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Numerical study of effect of pressure gradient on stability of an incompressible boundary layer

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The effect of pressure gradient on the stability of two dimensional, laminar, incompressible boundary layer flow has been investigated numerically using filtered and orthonormalized integration techniques and the results have been compared with Schlichting's calculations.

The pressure gradient exerts an important influence on the stability of boundary layer flow. The effect of pressure gradient on the stability of boundary layer flow has been theoretically investigated by Schlichting[1,2] and Ulrich[2] using a classical asymptotic expansion method to obtain solution of the Orr–Sommerfeld equation, although this method encounters difficulty at the critical layer and is not popular nowadays. The advent of high speed digital computers has made the numerical scheme for solving the stability problems very popular. The recent success[3,4] of numerical methods in the solution of problems of the stability of boundary layer flows have made them widely accepted. The purpose of the present work has, therefore, been to numerically investigate the effect of pressure gradient on the stability of two-dimensional, laminar, incompressible boundary layer flow, and thus to have a check on the earlier analytical method due to Schlichting.

The dimensionless Orr–Sommerfeld equation for the present problem is

$$(U-C)(\phi''-\alpha^2\phi)-U''\phi = -\frac{i}{\alpha R}(\phi^{iv}-2\alpha^2\phi''+\alpha^4\phi), \qquad (1)$$

where the velocities have been divided by the free stream velocity U_m and the distance y, perpendicular to the free stream direction, has been divided by the boundary layer thickness δ; $U(y)$ is the mean velocity in the boundary layer flow; $C=C_r+iC_i$, C_r being the wave velocity and C_i being the amplification factor; α is the wavenumber, ϕ is the amplitude of disturbance, and R is the Reynolds number $(U_m\delta/\nu)$, ν being the kinematic viscosity.

The corresponding boundary conditions are

$$y=0: \quad \phi=\phi'=0,$$
$$y=1: \quad \phi=\phi'=0. \qquad (2)$$

The influence of a pressure gradient on stability manifests itself through the form of the mean velocity profile $U(y)$. A sixth-order polynomial for the mean velocity, giving one parameter family of velocity profiles satisfying the momentum equation and the first compatibility condition at the wall, has been assumed. These velocity profiles are known to be sufficiently accurate for favorable pressure gradients and moderately large ($\Lambda < -6$) adverse pressure gradients and are expressed in dimensionless form as

$$U(y)=F(y)+\Lambda G(y), \qquad (3)$$

where

$$\Lambda=\frac{\delta^2}{\nu}\frac{dU_m}{dx},$$

$$F(y)=2y-5y^4+6y^5-2y^6,$$

and

$$G(y)=0.2y-0.5y^2+y^4-y^5+0.3y^6.$$

The positive value of the pressure gradient parameter, Λ, indicates a favorable pressure gradient while an adverse pressure gradient is expressed by a negative value of Λ.

For the solution of Eq. (1), which is in terms of the dimensionless distance y (based on the boundary layer thickness δ), the values of U and U'' can easily be calculated from Eq. (3). The accuracy of these values depends on the accuracy with which a sixth-order polynomial, used in the present case, can approximate the more accurate Falkner–Skan type of profiles. The advantage of the present method is, however, the ease with which the profiles of the velocity and the velocity derivatives can be obtained as explicit functions of a single shape parameter Λ. Equation (1) can no doubt easily be transformed into a very similar equation in terms of the Falkner–Skan local similarity coordinate $\eta=(y/x)[0.5(m+1)Re_x]^{0.5}$, but the determination of the profiles of the velocity and the velocity derivatives have to be done separately for each value of either the pressure parameter $\bar{\beta}=(2x/U)\,dU/dx$, or the Falkner–Skan parameter $\beta=2m/(m+1)$, in which m determines the distribution of the free-stream velocity $U\sim x^m$. While in the present case the main emphasis is on comparison

TABLE I. Critical Reynolds numbers based on displacement thickness δ^* for various values of pressure gradient parameter.

Λ	Present calculations	Due to Schlichting
+6.0	7 400	7 400
+4.0	4 500	4 460
+2.0	1 730	1 700
+0.0	540	645
−1.0	390	385
−3.0	210	193
−5.0	128	122

with Schlichting, the neutral stability curves and the critical Reynolds number for different positive and negative values of the shape parameter Λ, this parameter is also used to avoid the very tedious calculation of the profiles of velocity and velocity derivatives in local similarity coordinates as a function of β or $\bar{\beta}$. Note that the values of these and η_δ, which is the value of η at which the flow velocity approaches the 99% of the free-stream velocity (definition of the boundary layer thickness), are mostly available in the literature for selected positive values of β or $\bar{\beta}$, as for example by Back,[5] who calculated these in a slightly different form for $\bar{\beta}$ from 0 to 20. The relations between Λ, β, and $\bar{\beta}$ are given by the equation:

$$\Lambda = \eta_\delta^2\beta = 2\eta_\delta^2\bar{\beta}/(2-\bar{\beta}). \tag{4}$$

It is, therefore, evident that comparison with other authors regarding stability calculations in local similarity coordinates as a function of β or $\bar{\beta}$ is possible in case the data are available on η_δ at these values of β or $\bar{\beta}$.

Obremski *et al.*[6] have investigated the instability of the flow and the initial Reynolds number for several different types of profiles, among others for the Falkner–Skan type of profiles. A comparison with their result is done to examine whether the approximation of U'' in the present note is sufficiently accurate with respect to the Falkner–Skan type of profile. Their definition of η_δ, however, is the value of η at which the flow velocity approaches by 99.99% the free stream velocity. In addition, they define Reynolds number with respect to a characteristic length, which is a somewhat complicated function of the displacement thickness used here. A comparison of the present results for $\Lambda = 0$ with those of Obremski *et al.*[6] shows good agreement in the value of the critical Reynolds number.

The neutral stability curves for various values of the pressure gradient parameter were obtained by solving the Orr–Sommerfeld equation numerically. For this purpose, two programs were developed—one based on the Kaplan filtering technique[7] and the other based on the technique of orthonormalized[8] integration. While these techniques are described in detail elsewhere, the principal features of these techniques follow from the fact that in the Orr–Sommerfeld equation there can be a linear combination of four independent solutions, in which two are rapidly growing and two are slowly growing solutions. Only the two slowly growing solutions, whose initial values at the outer edge of the boundary layer can easily be estimated, are found by numerical integration, for example by the Runge–Kutta method, for several steps toward the wall before either of the techniques are used to reduce or eliminate the parasitic

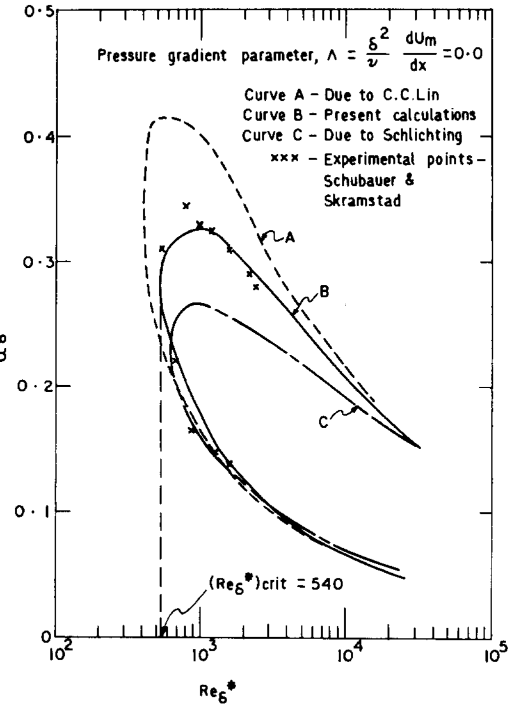

FIG. 1. Comparison of neutral stability curves at zero pressure gradient.

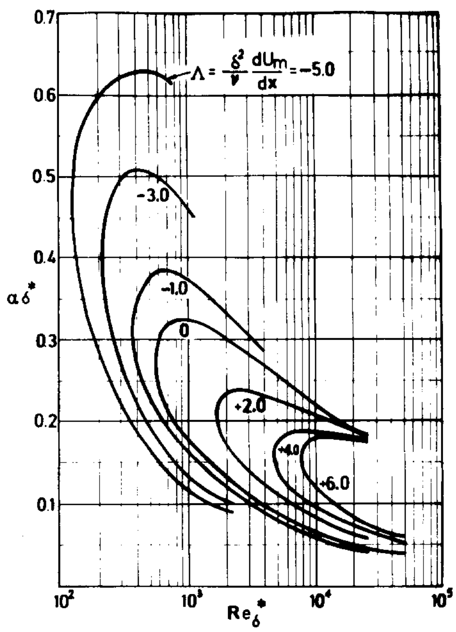

FIG. 2. Neutral stability curves for various values of pressure gradient parameter.

computation round-off errors. In the filtration technique several stored values of third and fourth derivatives of slowly growing solutions are used, as well as one rapidly growing solution, from which a very good approximation of the coefficient for linear combination is obtained. In the orthonormalization procedure, orthonormalization is done for the two linear independent solutions. The boundary conditions at the wall are satisfied by keeping C_r constant and changing the values of α and Re. Although the method based on orthonormalized integration would be sufficient for the present work, the method based on filtered integration, being faster, was employed at lower Reynolds number ($R < 10\,000$) to effect a saving in computer time. Evidently, the technique of filtered integration fails at higher

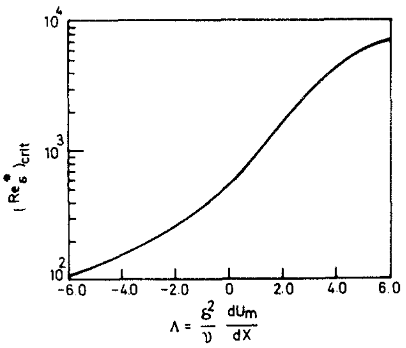

FIG. 3. Variation of critical Reynolds number with pressure gradient parameter.

Reynolds numbers because of difficulties in filtering out the very rapid growth of parasitic errors. The method could be used for comparatively high velocities by using Reynolds numbers based on the displacement thickness, instead of on the usual boundary layer thickness. The programs were run on an IBM 360/44 computer. The calculation of one point on the neutral stability curve, at fairly small Reynolds number, using filtered integration, takes about 2 min and requires four iterations, while with orthonormalized integration the same calculation takes about 3 min and requires six iterations.

The critical Reynolds numbers for various values of the pressure gradient parameter obtained by the present calculations have been presented in Table I and they compare favorably with Schlichting's results. Further, the critical Reynolds number for zero pressure gradient is approximately the same as that given in Fig. 2 of Ref. 9, which was obtained for the Falkner-Skan type of velocity profile. Obremski *et al.*[6] give a value of critical Reynolds number of 520 at zero pressure gradient.

The neutral stability curves obtained by various methods for the zero-pressure gradient case have been compared, in Fig. 1, with other authors,[10] and Fig. 2 represents the neutral stability curves for various values of the pressure gradient parameter. A plot of the critical Reynolds number as a function of the pressure gradient parameter is shown in Fig. 3, which shows the well-known result that a favorable pressure gradient stabilizes the flow, while an adverse pressure gradient has a destabilizing effect.

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