in \mathbb{Z}_p^*$

By Proposition 4.3 it is enough to study the cycle structure of $\pi^{-1} \circ \pi_{(0,1)}$. For $b \in \mathbb{Z}_p^*$, the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(0,b)}$ can be obtained by the symbol transformation: replacing (c,d) by (bc, bd) in the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(0,1)}$.

We have

$$\pi_{(0,1)}^{(0)}\big((c,d)\big) = (xc, y(1+d)).$$

Then we can see that

$$\pi_{(0,1)}^{(0)} = C_0 C_1 \cdots C_{p-1} C_p,$$

where the $p+1$ cycles are given by

$$C_j = \big((jx^2,0)\,(jx^3,y)\,(jx^4,y+y^2)\cdots(jx,y+\cdots+y^{p-2})\big) \quad \text{for } j \in \mathbb{Z}_p$$
$$C_p = \big((1,x')\,(x,x')\,(x^2,x')\cdots(x^{p-2},x')\big), \tag{2}$$

and it is evident that they are of length $p-1$. The missing element $(0,x')$ is fixed by $\pi_{(0,1)}^{(0)}$.

We get $\pi_{(0,1)}^{(1)}\big((c,d)\big) = (c,d)$ since $a = 0$, and so $\pi_{(0,1)}^{(1)}$ is the identity map. We have

$$\pi_{(0,1)}^{(2)}\big((c,d)\big) = \begin{cases} (c+x^2, 0) & \text{if } d = 0 \\ (c,d) & \text{otherwise} \end{cases}$$

So it has only one cycle given by

$$\pi_{(0,1)}^{(2)} = C^{(2)} = \big((0,0)\,(x^2,0)\,(2x^2,0)\cdots((p-1)x^2,0))\big),$$

it is of length p and all other elements are fixed.

Therefore we have

$$\pi^{-1} \circ \pi_{(0,1)} = \pi_{(0,1)}^{(2)} \circ \pi_{(0,1)}^{(1)} \circ \pi_{(0,1)}^{(0)} = C^{(2)} C_0 C_1 \cdots C_{p-1} C_p.$$

Observe that an element $(kx^2, 0)$ of the cycle $C^{(2)}$ appears exactly in one cycle C_j (when $j = k$). So the product $C^{(2)} C_0 C_1 \ldots C_{p-1}$ will form a single cycle of length $p(p-1)$, and Figure 5 shows how the cycles are joined together to form a single cycle.

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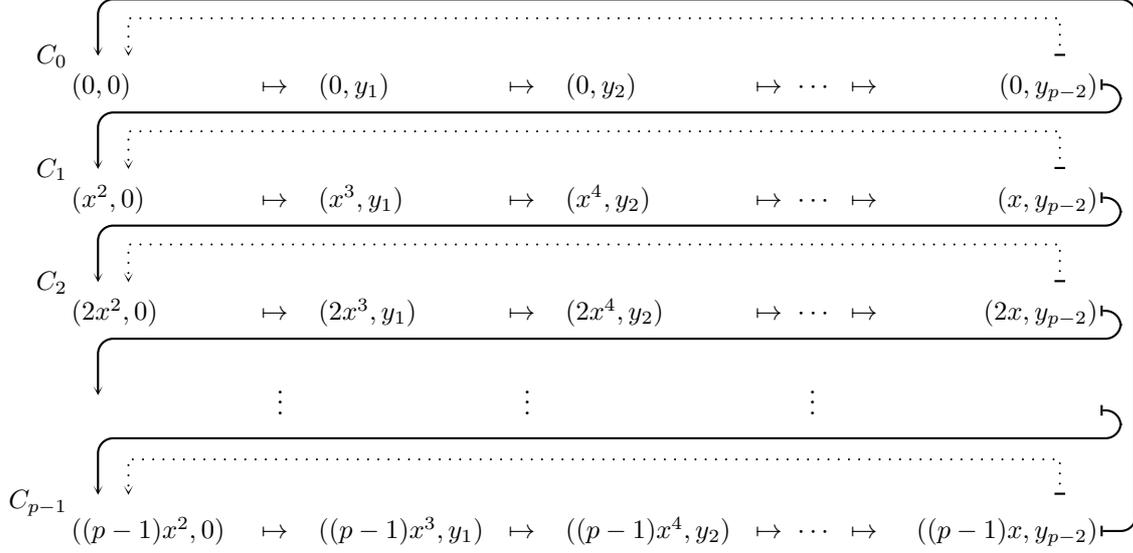
Figure 5: The cycle formed by the product $C^{(2)}C_0C_1\cdots C_{p-1}$

Let F_1 denote the cycle formed by the product $C^{(2)}C_0C_1\ldots C_{p-1}$ and let $F_2 = C_p$. Thus we have

$$\pi^{-1} \circ \pi_{(0,1)} = F_1 F_2$$

is a product of two disjoint cycles. Therefore for $b \in \mathbb{Z}_p^*$, the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(0,b)}$ can be given by

$$\pi^{-1} \circ \pi_{(0,b)} = \bar{F}_1 \bar{F}_2,$$

where the cycles $\bar{F}_1$ and $\bar{F}_2$ are obtained by the symbol transformation: replacing (c,d) by (bc, bd) in F_1 and F_2 respectively. From (2) we can see that the cycle $\bar{F}_2$ can be given by

$$\bar{F}_2 = \left((b, bx')(bx, bx')(bx^2, bx')\cdots(bx^{p-2}, bx')\right) = \left((1, bx')(x, bx')(x^2, bx')\cdots(x^{p-2}, bx')\right)$$

Let $bx' = x^s$ for some $s \in \{0, 1, \ldots, p-2\}$ and observe that the cycle $\bar{F}_2$ contains exactly two elements of $\mathcal{I}^{(2)}$, namely, (x^s, bx') of the form (z, z) and (x^{s-1}, bx') of the form (z, zx). The other $(p-3)$ elements of $\bar{F}_2$ belong to $\mathcal{I}^{(1)}$. Evidently, elements from both $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ appear in the cycle $\bar{F}_1$.

Case: $a \in \mathbb{Z}_p^*$ and $b = 0$

By Proposition 4.3 it is enough to study the cycle structure of $\pi^{-1} \circ \pi_{(1,0)}$. For $a \in \mathbb{Z}_p^*$, the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(a,0)}$ can be obtained by the symbol transformation: replacing (c,d) by (ac, ad) in the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(1,0)}$.

We have

$$\pi_{(1,0)}^{(0)}\left((c,d)\right) = (x(1+c), yd).$$

Then we can see that

$$\pi_{(1,0)}^{(0)} = C_0 C_1 \cdots C_{p-1} C_p.$$

where the $p+1$ cycles are given by

$$\begin{aligned}
C_j &= \left((x, jy)(x + x^2, jy^2)\cdots(x + \cdots + x^{p-2}, jy^{p-2})(0, j)\right) \quad \text{for } j \in \mathbb{Z}_p \\
C_p &= \left((y', 1)(y', y)(y', y^2)\cdots(y', y^{p-2})\right),
\end{aligned} \tag{3}$$

and it is evident that they all are of length $p-1$. The missing element $(y', 0)$ is fixed by $\pi_{(1,0)}^{(0)}$.

We have

$$\pi^{(1)}_{(1,0)}((c,d)) = \begin{cases} (c, y+d) & \text{if } c = x \\ (c,d) & \text{otherwise} \end{cases}$$

So it has only one cycle given by

$$\pi^{(1)}_{(1,0)} = C^{(1)} = \big((x,0)\,(x,y)\,(x,2y)\cdots(x,(p-1)y))\big),$$

it is of length p and all other elements are fixed. We get $\pi^{(2)}_{(1,0)}((c,d)) = (c,d)$ since $b = 0$, and so $\pi^{(2)}_{(1,0)}$ is the identity map. Therefore we have

$$\pi^{-1} \circ \pi_{(1,0)} = \pi^{(2)}_{(1,0)} \circ \pi^{(1)}_{(1,0)} \circ \pi^{(0)}_{(1,0)} = C^{(1)}C_0C_1\cdots C_{p-1}C_p.$$

Observe that an element (x, ky) of the cycle $C^{(1)}$ appears exactly in one cycle C_j (when $j = k$). So the product $C^{(1)}C_0C_1\dots C_{p-1}$ will form a single cycle of length $p(p-1)$, and Figure 6 shows how the cycles are joined together to form a single cycle.
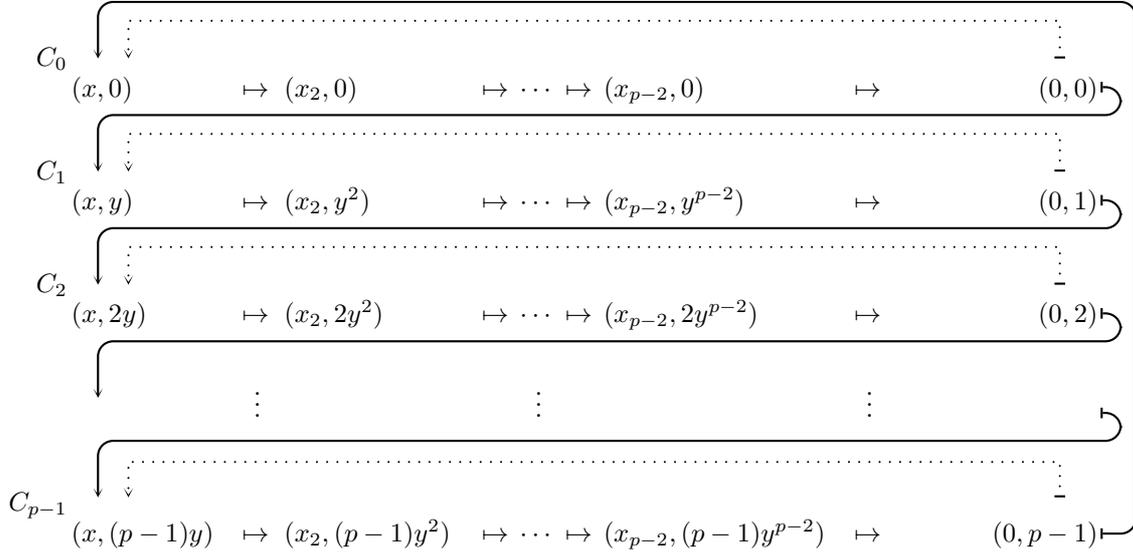
Figure 6: The cycle formed by the product $C^{(1)}C_0C_1\cdots C_{p-1}$

Let F_1 denote the cycle formed by the product $C^{(1)}C_0C_1\dots C_{p-1}$ and let $F_2 = C_p$. Thus we have

$$\pi^{-1} \circ \pi_{(1,0)} = F_1F_2$$

is a product of two disjoint cycles. Therefore for $a \in \mathbb{Z}_p^*$, the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(a,0)}$ can be given by

$$\pi^{-1} \circ \pi_{(a,0)} = \bar{F}_1\bar{F}_2,$$

where the cycles $\bar{F}_1$ and $\bar{F}_2$ are obtained by the symbol transformation: replacing (c,d) by (ac, ad) in F_1 and F_2 respectively. From (3) we can see that the cycle $\bar{F}_2$ can be given by

$$\bar{F}_2 = \big((ay',a)\,(ay',ay)\,(ay',ay^2)\cdots(ay',ay^{p-2}))\big) = \big((ay',1)\,(ay',y)\,(ay',y^2)\cdots(ay',y^{p-2}))\big)$$

Let $ay' = y^s$ for some $s \in \{0,1,\dots,p-2\}$ and observe that the cycle $\bar{F}_2$ contains exactly two elements of $\mathcal{I}^{(2)}$, namely, (ay', y^s) of the form (z, z) and (ay', y^{s-1}) of the form (z, zx). The other $(p-3)$ elements of $\bar{F}_2$ belong to $\mathcal{I}^{(1)}$. Evidently, elements from both $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ appear in the cycle $\bar{F}_1$.

Case: $a \in \mathbb{Z}_p^*$ and $b \in \mathbb{Z}_p^*$

By Proposition 4.3 it is enough to study the cycle structure of $\pi^{-1} \circ \pi_{(1,b)}$ for $b \in \mathbb{Z}_p^*$. For $a \in \mathbb{Z}_p^*$, the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(a,ab)}$ (observe that all the elements of this case are covered) can be obtained by the symbol transformation: replacing (c,d) by (ac, ad) in the disjoint cycle decomposition of $\pi^{-1} \circ \pi_{(1,b)}$.

We have

$$\pi_{(1,b)}^{(0)}\big((c,d)\big) = (x(1+c), y(b+d)).$$

Then we can see that

$$\pi_{(1,b)}^{(0)} = C_0 C_1 \cdots C_{p-1} C_p.$$

where the $p+1$ cycles are given by

$$
\begin{aligned}
C_j &= \big((x, (b+j)y)\, (x + x^2, (y + y^2)b + jy^2) \cdots (x_{p-2}, y_{p-2}b + jy^{p-2})\, (0, j)\big) \quad \text{for } j \in \mathbb{Z}_p \\
C_p &= \big((y', 0)\, (y', yb)\, (y', (y + y^2)b) \cdots (y', y_{p-2}b)\big),
\end{aligned}
\tag{4}
$$

and it is evident that they all are of length $p-1$. The missing element (y', bx') is fixed by $\pi_{(1,b)}^{(0)}$.

Note that the cycle C_j for $j = bx'$ is given by

$$C_{bx'} = \big((x_1, bx')\, (x_2, bx') \cdots (x_{p-2}, bx')\, (0, bx')\big) \tag{5}$$

and all the elements of $C_{bx'}$ are of the form (c, bx'), where $c\, (\neq y') \in \mathbb{Z}_p$, and also all the elements of C_p are of the form (y', d), where $d\, (\neq bx') \in \mathbb{Z}_p$. As in the previous two cases, we will show that the permutation

$$\pi^{-1} \circ \pi_{(1,b)} = \pi_{(1,b)}^{(2)} \circ \pi_{(1,b)}^{(1)} \circ \pi_{(1,b)}^{(0)} = F_1 F_2$$

can be written as a product of two disjoint cycles. Moreover, we will show that all the elements of the cycle $C_{bx'}$ appear in F_1 and all the elements of the cycle C_p appear in F_2, which is the crucial point in our proof.

We have

$$\pi_{(1,b)}^{(1)}\big((c,d)\big) = \begin{cases} (x, y+d) & \text{if } c = x \\ (c, d) & \text{otherwise} \end{cases}$$

So it has only one cycle given by

$$C^{(1)} = \big((x,0)\, (x,y)\, (x, 2y) \cdots (x, (p-1)y))\big),$$

it is of length p and all other elements are fixed.

Observe that an element (x, ky) of the cycle $C^{(1)}$ appears exactly in one cycle C_j (when $b + j = k$). So the product $C^{(1)} C_0 C_1 \cdots C_{p-1}$ will form a single cycle of length $p(p-1)$ as shown in Figure 7, and let us denote it by $\hat{C}$. Then we get

$$\pi_{(1,b)}^{(1)} \circ \pi_{(1,b)}^{(0)} = C^{(1)} C_0 C_1 \cdots C_{p-1} C_p = \hat{C} C_p.$$

We have

$$\pi_{(1,b)}^{(2)}\big((c,d)\big) = \begin{cases} (c + x^2 b, 0) & \text{if } d = 0 \\ (c, d) & \text{otherwise} \end{cases}$$

So it has only one cycle given by

$$C^{(2)} = \big((0,0)\, (bx^2, 0)\, (2bx^2, 0) \cdots ((p-1)bx^2, 0))\big),$$

it is of length p and all other elements are fixed. Therefore we have

$$\pi^{-1} \circ \pi_{(1,b)} = \pi_{(1,b)}^{(2)} \circ \pi_{(1,b)}^{(1)} \circ \pi_{(1,b)}^{(0)} = C^{(2)} C^{(1)} C_0 C_1 \cdots C_{p-1} C_p = C^{(2)} \hat{C} C_p.$$

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Figure 7: The cycle formed by the product $C^{(1)}C_0C_1\cdots C_{p-1}$

Observe that the cycle $C^{(2)}$ contains all the elements of the form $(c,0)$, where $c \in \mathbb{Z}_p$. Note also that the cycle C_j for $j(\neq bx') \in \mathbb{Z}_p$ contains exactly one element of the form $(c,0)$ given by $(-\frac{jx}{b},0)$ (see that for some $i \in \{1,2,\ldots,p-2\}$, if the second component satisfies $(by_i + jy^i) = 0$ then the first component $x_i = -\frac{jx}{b}$), and the other element $(-x'x,0) = (y',0)$ appears in the cycle C_p. So all the elements of $C^{(2)}$ except $(y',0)$ appear in the cycle $\hat{C} = C_0C_1\cdots C_{p-1}$, and the order in which they appear can also be seen from the figure below.

$$(0,0) \mapsto \cdots \mapsto (-\tfrac{x}{b},0) \mapsto \cdots \mapsto (-\tfrac{2x}{b},0) \mapsto \cdots \mapsto (-\tfrac{(bx'-1)x}{b},0) = (y'+\tfrac{x}{b},0)$$

$$(-\tfrac{(p-1)x}{b},0) \leftarrow \cdots \leftarrow (-\tfrac{(bx'+2)x}{b},0) \leftarrow \cdots \leftarrow (-\tfrac{(bx'+1)x}{b},0) = (y'-\tfrac{x}{b},0)$$

Figure 8: Order of the elements of the form $(z,0)$ in $\hat{C}$

Looking at only the elements of the form $(c,0)$ in the representation of $\hat{C}$, observe that the element $(-\frac{jx}{b},0)$ for $j(\neq bx') \in \mathbb{Z}_p$ is followed by $(-\frac{(j+1)x}{b},0)$ when $j \neq bx'-1$, and in the case when $j = bx'-1$, i.e., $(-\frac{(bx'-1)x}{b},0)$ is followed by $(-\frac{(bx'+1)x}{b},0)$, a jump of $-\frac{2x}{b}$ in the first component. Keeping in view how the composition is worked out, we can see the following: suppose that an element (c',d') is mapped to $(c,0)$ in $\pi^{(1)}_{(1,b)} \circ \pi^{(0)}_{(1,b)} = \hat{C}C_p$, then the element (c',d') will be mapped to $(c+x^2b,0)$ in $\pi^{(2)}_{(1,b)} \circ \pi^{(1)}_{(1,b)} \circ \pi^{(0)}_{(1,b)} = C^{(2)}\hat{C}C_p$; the product by $C^{(2)}$ on the left with $\hat{C}C_p$ does not effect the mapping of other elements that are not mapped to elements of the form $(c,0)$ in $\hat{C}C_p$. Intuitively, the product by $C^{(2)}$ on the left with $\hat{C}C_p$ permutes the expressions: from an element of the form $(c,0)$ to the next first element (c',d') that is mapped to an element of the form $(c,0)$ in the representation of $\hat{C}$ and C_p. Having this in mind, let us now look at the cycles in the disjoint cycle decomposition of $\pi^{(2)}_{(1,b)} \circ \pi^{(1)}_{(1,b)} \circ \pi^{(0)}_{(1,b)} = C^{(2)}\hat{C}C_p$.

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The following summations in (6) are useful to check the cycles of the product $C^{(2)}\hat{C}C_p$.

Note that $x^2b \neq \frac{x}{b}$ for any $b \in \mathbb{Z}_p^*$ since otherwise $xb^2 = 1$, a contradiction to the fact that x is a generator of $\mathbb{Z}_p^*$ and it can not be a square. Let $t, 1 \leq t \leq p-1$, be the inverse of $(xb^2 - 1)$ in $\mathbb{Z}_p^*$. Then we can see that t is the smallest positive integer satisfying

$$tx^2b - (t+1)\frac{x}{b} = 0 \quad \text{and} \quad (p-t)x^2b - (p-t-1)\frac{x}{b} = 0. \tag{6}$$

Let F_1 be the cycle containing the element $(-\frac{(bx'-1)x}{b}, 0) = (y' + \frac{x}{b}, 0)$ in the product $C^{(2)}\hat{C}C_p$ and it can be seen from the figure below. Clearly, the cycle F_1 contains t elements of the form $(c, 0)$.

$$\overbrace{(y' + \tfrac{x}{b}, 0) \mapsto \cdots \mapsto (x, bx') \mapsto \cdots \mapsto (x_{p-2}, bx')}^{\text{elements of } C_{bx'}} \mapsto (0, bx') \mapsto \cdots \mapsto (y' + \tfrac{x}{b} - \tfrac{2x}{b} + x^2b, 0)$$

$$(y' + \tfrac{x}{b} - \tfrac{tx}{b} + (t-1)x^2b, 0) \leftarrow \cdots \leftarrow (y' + \tfrac{x}{b} - \tfrac{4x}{b} + 3x^2b, 0) \leftarrow \cdots \leftarrow (y' + \tfrac{x}{b} - \tfrac{3x}{b} + 2x^2b, 0)$$
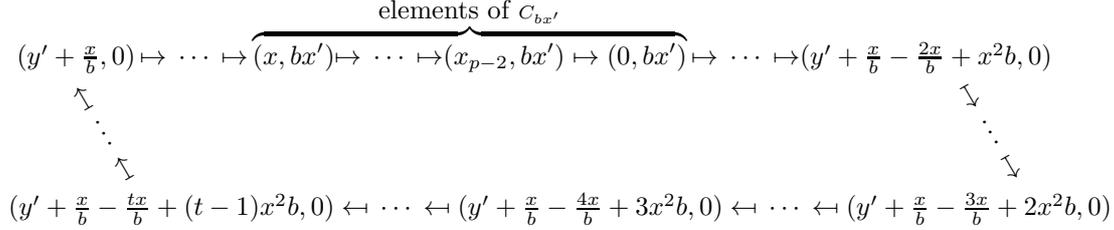
Figure 9: The cycle F_1 containing the element $(y' + \frac{x}{b}, 0)$ in the product $C^{(2)}\hat{C}C_p$

Let F_2 be the cycle containing the element $(y', 0)$ in the product $C^{(2)}\hat{C}C_p$ and it can be seen from the figure below. Clearly, the cycle F_2 contains the other $(p-t)$ elements of the form $(c, 0)$.

$$\overbrace{(y', 0) \mapsto \cdots \mapsto (y', y_{p-2}b)}^{\text{elements of } C_p} \mapsto (y' + x^2b, 0) \mapsto \cdots \mapsto (y' - \tfrac{x}{b} + 2x^2b, 0)$$

$$(y' - \tfrac{(p-t-1)x}{b} + (p-t)x^2b, 0) \leftarrow \cdots \leftarrow (y' - \tfrac{3x}{b} + 4x^2b, 0) \leftarrow \cdots \leftarrow (y' - \tfrac{2x}{b} + 3x^2b, 0)$$
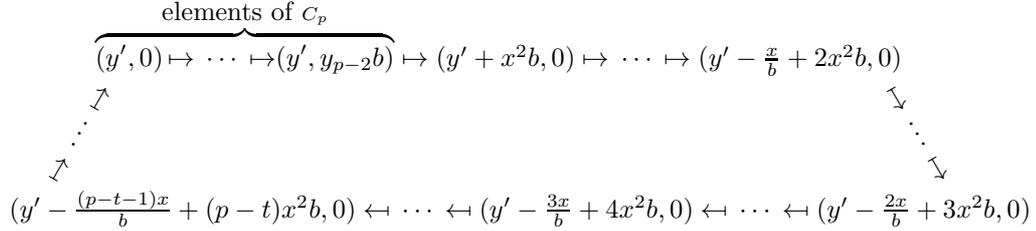
Figure 10: The cycle F_2 containing the element $(y', 0)$ in the product $C^{(2)}\hat{C}C_p$

Note that all the elements of the form $(c, 0)$ appear in the two cycles F_1 and F_2, and so these cycles must contain all the elements of $\mathcal{I}$ except (y', bx') which is the fixed element in $\pi_{(1,b)}^{(1)} \circ \pi_{(1,b)}^{(0)} = \hat{C}C_p$. The element (y', bx') is also fixed by $\pi_{(1,b)}^{(2)}$. Therefore we get

$$\pi_{(1,b)}^{(2)} \circ \pi_{(1,b)}^{(1)} \circ \pi_{(1,b)}^{(0)} = C^{(2)}\hat{C}C_p = F_1 F_2.$$

Note also that F_1 contains all the elements of $C_{bx'}$ and F_2 contains all the elements of C_p. Finally, for $a \in \mathbb{Z}_p^*$, we get that

$$\pi^{-1} \circ \pi_{(a,ab)} = \pi_{(a,ab)}^{(2)} \circ \pi_{(a,ab)}^{(1)} \circ \pi_{(a,ab)}^{(0)} = \bar{F}_1 \bar{F}_2,$$

where the cycles $\bar{F}_1$ and $\bar{F}_2$ are obtained by the symbol transformation: replacing (c, d) by (ac, ad) in the cycles F_1 and F_2 respectively.

From (4) we can see that the cycle $\bar{F}_2$ contains elements of the form (ay', d) for all $d \in \mathbb{Z}_p$ except may be $d = abx'$. So it must contain at least one element of the form (z, z) or (z, zx) which belong to $\mathcal{I}^{(2)}$. Since F_2 contains all the elements of C_p in the order as mentioned in (4), in the corresponding part of the cycle $\bar{F}_2$, at most three elements of $\mathcal{I}^{(2)}$ can appear; the other elements (at least $p-3$) belong to $\mathcal{I}^{(1)}$. Evidently, the cycle $\bar{F}_1$ contains elements from both

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$\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$. Hence the proof is completed.

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