Journal of Combinatorial Theory, Series A 120 (2013) 600–622

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Notes on higher-dimensional partitions

Suresh Govindarajan

Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India

ARTICLE INFO

Article history:
Received 23 April 2012
Available online 24 November 2012

Keywords:
Higher-dimensional partitions
Skew Ferrers diagrams
Exact enumeration

ABSTRACT

We show the existence of a series of transforms that capture several structures that underlie higher-dimensional partitions. These transforms lead to a sequence of matrices whose entries are given combinatorial interpretations as the number of particular types of skew Ferrers diagrams. The end result of our analysis is the existence of a matrix, that we denote by F, which implies that the data needed to compute the number of partitions of a given positive integer is reduced by a factor of half. The number of spanning rooted forests appears intriguingly in a family of entries in the matrix, F. Using modifications of an algorithm due to Bratley–McKay, we are able to directly enumerate entries in some of the matrices. As a result, we have been able to compute numbers of partitions of positive integers $\leqslant 26$ in any dimension.

1. Introduction

An unrestricted d-dimensional partition of n is a collection of n points (nodes) in $\mathbb{Z}_+^{d+1}$ satisfying the following property: if the collection contains a node $\boldsymbol{a} = (a_1, a_2, \ldots, a_{d+1})$, then all nodes $\boldsymbol{x} = (x_1, x_2, \ldots, x_{d+1})$ with $0 \leqslant x_i \leqslant a_i$, $\forall i = 1, \ldots, d+1$ also belong to it [1,2]. Let $p_d(n)$ denote the number of distinct such partitions. Denote by $P_d(q)$, the generating function of unrestricted d-dimensional partitions: $(p_d(0) = 1)$

$$P_d(q) = \sum_{n=0}^{\infty} p_d(n) q^n.$$

E-mail address: suresh@physics.iitm.ac.in.

0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2012.11.005

There exist explicit formulae for the generating functions for $d = 1$ and $d = 2$ due to Euler and MacMahon respectively [3]. However, no such formulae exist for $d > 2$ as an inspired guess of MacMahon was subsequently proven to be false [1]. It appears that there is no simple formula and one has to take recourse to brute force enumeration. Given that asymptotically one has [4–6]

$$\log p_d(n) \sim n^{d/d+1},$$

it is easy to see that the numbers grow exponentially fast and naive enumeration is not the way to go.

The first serious attempt at direct enumeration of partitions in any dimension is due to Atkin et al. [1] based on an algorithm due to Bratley and McKay [7]. Knuth provided another algorithm that enumerates numbers of topological sequences which can be used, in principle, to generate numbers of partitions in any dimension [8]. Both algorithms are highly recursive and easily implemented on a computer.

This paper attempts to find structures in the enumeration of partitions and come up with refinements in their enumeration. Such refinements when cleverly combined with computer-based enumeration should in principle enable one to enumerate partitions of integers below some maximum value in any dimension. The maximum value turns out be 25 in our case though we believe that, with some effort, this number can be pushed to around 30.

Our refinements begin with the result of Atkin et al. who showed that the binomial transform of $p_d(n)$ leads to a lower-triangular matrix that we denote by $A = (a_{n,r})$

$$p_d(n) = \sum_{r=0}^{d+1} \binom{d+1}{r} a_{n,r}. \tag{2.1}$$

This transform implies that in order to compute partitions of a positive integer n in any dimension, we need to only compute $(n-1)$ numbers that make up a particular row of the matrix A. We show the existence of another triangular matrix, that we denote by $F = (f_{n,x})$, as a transform of the matrix A with fewer entries

$$a_{m+r+1,r} = \sum_{x=0}^{r} \sum_{p=x}^{m} \binom{r}{x} \binom{\binom{r-x}{2}}{m-p} f_{p+x+1,x}. \tag{2.10}$$

Our result is that we need only $[(n-1)/2]$ independent numbers i.e., roughly half of the initial estimate to determine partitions of n in any dimension. We illustrate the gain by explicitly displaying the first eleven rows of the matrices, A and F

$$A = \begin{pmatrix}
1 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 3 & 1 \\
0 & 1 & 5 & 6 & 1 \\
0 & 1 & 9 & 18 & 10 & 1 \\
0 & 1 & 13 & 44 & 49 & 15 & 1 \\
0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\
0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\
0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\
0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix},$$

$$F = \begin{pmatrix} 1 & & & & & \\ 0 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 3 & & & \\ 0 & 1 & 7 & & & \\ 0 & 1 & 11 & 16 & & \\ 0 & 1 & 18 & 58 & & \\ 0 & 1 & 26 & 135 & 125 & \\ 0 & 1 & 38 & 293 & 618 & \\ 0 & 1 & 52 & 574 & 1927 & 1296 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The matrix, F is, in a sense, the end-point of a sequence of transforms and matrices that we introduce. We also provide combinatorial interpretations for the various matrices that appear as a result of these transforms. This enables us to modify the Bratley–McKay (BM) algorithm to directly enumerate the matrix A that we mentioned earlier and a second matrix, C that we define in the sequel. As we discuss in Appendix A, similar refinements can be carried out for partitions restricted in a box.

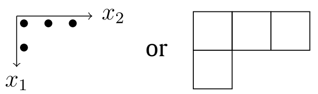
2. Structures in higher-dimensional partitions

2.1. Ferrers diagrams and permutation symmetry

A Ferrers diagram represents the partition as a $(d+1)$-dimensional arrangement of nodes. For instance, the following one-dimensional partition of 4 (corresponding to $3+1$)

$$\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right\} \quad \text{or} \quad \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{pmatrix} \quad \text{in compressed form,}$$

is represented by the following two-dimensional Ferrers diagram or as a Young diagram where we replace the nodes by squares (more generally, hypercubes).



There is a natural action of S_{d+1} on the $(d+1)$-dimensional Ferrers diagram – this corresponds to permuting the $(d+1)$ coordinates. For one-dimensional partitions, this is referred to as conjugation. The symmetry group of a d-dimensional partition is the *largest* sub-group of S_{d+1} that acts trivially on the corresponding Ferrers diagram.

2.2. The intrinsic dimension

Typically, one is interested in the asymptotic behavior of $p_d(n)$ for large number of nodes n while keeping the dimension d fixed. However, one may ask about what happens to $p_d(n)$ if we keep the number of nodes. i.e., n, fixed and keep increasing d. It is easy to see that when $d > n+1$, all the nodes of the Ferrers diagram (FD) necessarily lie in some r-dimensional hyperplane with $r < d$. This motivates the following definition (implicitly present in Atkin et al. [1]).

Definition 2.1. Given a Ferrers diagram, let it be contained in an r-dimensional hyperplane but not in any $(r-1)$-dimensional hyperplane. The intrinsic dimension (i.d.) of the Ferrers diagram is defined to be r.

Note that such an r-dimensional hyperplane is given by setting $d+1-r$ coordinates to zero. Any permutation of the $d+1-r$ coordinates (that are set to zero to obtain the hyperplane containing the nodes) does not change the Ferrers diagram. It is thus easy to see that the symmetry of a Ferrers diagram in $(d+1)$ dimensions of i.d. r is necessarily of the form $H \times S_{d+1-r}$ where $H \subseteq S_r$. We shall (somewhat loosely) call H, the symmetry of the Ferrers diagram.

Let two d-dimensional partitions be equivalent if their Ferrers diagrams are related by an S_{d+1} action. It is easy to see that all d-dimensional partitions belonging to such an equivalence class have the same intrinsic dimension. Further, given a $(d+1)$-dimensional Ferrers diagram with symmetry H and i.d. r, the number of Ferrers diagrams in its equivalence class is given by the order of the coset $S_{d+1}/(H \times S_{d+1-r})$ i.e.,

$$\frac{(d+1)!}{(d+1-r)! \times \operatorname{ord}(H)} = \binom{d+1}{r} \times \frac{r!}{\operatorname{ord}(H)}.$$

Definition 2.2. A Ferrers diagram is said to be strict when its intrinsic dimension equals its dimension.

Given a $d+1$-dimensional Ferrers diagram of i.d. r, it is useful to drop the $(d+1-r)$ dimensions that are orthogonal to the hyperplane containing the nodes thus obtaining a strict FD. The symmetry of the strict Ferrers diagram is now $H \subseteq S_r$.

Definition 2.3. A generalized Ferrers diagram (gFD) refers to the equivalence class of strict Ferrers diagrams obtained by the action of S_r on a given strict Ferrers diagram of i.d. r.

Definition 2.4. The weight of a gFD of i.d. r and symmetry $H \subseteq S_r$ is defined to be $r!/\operatorname{ord}(H)$.

Since $H \subseteq S_r$, Lagrange's theorem implies that the weight, $r!/\operatorname{ord}(H)$, is a positive non-zero integer. The number of strict FD's in a gFD of i.d. r and symmetry group H is $r!/\operatorname{ord}(H)$. To an equivalence class of a given Ferrers diagram, we associate three numbers: the number of nodes n, the i.d. r, and the weight, w. An important observation is that there exist *no* Ferrers diagram with n nodes and i.d. $r \geqslant n$ – this follows from noting that one needs at least $r+1$ nodes to create a Ferrers diagram of i.d. r. We see that the number of d-dimensional partitions is thus given by

$$\begin{aligned} p_d(n) &= \sum_{r=0}^{n-1} \binom{d+1}{r} \sum_{\lambda \vdash (n,r)} 1 \\ &= \sum_{r=0}^{n-1} \binom{d+1}{r} \sum_{[\lambda] \vdash (n,r)} w\big([\lambda]\big) \\ &:= \sum_{r=0}^{n-1} \binom{d+1}{r} a_{n,r}, \end{aligned}$$

wherein the first line, $\lambda \vdash (n,r)$, indicates that we sum over all strict FD's, λ, with n nodes and i.d. r. In the second line, $[\lambda] \vdash (n,r)$, indicates that we now sum over gFD's i.e., over equivalence classes, $[\lambda]$, of strict partitions λ with n nodes and i.d. r. Note that $a_{n,r}$ has no dependence on d and counts the numbers of strict Ferrers diagrams with n nodes and i.d. r. We shall provide a second, and more useful, combinatorial description of $a_{n,r}$ later.

2.3. The first transform

We extend $a_{n,r}$ into a lower-triangular matrix, that we denote by A, by setting $a_{n,r} = 0$ when $r \geqslant n$. Thus, we obtain the matrix $A = (a_{n,r})$ for $n = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$. With this definition, we can rewrite the above equation as

$$\boxed{p_d(n) = \sum_{r=0}^{d+1} \binom{d+1}{r} a_{nr}}. \tag{2.1}$$

To our knowledge, the above observation first appeared in a paper by Atkin et al. [1]. Thus the $p_d(n)$, for a fixed value of n, corresponds to the binomial transform of the n-th row of the matrix A. It is easy to see that $a_{n,0} = \delta_{n,1}$. The lower triangular nature of A implies that only $n-1$ numbers, $a_{n,1}, a_{n,2}, \ldots, a_{n,(n-1)}$ determine $p_d(n)$ for any d. The matrix A appears in the OEIS as sequence number A119271 [9]. The inverse binomial transform is given by

$$\boxed{a_{n,r} = \sum_{d=0}^{r-1} (-1)^{d+r+1} \binom{r}{d+1} p_d(n) \quad \text{for } n \geqslant r+1},$$

with $p_0(n) = 1$. Of course, $a_{nr} = 0$ when $n < r+1$ reflecting the lower-triangular nature of the matrix. Suppose we know all partitions of $n_{\max}$ up to $d_{\max}$. This determines the first $n_{\max}$ rows and $(d_{\max}+1)$ columns of the matrix A.

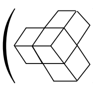
For low values of n, we can explicitly compute the entries in the A-matrix by listing the gFD's and working out their weights as we do below

$$p_d(2) = \binom{d+1}{1} w(\square\square) = \binom{d+1}{1},$$

$$p_d(3) = \binom{d+1}{1} w(\square\square\square) + \binom{d+1}{2} w\left(\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}\right) = \binom{d+1}{1} + \binom{d+1}{2},$$

$$p_d(4) = \binom{d+1}{1} w(\square\square\square\square) + \binom{d+1}{2} w\left(\begin{smallmatrix}\square\square\square\\ \square\end{smallmatrix}\right) + \binom{d+1}{2} w\left(\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}\right)$$

$$+ \binom{d+1}{3} w\left(\; \right)$$



$$= \binom{d+1}{1} + 3\binom{d+1}{2} + \binom{d+1}{3}.$$

The first few rows of the A-matrix are as follows (see also [10])

$$A = \begin{pmatrix}
1 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 3 & 1 \\
0 & 1 & 5 & 6 & 1 \\
0 & 1 & 9 & 18 & 10 & 1 \\
0 & 1 & 13 & 44 & 49 & 15 & 1 \\
0 & 1 & 20 & 97 & 172 & 110 & 21 & 1 \\
0 & 1 & 28 & 195 & 512 & 550 & 216 & 28 & 1 \\
0 & 1 & 40 & 377 & 1370 & 2195 & 1486 & 385 & 36 & 1 \\
0 & 1 & 54 & 694 & 3396 & 7603 & 7886 & 3514 & 638 & 45 & 1 \\
0 & 1 & 75 & 1251 & 7968 & 23\,860 & 35\,115 & 24\,318 & 7484 & 999 & 55 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Definition 2.5. Consider a pair of FD's (λ, μ) such that $\mu \subseteq \lambda$. Then, a skew Ferrers diagram is the set of nodes $\lambda \setminus \mu$.

One can think of the entries in the A-matrix as counting skew Ferrers diagrams obtained by deleting the node at the origin $(0,0,\ldots,0)^T$ that is contained in any Ferrers diagram. Then, $a_{n,r}$ is the number of strict FD's of dimension r obtained by adding $(n-1)$ nodes to the node at the origin. One sets $a_{1,0}=1$.

2.4. A combinatorial interpretation

We will now provide another combinatorial interpretation for the numbers a_{nr} that make up the lower-triangular matrix A. We begin with the observation that $a_{r+1,r}=1$ – this follows because there is a unique FD of i.d. r containing $r+1$ nodes. The coordinates are given in the following $r\times(r+1)$ matrix[1]

$$\mu_r := \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & 1 \end{pmatrix}. \tag{2.2}$$

This FD has maximal symmetry S_r and weight 1.

Remark. Every FD with intrinsic dimension r necessarily contains μ_r. This implies that an FD with n nodes and i.d. r can be obtained by adding $m=n-r-1$ additional nodes to μ_r. This leads to the following combinatorial interpretation for $a_{m+r+1,r}$.

Proposition 2.6. *$a_{m+r+1,r}$ is the number of strict Ferrers diagrams with i.d. r obtained by adding m nodes to the standard Ferrers diagram, μ_r.*

Let λ be an FD that contributes to $a_{n,r}$. Its symmetry group $H\subseteq S_r$ – this implies that there will be $r!/\operatorname{ord}(H)=w(\lambda)$ distinct FD's obtained from it by the action of S_r.

So far we have completely determined the first 26 rows of the A-matrix. The entries have been determined by combining several methods: (i) taking the inverse binomial transform of known numbers for higher-dimensional partitions, (ii) by direct enumeration using the combinatorial interpretation, (iii) by determining another matrix, C, that we introduce later and (iv) finally by determining the matrix F. It is important to note that the numbers, when available, from the different methods agree. Further, none of the conjectural formulae are used in determining the entries.

2.5. The second transform

Definition 2.7. Let λ be an FD of i.d. r and consider the skew FD $\lambda\setminus\mu_r$. Let the nodes of the skew FD be contained in an x-dimensional hyperplane (obtained by setting $r-x$ coordinates to zero) but not in any $(x-1)$-dimensional hyperplane. The reduced dimension (r.d.) of the FD λ is said to be x.

Clearly the reduced dimension of an FD is always less than or equal to its intrinsic dimension. The symmetry of an FD with i.d. r and r.d. x is necessarily of the form $H\times S_{r-x}\subset S_r$. Then, one has

$$a_{m+r+1,r} = \sum_{x=0}^{r} \binom{r}{x} c_{m,x}, \tag{2.3}$$

where the binomial term $\binom{r}{x}$ gives the number of ways x coordinates can be chosen in dimension r and $c_{0,0}=1$ and $c_{m,0}=c_{0,m}=0$ for $m>0$.

(1) The coefficients $c_{m,x}$ are clearly independent of the i.d. r as they are related to the skew FD's with m nodes and r.d. x.
(2) We say that a skew FD is *strict* if its dimension and r.d. are the same.

[1] Recall that each column is the coordinate of a node and thus there are $(r+1)$ columns and r rows.

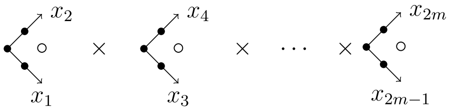
(3) Let us denote the equivalence class of strict skew Ferrers diagrams, $\lambda \setminus \mu_x$, under the S_x action as an sFD. All skew FD's in an sFD will have identical reduced and intrinsic dimensions. Thus, given such a skew Ferrers diagram with symmetry $H \subseteq S_x$, its equivalence class will contain $\frac{x!}{\mathrm{ord}(H)}$ distinct skew Ferrers diagrams.
(4) The $c_{m,x}$ are non-negative integers since they count the number of strict skew FD's with m nodes and r.d. x.
(5) For fixed m, one can see that the maximum value of r.d. with m nodes is $2m$. This enables us to convert the above equation into a second binomial transform

$$\boxed{a_{m+r+1,r} = \sum_{x=0}^{2m} \binom{r}{x} c_{m,x}}, \tag{2.4}$$

where we extend $c_{m,x}$ into a matrix, $C = (c_{m,x})$, by setting $c_{m,x} = 0$ for $x > 2m$.
(6) For fixed m, we can consider $a_{m+r+1,r}$ as a function of r. The function $g_m(r) := 2m!!a_{m+r+1,r}$ is a polynomial of degree $2m$, conjecturally with integer coefficients, in the variable r and $g_m(0) = 0$ for $m > 0$.
(7) We have directly determined eleven rows ($m \in [0, 10]$) of the C-matrix. The first few rows of the C-matrix are

$$C = \begin{pmatrix} 1 & & & & & & & & & & \\ 0 & 1 & 1 & & & & & & & & \\ 0 & 1 & 3 & 6 & 3 & & & & & & \\ 0 & 1 & 7 & 20 & 46 & 45 & 15 & & & & \\ 0 & 1 & 11 & 61 & 198 & 480 & 645 & 420 & 105 & & \\ 0 & 1 & 18 & 138 & 706 & 2508 & 6441 & 10\,395 & 9660 & 4725 & 945 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

It is easy to see that there is only one sFD with m nodes and r.d. $2m$. In the picture below, the m nodes of the sFD are indicated by open circles. The filled circles indicate the nodes of μ_{2m} that must be added to the sFD to obtain an FD.



(2.5)

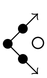
The symmetry of the skew FD is $(S_m \ltimes \mathbb{Z}_2^m)$ and thus $c_{m,2m}$ is the dimension of the coset i.e.,

$$c_{m,2m} = \frac{\dim(S_{2m})}{\mathrm{ord}(S_m \ltimes \mathbb{Z}_2^m)} = \frac{(2m)!}{(2m)!!} = (2m-1)!!.$$

Definition 2.8. A skew FD of i.d. r is said to be reducible if a proper subset of its nodes is contained in a d-dimensional hyperplane (obtained by setting $r - d$ coordinates to zero) with $d < r$ and the nodes not in the proper subset lie in the orthogonal $(r - d)$-dimensional hyperplane (obtained by setting the other d coordinates to zero).

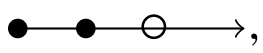
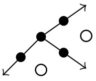
Definition 2.9. We say that an FD, λ, of i.d. r is reducible if the skew FD, $\lambda \setminus \mu_r$ is reducible.

Thus a reducible sFD has multiple *components* consisting of non-intersecting proper subsets of its nodes lying in mutually orthogonal hyperplanes. Thus the sFD given in Eq. (2.5) is reducible with m components each of which is isomorphic to the irreducible sFD σ_2 defined as follows:

$\sigma_2 =$  $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$ (2.6)

We can thus write the sFD (2.5) as $\sigma_2 \times \sigma_2 \times \cdots \times \sigma_2 = \sigma_2^m$.

Similarly, one has two distinct sFD's with $x = 2m - 1$ and the two sFD's are reducible containing σ_2^n (for some suitable value of n) as one of the components and the other component are the following two irreducible sFD's that contribute to $c_{1,1}$ and $c_{2,3}$ respectively

$\sigma_1 =$  , $\sigma_3 =$  $= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, (2.7)

(a) (b)

where we have called the second sFD σ_3 – it has two nodes and has r.d. 3. In other words, $c_{2m,2m-1}$ has contributions from two sFD's – one of the form $\sigma_2^{(m-1)} \times \sigma_1$ and the other of the form $\sigma_2^{(m-2)} \times \sigma_3$. Studying the symmetries of these two sFD's with r.d. $(2m-1)$, one obtains

$$c_{m,2m-1} = \frac{(2m-1)!}{(2m-2)!!} + \frac{(2m-1)!}{2(2m-4)!!} = m \times (2m-1)!!.$$

Clearly, such a diagrammatic method will enable one to write further formulae (we will provide a few more in Appendix A) for $c_{m,x}$. However, it can get tricky to find all possible diagrams. Keeping this in mind, we make the following definition.

Definition 2.10. The density, ρ, of an sFD with m nodes and r.d. x is $\rho := m/x$.

The density of an sFD is always greater than or equal to $\frac{1}{2}$ since $c_{m,x} = 0$ when $x > 2m$.

Proposition 2.11. *When its density is in the range $(\frac{1}{2}, \frac{2}{3})$, an sFD with m nodes and r.d. x is necessarily reducible and one of its components is the sFD, $(\sigma_2)^n$, for some $n \geqslant n_{\min} = 2x - 3m$.*

The result follows from Proposition 2.15 that we prove later. When $\rho < 2/3$, the proposition implies it is impossible to construct an sFD that does not contain σ_2 as a component. The first new sFD, σ_3, appears at $\rho = \frac{2}{3}$. The minimum value of n is fixed by the condition that the density of the sFD goes past or equals $\frac{2}{3}$ after deleting the nodes that appear in $(\sigma_2)^n$ i.e., it is smallest value of n such that

$$\frac{m-n}{x-2n} \geqslant \frac{2}{3} \quad \Rightarrow \quad n \geqslant 2x - 3m.$$

2.6. The third transform

Proposition 2.11 suggests that in counting the skew FD's that contribute to $c_{m,x}$, we can remove components isomorphic to σ_2 in reducible skew FD's and only count skew FD's that do not contain any σ_2 components. This motivates the next transform where we introduce a new matrix $D = (d_{m,x})$

$$\boxed{c_{m,x} = \sum_{y=y_{\min}}^{m} \frac{x!}{(2y)!!(x-2y)!} d_{m-y,x-2y}}, \tag{2.8}$$

with $d_{0,0} = 1$, $d_{m,0} = d_{0,m} = 0$ for $m > 0$ and $y_{\min} = 2x - 3m$. The pre-factor in the transform is determined by the order of the symmetry of σ_2^y which is $2^y y! = (2y)!!$.

(1) $d_{m,x}$ counts the number of skew FD's with m nodes and r.d. x not containing σ_2 as a component. Thus it is non-negative.
(2) Proposition 2.11 implies that $d_{m,x} = 0$ when $m/x > 2/3$. This is stronger than the condition $m/x > 1/2$ implied by the property of the C-matrix.

(3) It is useful to rewrite the transform as follows:

$$c_{m,2m-z}=\sum_{y=\lceil z/2\rceil}^{2z}\frac{(2m-z)!}{(2m-2y)!!(2y-z)!}d_{y,2y-z}.$$

In this form, one sees that completely determining row z of the D-matrix leads to a nice compact formula for $c_{m,2m-z}$. The D-matrix clearly contains fewer terms than the C-matrix since $d_{m,x}=0$ when $\rho<2/3$.

(4) To illustrate the transform, consider $c_{m,2m-1}$ which we have already computed. One sees that

$$\begin{aligned}c_{2m,2m-1}&=\sum_{y=1}^{2}\frac{(2m-1)!}{(2m-2y)!!(2y-1)!}d_{y,2y-1}\\&=\frac{(2m-1)!}{(2m-2)!!}d_{1,1}+\frac{(2m-1)!}{3!(2m-4)!!}d_{2,3}.\end{aligned}\tag{2.9}$$

It is easy to see that $d_{1,1}=1$ as there is precisely one sFD, σ_1 and $d_{2,3}=3$ as there are three inequivalent diagrams under the action of S_3 on the sFD, σ_3.

(5) When $\rho=2/3$, there is only one sFD, σ_3^m, that contributes to $d_{2m,3m}$. This implies that

$$d_{2m,3m}=\frac{(3m)!}{m!2^m},\quad m=1,2,3,\ldots.$$

The first few rows of the matrix D are

$$D=\begin{pmatrix}1&&&&&&&\\0&1&&&&&&\\0&1&3&3&&&&\\0&1&7&17&28&&&\\0&1&11&58&156&295&90&\\0&1&18&135&640&1913&3786&2310\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}.$$

2.7. The final transform

The main advantage of the D-matrix is that it contains fewer terms than the C-matrix. Using it, we have arrived at formulae for $c_{m,2m-z}$ for $z=2,3,4,5$ analogous to the one in Eq. (2.9) that can be obtained, in principle, from the matrix D. Can we do better? We saw that as the density increased from 1/2 to 2/3, only one irreducible diagram appears. At $\rho=\frac{3}{4}$, two new sFD's appear. They are

$$\sigma_{4a}=\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\end{pmatrix},\qquad\sigma_{4b}=\begin{pmatrix}1&1&1\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

In fact, one can define another transform that removes reducible components of type σ_3 from sFD's that contribute to the matrix, D, for $\rho\in(2/3,3/4)$. The next proposition will enable to do this and a lot more by removing a whole family of reducible components that necessarily appear in sFD's with $\rho<1$.

Definition 2.12. Let $\mathcal{D}:=\bigcup_r\mathcal{D}_r$, where $\mathcal{D}_r$ denotes the set of strict Ferrers diagrams of dimension r consisting only of nodes of the form $(1,1,0,\ldots,0)^T$ or its S_r images in addition to the nodes present in μ_r.

We say, somewhat loosely, that a strict skew FD, σ of r.d. x is in $\mathcal{D}$ if the FD $\mu_x \cup \sigma \in \mathcal{D}$. One can show that σ_2, σ_3 and $\sigma_{4a/b}$ are the only irreducible strict skew Ferrers diagrams in dimensions 2, 3 and 4 respectively that appear in $\mathcal{D}$.

Let $e_{m,r}$ denote the number of Ferrers diagrams in $\mathcal{D}$ obtained by adding m nodes to μ_r. It is easy to see that $e_{m,x} = \binom{\binom{x}{2}}{m}$ as there are $\binom{x}{2}$ possible nodes from which we need to choose m nodes. We define a new transform that removes reducible components that are in $\mathcal{D}$

$$\begin{aligned} a_{m+r+1,r} &= \sum_{x=1}^{r}\sum_{p=0}^{m}\binom{r}{x} e_{m-p,r-x} f_{p+x+1,x} \\ &= \sum_{x=1}^{r}\sum_{p=0}^{m}\binom{r}{x}\binom{\binom{r-x}{2}}{m-p} f_{p+x+1,x}, \end{aligned} \tag{2.10}$$

where in the second line we use the explicit formula for $e_{m,x}$ and $f_{1,0} = 1$, $f_{n,0} = f_{1,n-1} = 0$ for $n > 1$. In the first line, a typical term in the summation on the right hand side consists of reducible strict FD's with the component in $\mathcal{D}$ having i.d. $r-x$ and $(m-p)$ nodes added to μ_{r-x} and the other component consisting of an strict FD with no reducible component in $\mathcal{D}$, i.d. and r.d. x and p nodes added to μ_x – their number is counted by $f_{p+x+1,x}$. The binomial factor $\binom{r}{x}$ is the number of ways one can choose x dimensions occupied by the FD's contributing to $f_{p+x+1,x}$. The above formula defines a new matrix $F = (f_{n,r})$. The entry $f_{r+m+1,r}$ is the *the number of strict FD's of i.d. r obtained by adding m nodes to μ_r and does not contain any reducible components that are in $\mathcal{D}$*. Such an FD must necessarily have r.d. also equal to r, else it will necessarily have a reducible component isomorphic to μ_{r-x} if its r.d. is x.

It is easy to see that $f_{r+1,r} = 0$. The only contribution to $a_{r+1,r}$ is the unique FD μ_r which is $\mathcal{D}$. Similarly, $f_{r+2,r} = 0$ when $r > 1$ as the only contribution to $a_{r+2,r}$ is of the form $\sigma_1 \times \sigma_2^{r-1}$. One also has $f_{3,1} = 1$ with σ_1 being the unique FD contributing to it. The next proposition shows the advantage of defining the F-matrix.

Proposition 2.13. $f_{m+r+1,r} = 0$ *when* $r > m$.

Proof. Let λ be an FD of i.d. r with $m+r+1$ nodes that contributes to $f_{m+r+1,r}$. Consider the skew FD, $\lambda \setminus \mu_r$ – it has m nodes. It must be a strict skew FD otherwise it has an irreducible component isomorphic to μ_x for some $x < r$. Thus, the proposition implies that there are no strict skew FD's with density $\rho = m/r < 1$.

We can also assume that the skew FD is irreducible – if it is reducible, it must necessarily have at least one irreducible component with density < 1 and we can focus on (proving the non-existence) such irreducible components. Our goal is thus reduced to proving that there are no irreducible strict skew FD's with density < 1.

Definition 2.14. Let us call the nodes obtained by all permutations of the coordinates of the node $(1,1,0,\dots,0)^T$ as nodes of type 1. Similarly, call the nodes obtained by permuting coordinates of $(2,0,\dots,0)^T$ as type 2. Nodes of type 3 are nodes that are not of type 1 or 2.

Examples of type 3 nodes include $(1,1,1,0,\dots,0)^T$ and $(3,0,\dots,0)$. Such nodes cannot be added to the FD μ_r without including supporting nodes of type 1 and 2. The addition of nodes of type 3, when possible, never increases the r.d. of an FD and therefore increases the density. Thus, given an FD λ (of i.d. r and r.d. r) containing type 3 nodes, we can form a new FD λ' with the same r.d. but lower density by removing all nodes of type 3. Further, if $\lambda \setminus \mu_r$ is irreducible, $\lambda' \setminus \mu_r$ is also irreducible. The skew FD $\lambda' \setminus \mu_r$ thus consists of nodes of type 1 and type 2. If it consists of only nodes of type 1, then $\lambda' \in \mathcal{D}$. Thus, we only need to consider irreducible strict skew FD's containing at least one node of type 2.

For the rest of the discussion, let λ' be an FD such that $\lambda' \setminus \mu_r$ is an irreducible strict skew FD containing only nodes of type 1 and at least one node of type 2. It is easy to see that removing a node of type 2 does not affect the irreducibility of the skew FD. Further, it does not reduce the r.d. as the only way a type 2 node can reduce the r.d. of a skew FD is when it appears as a part of a reducible component isomorphic to σ_1. Thus, we can delete all type 2 nodes to obtain a new FD λ'' that is irreducible and contains only type 1 nodes. Again, it is easy to see that $\rho(\lambda'') \leqslant \rho(\lambda')$. Further $\lambda'' \in \mathcal{D}$. Thus, one has the sequence

$$\rho\big(\lambda''\big) \leqslant \rho\big(\lambda'\big) \leqslant \rho(\lambda).$$

Let $\lambda' \setminus \mu_r$ have $(r-1)$ nodes so that its density is just below one and contain z nodes of type 2. Then, $\lambda'' \setminus \mu_r$ will have $(r-1-z)$ nodes and be irreducible. The next proposition shows that such a λ'' does not exist. Hence, there exists *no* FD λ'' and hence no FD λ' with density <1. □

Proposition 2.15. *The only strict FD's in $\mathcal{D}$ of i.d. r such that the skew FD $\lambda \setminus \mu_r$ is strict and irreducible with density less than* 1 *necessarily have $\rho = \frac{r-1}{r}$.*

Proof. Let us assume that $\lambda \setminus \mu_r$ has $(r-2)$ nodes and is irreducible. Let us try to construct such a strict skew FD and we will see that there are not enough nodes. Start by putting the first type 1 node in the x_1x_2 plane. The irreducibility condition implies that the second node must be either in the x_1x_α or x_2x_α plane where α is not 1 or 2. The key point is that the additional node must contain one of the used up coordinates, x_1 or x_2 in this case and a new coordinate so that irreducibility is maintained. Clearly, such a process needs $(r-1)$ nodes to get an irreducible skew FD $\lambda \setminus \mu_r$ with r.d. r. This is impossible. Hence, there exists *no* irreducible skew FD λ with density $\frac{r-2}{r}$. It is easy to extend the argument to exclude even lower densities. Thus, the only possibility that is not ruled out is to have strict skew FD's with $(r-1)$ nodes with r.d. r – these have density $\frac{r-1}{r}$. □

2.7.1. Properties of the F-matrix

(1) The most important property is the one implied by Proposition 2.13 which says that the F-matrix is lower triangular with $f_{n,r}=0$ when $r>(n-1)/2$. For fixed value of n, the F-matrix has far fewer terms (roughly half) than the corresponding row in the A-matrix. We have determined the first 26 rows of the matrix F.

(2) It turns out that there are other transforms that also lead to matrices with fewer entries like the matrix F. See for instance, the box transform that we consider in Appendix A. However, their relationship to the matrix A is not as simple as Eq. (2.10). The simplicity of Eq. (2.10) is what picks out the matrix F as special.

(3) We can also use this idea to refine the counting problem associated with the matrix C. Let $C^{\mathcal{D}} = (c^{\mathcal{D}}_{m,x})$ denote the contributions to the matrix C that arise from FD's that are in $\mathcal{D}$. Since the set $\mathcal{D}_r$ is invariant under S_r, it is easy to see that $C^{\mathcal{D}}$ is given by the transform

$$\binom{\binom{x}{2}}{m} = \sum_{x=0}^{2m} \binom{r}{x} c^{\mathcal{D}}_{m,x}.$$

Then, we can define $\widetilde{C} = (\widetilde{c}_{m,x})$ by removing contributions that arise from reducible parts that are isomorphic to contributions to $C^{\mathcal{D}}$. Then, one has

$$c_{m,x} = \widetilde{c}_{m,x} + c^{\mathcal{D}}_{m,x} + \sum_{y=1}^{x-1}\sum_{p=1}^{m-1} \binom{x}{y} c^{\mathcal{D}}_{m-p,x-y}\widetilde{c}_{p,y}.$$

Given a strict skew FD that contributes to $\widetilde{c}_{m,x}$, it is easy to see that there is a unique FD obtained by adding nodes in μ_x to the skew FD. Further, this FD must contribute to the entry $f_{m+x+1,x}$ in the matrix F. Since the converse also holds i.e., given a strict FD of i.d. x that contributes to the matrix F, the skew FD obtained by deleting nodes in μ_x gives a skew FD that contributes to $\widetilde{C}$. Thus, one has

$$\widetilde{c}_{m,x} = f_{m+x+1,x}.$$

We observe numerically that $f_{2m+1,m} = (m+1)^{m-1}$ for $m = 0, 1, 2, \dots, 12$. That it holds for all m follows from Proposition 2.16, which defines and provides a formula for $f_{2m+1,m}(\alpha)$, since $f_{2m+1,m} = \sum_\alpha f_{2m+1,m}(\alpha)$. These numbers appear in the sequence numbered A000272 in the OEIS [9]. The next proposition presents a further refinement. We need a few definitions which we briefly state. A graph, consisting of vertices and undirected edges, with no cycles is called an acyclic graph or a forest. A forest may consist of disconnected components and is called a tree if it has only one connected component. A rooted tree is one with a marked/special vertex (called the root) while a rooted forest is one in which every component is rooted. A spanning forest is any subgraph that is both a forest (contains no cycles) and spanning (includes every vertex) [11,12].

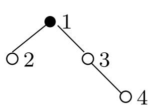
Proposition 2.16. *Let α be the number of nodes of type 2 contained in an FD that contributes to $f_{2m+1,m}$. Let $f_{2m+1,m}(\alpha)$ denote the total number of such Ferrers diagrams. Then, $f_{2m+1,m}(\alpha)$ is the number of spanning rooted forests on m vertices and α components. It follows from a result due to Cayley on the numbers of spanning rooted forests that* [13]

$$f_{2m+1,m}(\alpha) = \binom{m-1}{\alpha-1} m^{m-\alpha}.$$

Proof. We will provide a bijective map relating FD's that contribute to $f_{2m+1,m}(\alpha)$ to spanning rooted forests on m vertices and α components. There is a natural action of S_m on both sides – on the FD side, it corresponds to permuting the m coordinates and on the rooted forest side, it corresponds to relabeling the m nodes. We identify these two groups.

Given a skew FD that contributes to $f_{2m+1,m}(\alpha)$, we can construct a graph with m vertices labeled from $(1, \dots, m)$ as follows. The type 2 nodes become root vertices carrying the label of the non-vanishing coordinate. Thus if a type 2 node has non-vanishing j-th coordinate, assign it the label j. Add $(m-\alpha)$ vertices and label them with the unused labels. Every type 1 vertex has two non-vanishing coordinates, say the j-th and k-th coordinates. Assign an edge that connects vertex j to vertex k. Repeat for all type 1 nodes. In this process, there are as many components as there are type 2 nodes. Thus the graph is a spanning rooted forest on m vertices and α components. The following example illustrates the map for $m = 4$ and $\alpha = 1$. The root vertex is shown by a filled circle.

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow$$

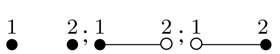


To prove the converse statement, given a spanning rooted forest with m vertices and α components, we need to construct an FD that contributes to $f_{2m+1,m}(\alpha)$. This is easy to do . Pick the root vertices and assign them to type 2 nodes whose non-vanishing coordinate decided by the label of the vertex. Next assign to all edges a type 1 node that has non-vanishing coordinates at precisely the locations decided by the labels of the vertices it connects. We thus recover the skew FD. □

Example. We know that $f_{5,2} = 3$. The three skew FD's are

$$\sigma_1^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

Note that there are two equivalence classes of skew FD's. Under S_2 action as the second and third skew FD's get mapped to each other.



3. Other matrices

3.1. New matrices

So far, we have considered transforms that lead to new matrices, A, C, D, F, all of which have non-negative entries since they all count numbers of skew Ferrers diagrams. We will now provide two other transforms that are partly conjectural and lead to matrices that are not positive definite – we denote the entries with Greek letters to remind us of this. We begin by expanding the entries in the matrix, A, as follows. Let

$$a_{m+r+1,r} = \sum_{z=0}^{2m} \alpha_{m,z} \frac{r^{2m-z}}{(2m)!!},$$

with $\alpha_{m,0} = 1$ for $m \geqslant 0$ and $\alpha_{m,2m} = 0$ for $m > 0$. The above transform provides the entries for another triangular matrix, $\alpha_{m,z}$, that denote by α by setting $\alpha_{m,z} = 0$ for $z > 2m$. One can explicitly relate the $\alpha_{m,z}$ to the entries in the C-matrix using Stirling numbers of the first kind.

Conjecture 3.1. *The entries of the matrix α, i.e., $\alpha_{m,z}$, are all integers.*

This is true for the first ten rows and appears to hold for the first eleven rows which have been determined using conjectures.

The second conjecture introduces a new matrix, that we denote by β, and its associated transform. It has been determined experimentally and verified to hold to the extent possible.

Conjecture 3.2. *The matrix α admits the following decomposition*

$$\alpha_{m,z} = \sum_{y=0}^{\lfloor z/2 \rfloor} \binom{m}{z-y} \beta_{z,y},$$

with $\beta_{0,0} = 1$ and $\beta_{2y,y} = 0$ for all $y > 0$.

By setting $\beta_{z,y} = 0$ for $y > \lfloor z/2 \rfloor$, this becomes the binomial transform

$$\alpha_{m,z} = \sum_{y=0}^{m} \binom{m}{z-y} \beta_{z,y}.$$

The inverse transform is

$$\beta_{z,y} = \sum_{m=0}^{z-y} (-1)^{m+z-y} \binom{z-y}{m} \alpha_{m,z}.$$

We now state a conjecture of Meeussen that fixes one of the coefficients.

Conjecture 3.3 *(Meeussen).*

$$\beta_{n,0} = H_n\left(\frac{1}{2}\right),$$

where $H_n(x)$ is the n-th Hermite polynomial.

Recall that the matrix α has $2m$ non-zero entries in the m-th row. The matrix β has fewer terms, roughly half the entries in the matrix α. We were able to determine eleven rows of the matrices, α and C, using the matrix β. Ten of these rows were verified through other means. This was our main motivation in searching for and finding the combinatorial problem that eventually lead to the matrix F.

3.2. The B-matrix

We now construct another lower triangular matrix $B = (b_{n,r})$ with $n = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$ and $b_{n,0} = 1$

$$p_d(n) = \sum_{r=0}^{n-1} \binom{d}{r} b_{n,r} = 1 + \sum_{r=1}^{n-1} \binom{d}{r} b_{n,r}. \tag{3.1}$$

The matrix B appears in the OEIS as sequence number A096806. Using Pascal's identity

$$\binom{d+1}{r} = \binom{d}{r} + \binom{d}{r-1},$$

we can relate the matrix B to A. Thus, one has the relation

$$b_{n,r} = a_{n,r} + a_{n,r+1}.$$

One can easily show that $b_{n,n-1} = 1$ using the above formula and known properties of the matrix A. The first six rows of B have been determined explicitly, for instance, in Andrews' book on Partitions [2]. It is easy to check that the above relation holds for all six rows.

3.3. Hanna's matrix

Conjecture 3.4 *(Hanna). There exists a lower-triangular matrix $T = (\tau_{ij})$ (with $i, j = 0, 1, 2, \ldots$) with integral entries and ones on its diagonal such that*

$$p_d(n) = \sum_{j=0}^{n} (T^d)_{n,j}.$$

In other words, the sum of the n-th row of the d-th power of T gives the d-dimensional partition of n. This matrix appears in the OEIS as sequence A096651. Since $p_d(0) = 1$, we can set $\tau_{0,0} = 1$ and $\tau_{j,0} = 0$ for $j > 0$. For the rest of the discussion, we will consider $n > 0$ and can delete the zeroth row and column of the T-matrix as they no longer play a role. We shall however use the same symbol T to denote the modified matrix as it is easy to reconstruct the original T-matrix by adding back the zeroth row and column. We shall prove the existence as well as the integrality of the matrix T by constructing an explicit map that relates T to the matrix B (and hence A) that we considered in the previous section.

Proof of Conjecture 3.4. For $n \geqslant 1$, the Hanna conjecture can be written as

$$p_d(n) = \sum_{j=1}^{n} (T^d)_{n,j} = \sum_{x_1 \cdots x_d} \tau_{n,x_1} \tau_{x_1,x_2} \cdots \tau_{x_{d-1},x_d},$$

where $n \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_d \geqslant 1$. It obviously holds for $n = 1$ since $\tau_{11} = 1$. Using the fact that T has ones in its diagonal, we can simplify the above expression to

$$p_d(n) = 1 + \sum_{r=1}^{n-1} \binom{d}{r} \sum_{x_1 \cdots x_r} \tau_{n,x_1} \tau_{x_1,x_2} \cdots \tau_{x_{r-1},x_r}$$

with sum now running over all sequences of r positive non-zero integers $(x_1, \dots, x_r)$ such that $x_0 = n > x_1 > x_2 > \cdots > x_r \geqslant 1$. The combinatorial factor expresses the number of ways in which diagonal elements are chosen. Comparing the above equation with Eq. (3.1) implies the (potential) identity for $n > 1$ and $r \geqslant 1$

$$\sum_{x_1 \cdots x_r} \tau_{n,x_1} \tau_{x_1,x_2} \cdots \tau_{x_{r-1},x_r} = b_{n,r}, \tag{3.2}$$

with $n > x_1 > x_2 > \cdots > x_r \geqslant 1$. Let us assume that this relation holds for $n < m$ (for some $m > 1$) and that we have determined $m-1$ rows of T. Then, we can rewrite the above equation as

$$\sum_{1 \leqslant x < m} \tau_{m,x} b_{x,r-1} = b_{m,r} \quad \text{for } m > r \geqslant 1. \tag{3.3}$$

The above $(m-1)$ equations are linear equations in $(m-1)$ unknowns: $(\tau_{m,1}, \dots, \tau_{m,m-1})$ – these are the undetermined entries in the m-th row of T. Hence, they have a solution if the matrix (constructed using $b_{x,(r-1)}$) is invertible. The matrix is upper triangular with ones in its diagonal. Hence it is has determinant one and hence is invertible. This enables us to recursively determine all the entries in the matrix T. This proves the *existence* of T.

We shall inductively prove the integrality of the matrix T using more explicit details of Eq. (3.3). We begin with the equation for $r = m-1$ and it gives

$$\tau_{m,m-1} b_{m-1,m-2} = b_{m,m-1} \quad \Rightarrow \quad \boxed{\tau_{m,m-1} = 1},$$

where we have used $b_{m,m-1} = 1$ for $m \geqslant 1$. Next consider, $r = m-2$. This equation gives $\tau_{m,m-2} + \tau_{m,m-1} b_{m-1,m-3} = b_{m,m-2}$ which gives

$$\tau_{m,m-2} = b_{m,m-2} - \tau_{m,m-1} b_{m-1,m-3},$$

where we have used the fact that $\tau_{m,m-1}$ has been solved for and shown to be integral in the previous step. Note that this implies that $\tau_{m,m-2}$ is integral. Proceeding in this manner from $r = (m-1)$ to $r = 1$, we thus determine all the unknowns. A typical equation will take the form (reflecting the triangular nature of the equations)

$$\boxed{\tau_{m,m-r} = b_{m,m-r} - \sum_{x=m-r+1}^{m-1} \tau_{m,x} b_{x,m-r}},$$

for $r = 1, 2, \dots, m-1$. We assume that $\tau_{m,m-r'}$ is integral for all $r' < r$. Thus the right hand side is integral as it only contains integral terms. Hence $\tau_{m,m-r}$ is integral. This concludes the proof of integrality of the matrix T. □

We now state an unproven conjecture of Hanna and Meeussen.

Conjecture 3.5 *(Hanna–Meeussen). $m!\tau_{m+r+1,m}$ is a polynomial of degree m in r with integral polynomial coefficients.*

It is easy to show that $\tau_{m+r+1,m}$ is a polynomial of degree $2m-1$ in r using the properties of the matrix A. However, the above conjecture is stronger and seems to consistent with known data for $m = 0, 1, \dots, 11$.

4. Practical considerations

This section provides details on the exact enumeration of higher-dimensional partitions as well as the matrices defined in this paper. With access to high-performance computing getting easier in recent times, this is indeed an additional computational aspect that can and must be added to the theoretical discussion of the previous section. We will first discuss the algorithms that we used and then discuss exact enumerations as we carried out.

4.1. Algorithms for higher-dimensional partitions

There are two algorithms in the literature for computing higher-dimensional partitions. The first one is due to Bratley and McKay (the BM algorithm) [7] and the second one is due to Knuth [8] – both are more than 40 years old reflecting the lack of progress in this area. Both are highly recursive and provide distinct ways of exactly enumerating higher-dimensional partitions.

The BM algorithm. The partitions in any fixed dimension, say d, form a tree which we call the *partition* tree in $(d+1)$ dimensions[2] and which we denote by the symbol $\mathcal{T}_{d+1}$. Every node of the tree is the Ferrers diagram associated with a partition. The unique Ferrers diagram containing one point is the root node of the tree. New partitions can be formed by adding or deleting a point from the Ferrers diagram.[3] The children of the node are those partitions which are obtained by adding a point to its Ferrers diagram. The depth of a node is the number of points in the partition.

The BM algorithm recursively traverses the tree up to some fixed depth, say n, such that each node is visited precisely once. The heart of the algorithm is the routine called *part* that takes three arguments and is recursively called in the algorithm. Every time a node is visited, the partition is stored in an array called *current* and presented to user. If one is interested in only counting the number of partitions of an integer in a given dimension, if the current partition has m points, increment a suitable counter, call it $p_d(m)$, by one. At the end of the program, the counter thus contains the number of partitions of m.

The Knuth algorithm. Let $S_m = \mathbb{N}^m$ denote the set of points in the totally positive orthant in a hyper cubic lattice. Let $d_m(k)$ denote the number of topological sequences with index k (see [8,6] for definitions). Then a theorem due to Knuth [8] relates the numbers of topological sequences to numbers of partitions. To be precise, one has

$$p_m(n) = \sum_{k=0}^{n} d_m(k) p_1(n-k).$$

Since one-dimensional partitions are easily enumerated from the generating function, it is simple to generate $p_m(n)$ given $d_m(k)$ for all $k \leqslant n$. Knuth provided an algorithm to generate and count all topological sequences – he illustrated this method by generating numbers for the numbers of solid partitions for integers $\leqslant 28$. Recently, a parallelized version of this algorithm was used by the author and other collaborators to enumerate solid partitions of integers $\leqslant 68$ [6].

Remark. An important aspect of the BM algorithm is that its memory usage is of the order of nd bytes, where d is the dimension and n is the maximum depth. This is vastly superior to the Knuth algorithm, where a similar problem needs memory of the order of n^{d-1} bytes. However, when memory isn't an issue, our implementation of the Knuth algorithm typically takes less time than our implementation of the Bratley–McKay algorithm.

The modified BM algorithm. We begin with the observation that a suitably chosen sub-tree of the partition tree in r dimensions, $\mathcal{T}_r$ generates all partitions that contribute to the r-th column of the A-matrix i.e., $a_{n,r}$. The head node of this sub-tree is the Ferrers diagram μ_r defined in Eq. (2.2). The rest of the tree is generated by adding points to μ_r. Let us denote this sub-tree by $\mathcal{V}_r$ and the depth of this tree is clearly m where $m = n - r - 1$.

[2] Recall that the Ferrers diagram for a d-dimensional partition is a set of points in $d+1$ dimensions.

[3] To avoid confusion, in this section alone, we shall refer to nodes of a partition as points in the Ferrers diagram. This is to avoid confusion with the node of the tree.

The BM algorithm was designed to recursively traverse the partition tree visiting each node precisely once. The starting point of the algorithm is the root node whose Ferrers diagram consists of one point. Our idea is to change the initial configuration in the BM algorithm to the Ferrers diagram, μ_r and then call the recursive routine *part* with suitably chosen arguments.[4] For this modification to work correctly, the program should traverse the sub-tree $\mathcal{V}_r$ visiting each node precisely once to the chosen depth. This turned out to be easier as we experimentally observed that the sub-tree $\mathcal{V}_r$ appeared naturally in the original BM algorithm for low values of r. We then checked that the modified BM algorithm correctly generated entries in the A-matrix for $r \leqslant 10$. A careful analysis of the BM algorithm shows that this is indeed the case.

Thus, once we have the modified BM algorithm correctly traversing the sub-tree $\mathcal{V}_r$, we can do the following:

- Count the number of nodes at each depth – this gives the number $a_{m+r+r,r}$.
- At each node, numerically compute the reduced dimension, x of the Ferrers diagram. Then organizing the partitions by depth and r.d., we determine $\binom{r}{x} c_{m,x}$. The binomial prefactor is present since all $x \leqslant r$ will appear. This also implies that the algorithm is inefficient computationally for obtaining entries in the C-matrix.

A wish list of algorithms. As we just mentioned, the current algorithm to enumerate entries in the C-matrix is computationally inefficient as we generate $\binom{r}{x}$ partitions for each distinct contribution to $c_{m,x}$. It is also inefficient because we need to compute x for every given partition. Can we create a more efficient algorithm? The problem is that we do not have an elegant characterization of sFD's with r.d. equal to x. This is in contrast to what happened with the A-matrix. In that case, we could show that any FD that has i.d. r necessarily contains the FD μ_r. By using it as our initial configuration, we directly avoided configurations with smaller intrinsic dimension. For the C-matrix, we cannot avoid configurations that have smaller r.d. than the one of interest.

We do not have any algorithms for the matrices α and β as well as the matrices D, F. So far these have been computed only indirectly after the A- and C-matrices have been computed. However, Proposition 2.16 might be a good starting point to coming up with an algorithm that directly enumerates entries in the F-matrix.

4.2. Exact enumeration of higher-dimensional partitions

In order to evaluate higher-dimensional partitions for integers $\leqslant 26$ and dimensions $\leqslant 10$, we chose to use the Knuth algorithm does carry out our computations. There were no serious memory issues for dimensions $\leqslant 7$ and the Knuth algorithm worked well.

We needed to modify our computation when for dimensions 8, 9 and 10. The reduction in memory was done by counting topological sequences that fit into a box of size b. Then the memory requirement went down from n^{d-1} to b^{d-1}. For instance, when $n = 20$ and $b = 10$ (for $d = 10$), the memory usage went down by a factor of 2^9 and enabled us to keep our memory requirements in the 4–8 GB range as constrained by the IITM supercluster. However, some configurations are missed out as they do not fit into the box. Interestingly, one can show the error due to missed configurations is independent of box size when the index lies in the range $[b+1, 2b]$. This makes it easy to estimate the errors by comparing with known results at smaller values of b and then slowly increasing the value of b. This method was used, for instance, to determine the ten-dimensional partitions of 20 – this was carried out by using a box of size 11 with errors determined up to $k = b + 9$. This was one of the more difficult computations as it took a several months of computer time to first estimate the errors and then carry out the final run in the box. We were able to determine $p_d(n)$ for $n \leqslant 23$ and $d \leqslant 10$ and represent more than six months of computer time.

[4] We have determined that the correct call is $part(r+2, 0, \binom{r+1}{2})$. For comparison, the BM algorithm begins with the call $part(1, 0, 1)$. We thank Arun K. Jayaraman for implementing the BM algorithm as well as working out this modification.

4.3. Exact enumeration of the matrices A and C

The modified BM algorithm was used to generate the matrices A and C. The first eight rows of the matrix C have been completely determined. Two additional rows were determined using additional information from the D-matrix. We obtain

$$c_{m,2m-2} = \frac{(2m-2)!}{6(2m-4)!!}\big(3m^2 - m - 1\big),$$
$$c_{m,2m-3} = \frac{(2m-3)!}{6(2m-4)!!}\big(2m^4 - 6m^3 + 3m^2 + 3m + 4\big),$$
$$c_{m,2m-4} = \frac{(2m-4)!}{180(2m-6)!!}\big(15m^5 - 75m^4 + 95m^3 + 21m^2 + 88m + 42\big),$$
$$c_{m,2m-5} = \frac{(2m-5)!}{90(2m-6)!!}\big(258 - 167m - 80m^2 + 111m^3 - 174m^4 + 116m^5 - 31m^6 + 3m^7\big).$$

This determines all entries in the A-matrix of the form $a_{m+r+1,r}$ for $m = 0, \dots, 10$ for *all* values of r. We have determined the remaining entries for $a_{n,r}$ for $n \leqslant 23$ by using the BM algorithm when necessary. The entry $a_{23,11}$ was one of the longest runs and took about 880 hours of CPU time.

Using the β- and α-matrices as well as the Meeussen conjecture, we have also determined the 11-th row of the C-matrix. While none of these results were used in finally determining the entries in the A-matrix, there does not seem to be an inconsistency. This is only to be viewed as evidence for various conjectures.

4.4. Extracting the elements of the other matrices

All other matrices were obtained by using known numbers for the matrices A and C as we do not have an algorithm to directly enumerate them.

An improved implementation of the Bratley–McKay algorithm was provided to us recently by Prof. Bratley. This enabled us to enumerate a few more terms – in particular, we were able to enumerate rows 24, 25 and 26 up to and including $a_{26,12}$. This enabled us to completely determine 26 rows of the matrix F. This in turn determines all entries in 26 rows of the matrix A and hence determines partitions of 26 in any dimension. It also provides a check on the 23 rows of the matrix A which was independently determined.

5. Summary and conclusion

5.1. Summary of results

(1) Given a partition in any dimension, we have introduced two new attributes: its intrinsic dimension (i.d.) (see Definition 2.1) and its reduced dimension (r.d.) (see Definition 2.7).
(2) These two attributes lead to two new matrices, the A- and C-matrices (see Eqs. (2.1) and (2.4)) whose entries admit combinatorial interpretations. We propose a further refinement in the form of another matrix, D (see Eq. (2.8)).
(3) We show that the matrices C, D are the first in a series of transforms, the end-point of which leads to a matrix F (see Eq. (2.10)). The n-th row of this matrix has only $[(n-1)/2]$ non-zero entries (where $[x]$ is the integral part of x) and these entries determine the partitions of n in any dimension. This constitutes the main result of this paper.
(4) We see an intriguing relationship between the numbers of spanning rooted forests on m vertices and α components and a family of entries in the matrix F. This is Proposition 2.16.
(5) We conjecture the existence of two other matrices, α and β, with integer entries.
(6) We prove a conjecture of Hanna on the existence of a matrix that determines all higher-dimensional partitions.

(7) We propose a modification to an algorithm of Bratley and McKay that enables us to directly compute the A- and C-matrices. We compute the first 26 rows of the matrix F thereby obtaining partitions in all dimensions for integers $\leqslant 26$.
(8) Tables that provide the numerical results that we have obtained are available in the version of this paper posted on the arXiv [14].

5.2. Concluding remarks

We have shown the existence of several structures that lead to simplifications in the exact enumerations of higher-dimensional partitions. The combinatorial interpretations that we have provided have enabled us to come up with an algorithms to evaluate the A- and C-matrices. A few lines of code in Mathematica/Maple/Maxima/java can be used to store the A-matrix and compute $p_d(n)$ for $n \leqslant 26$ using the binomial transform in real time [14]. A working implementation of this is provided on the webpage:

http://www.physics.iitm.ac.in/~suresh/partitions.html.

We will be adding these numbers to the OEIS as well as providing modules for SAGE/Mathematica/Maxima.

It appears difficult to improve on our results which have determined all entries for the $n = 26$ row of the A-matrix. Further additions to the A-matrix will require new and efficient algorithms to directly enumerate either the C- or the F-matrix. Another approach would be a naive parallelization of the BM algorithm. We hope to be able to eventually determine partitions of integers less than 30 in any dimension in the future.

It might be that there are many further structures and refinements waiting to be discovered which might provide further simplifications in the computation of partitions of integers in any dimension. We hope our work provides impetus to work in this direction.

Acknowledgments

This work arose out my attempt to prove Hanna's conjecture. I thank Paul Hanna for drawing my attention to his conjecture. Paul Hanna and Wouter Meeussen have shown a passion for sequences that is highly infectious and have been a constant source of encouragement during my attempt at enumerating the various matrices. I am grateful to Arun K. Jayaraman, then an undergraduate student, who implemented the Bratley–McKay algorithm in C and modified it suitably based on my suggestions to enumerate the A- and C-matrices. I would like to thank Prof. Bratley for sending me his java implementation of the BM algorithm which showed me the merits of efficient programming among many other things. I thank Nicolas Destainville and Naveen Prabhakar for useful comments on an earlier draft of the manuscript. Last but not the least, I would like to thank the High Performance Computing Environment at IIT Madras (http://hpce.iitm.ac.in) for providing me access to the Leo and Vega superclusters where all the computations were carried out.

Appendix A. Ferrers diagrams in a symmetric box

Let us consider Ferrers diagrams of i.d. r that fit in a symmetric box of size b – points that lie within the box are such that all their coordinates take values in $(0, 1, \dots, b-1)$. Let us call them restricted Ferrers diagrams. It is easy to see that under the action of S_r that permutes the r-axes, FD's that fit in a box get mapped to FD's that also fit in the same box. Due to this property, we can construct analogs of the various matrices A, C, D, F for restricted FD's as well even though the total number of restricted FD's are finite. For instance, we have

$$p_d^{\mathrm{box}\, b}(n) = \sum_{r=0}^{d+1} \binom{d+1}{r} a_{n,r}^{\mathrm{box}\, b},$$

where $p_d^{\text{box}\,b}(n)$ is the number of FD's with n nodes that fit in a symmetric box of size b and $A^{\text{box}\,b} = (a_{n,r}^{\text{box}\,b})$. Similarly, we can define $C^{\text{box}\,b}$. The analog of the matrix A for restricted partitions not necessarily in a symmetric box has appeared in the work of Destainville et al. [15].

Let us focus on partitions that fit into a symmetric box of size two and denote the corresponding matrices in obvious notation:

$$A^{\text{box}\,2} = \left(a_{n,r}^{\text{box}\,2}\right), \qquad C^{\text{box}\,2} = \left(c_{m,x}^{\text{box}\,2}\right), \qquad D^{\text{box}\,2} = \left(d_{m,x}^{\text{box}\,2}\right) \quad \text{and} \quad F^{\text{box}\,2} = \left(f_{n,r}^{\text{box}\,2}\right).$$

We do not write out their relationships as they exactly mirror the corresponding formulae for unrestricted partitions.

Definition A.1. Let $\mathcal{B}$ be the set of strict Ferrers diagrams that fit in a symmetric box of size 2.

We say, somewhat loosely, that a strict skew FD, σ of r.d. x is in $\mathcal{B}$ if the FD $\mu_x \cup \sigma \in \mathcal{B}$. The only irreducible strict skew Ferrers diagrams at dimensions 2, 3 and 4 in $\mathcal{B}$ are σ_2, σ_3 and $\sigma_{4a/b}$ respectively. It is also easy to see that $\mathcal{D} \subset \mathcal{B}$.

The matrix $C^{\text{box}\,2} = (c_{m,x}^{\text{box}\,2})$ (for $m, x \geqslant 0$) has non-zero entries when $m \in [0, 2^x - x - 1]$ with $c_{0,0}^{\text{box}\,2} = 1$. Further, $c_{0,x}^{\text{box}\,2} = c_{x,0}^{\text{box}\,2} = 0$ for $x > 0$ and $c_{2^x-x-1,x}^{\text{box}\,2} = 1$. The maximum value of m, for fixed x, is obtained by considering the FD containing all nodes that are in the box. Such an FD has 2^x nodes in x dimensions and thus the corresponding skew FD has $2^x - x - 1$ nodes after deleting the nodes that lie in μ_x. Below, we provide the first few rows of the matrix, $C^{\text{box}\,2}$, for $m \in [1, 6]$

$$C^{\text{box}\,2} = \begin{pmatrix} 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 16 & 30 & 15 \\ 0 & 0 & 0 & 1 & 15 & 135 & 330 & 315 & 105 \\ 0 & 0 & 0 & 0 & 18 & 232 & 1581 & 4410 & 5880 & 3780 & 945 \\ 0 & 0 & 0 & 0 & 13 & 355 & 4000 & 23\,709 & 71\,078 & 116\,550 & 107\,100 & 51\,975 & 10\,395 \end{pmatrix}.$$

Extending the ideas that were used in defining the matrices D, F, we count *only* those skew FD's that do not contain skew FD's in $\mathcal{B}$ are reducible components. Let the matrix $\widehat{C}_{m,x}$ denote this reduced C-matrix that counts strict skew FD's of r.d. x with m nodes. Such FD's necessarily contain at least one node of type 2. Then one has the following relation that relates $\widehat{C}$ to the C:

$$c_{m,x} = \sum_{y=0}^{x} \sum_{p=0}^{m} \binom{x}{y} c_{m-p,x-y}^{\text{box}\,2} \widehat{c}_{p,y},$$

with $\widehat{c}_{0,0} = 1$ and $\widehat{c}_{0,x} = 0$ for $x > 0$. It is better to rewrite the above formula as follows:

$$c_{m,x} = \widehat{c}_{m,x} + c_{m,x}^{\text{box}\,2} + \sum_{y=1}^{x-1} \sum_{p=1}^{m-1} \binom{x}{y} c_{m-p,x-y}^{\text{box}\,2} \widehat{c}_{p,y}. \tag{A.1}$$

The first term in the right hand side of the above equation is the contribution from skew FD's that do no contain any reducible components in $\mathcal{B}$, the second term arises solely from terms that fit into a box of size 2. The last terms run over terms that contain reducible components in $\mathcal{B}$ but do not fit into a box of size 2. The next proposition shows that $\widehat{C}$ is a lower-triangular matrix with the m-th row containing m non-zero terms.

Proposition A.2. $\widehat{c}_{m,x} = 0$ *when* $x > m$ *or equivalently when the density* $\rho < 1$.

Proof. Since $\mathcal{D} \subset \mathcal{B}$ and all the irreducible strict skew FD's with density less than unity lie in $\mathcal{D}$, the above proposition follows from Proposition 2.13. □

$$\widehat{C} = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 7 & 16 \\ 0 & 1 & 11 & 57 & 125 \\ 0 & 1 & 18 & 135 & 602 & 1296 \\ 0 & 1 & 26 & 293 & 1911 & 7980 & 16\,807 \\ 0 & 1 & 38 & 574 & 5242 & 31\,860 & 127\,977 & 262\,144 \\ 0 & 1 & 52 & 1089 & 12\,972 & 106\,505 & 619\,872 & 2\,411\,416 & 4\,782\,969 \end{pmatrix}.$$

We observe that $\widehat{c}_{m,m} = (m+1)^{m-1}$.

We can carry out a similar refinement for strict FD's that contribute to the matrix A. One has

$$\boxed{a_{n,r} = \widehat{f}_{n,r} + \sum_{s=0}^{r-1} \sum_{p=s+1}^{n-1} \binom{r}{s} a^{\text{box}\,2}_{n-p+1,r-s} \widehat{f}_{p,s}}, \tag{A.2}$$

with $\widehat{f}_{1,0} = 1$ and $\widehat{f}_{n,0} = 0$ for $n > 0$. In order to interpret the first term, it is better to think of $a_{n,r}$ as the number of skew FD's obtained after removing the node at the origin of a strict FD. Then, the second term is the contribution from such skew FD's that do *not* contain reducible components that fit in a box of size two. A second equivalent definition in terms of m is as follows:

$$a_{m+r+1,r} = \widehat{f}_{m+r+1,r} + a^{\text{box}\,2}_{m+r+1,r} + \sum_{s=1}^{r-1} \sum_{p=0}^{m} \binom{r}{s} a^{\text{box}\,2}_{m-p+r-s+1,r-s} \widehat{f}_{p+s+1,s}.$$

It is easy to see that there is a bijective map that relates skew FD's that contribute to $\widehat{c}_{m,x}$ and those that contribute to $\widehat{a}_{m+x+1,x}$. The bijection follows by observing that if σ is a strict skew FD with m nodes and r.d. x, there is a unique FD (with i.d. and r.d. equal to x) obtained by adding the nodes in μ_x. Thus,

$$\widehat{f}_{m+x+1,x} = \widehat{c}_{m,x}.$$

It is easy to see using Proposition A.2 that for $\widehat{f}_{n,r} = 0$ when $r > n/2$. We define the matrix $\widehat{F} = (\widehat{f}_{n,r})$ for $n = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$. Further, we observe that $\widehat{f}_{2x+1,x} = c_{x,x} = (x+1)^{x-1}$. Below we reproduce the first eleven rows of the $\widehat{F}$-matrix. We reproduce the F-matrix alongside for comparison. The first instance where they differ is when $n = 8$ and $r = 3$ – this is precisely where the node $(1,1,1)^T$ that is not in $\mathcal{D}$ but present in $\mathcal{B}$ appears. As we go to higher values of n, an entry in $\widehat{F}$ will be generically smaller than the corresponding entry in F

$$\widehat{F} = \begin{pmatrix} 1 \\ 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 7 \\ 0 & 1 & 11 & 16 \\ 0 & 1 & 18 & 57 \\ 0 & 1 & 26 & 135 & 125 \\ 0 & 1 & 38 & 293 & 602 \\ 0 & 1 & 52 & 574 & 1911 & 1296 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 \\ 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 7 \\ 0 & 1 & 11 & 16 \\ 0 & 1 & 18 & 58 \\ 0 & 1 & 26 & 135 & 125 \\ 0 & 1 & 38 & 293 & 618 \\ 0 & 1 & 52 & 574 & 1927 & 1296 \end{pmatrix}.$$

The second row has only vanishing entries. That is because the only strict FD with two nodes fits in a box of size two. So the first non-vanishing contribution appears at $n = 3$, $r = 1$ if we ignore the $n = 1$, $r = 0$ term that is more or less part of the definition.

We can now revisit the problem of enumerating partitions of n in any dimension. We see that we need to enumerate the first n rows of the matrix $\widehat{F}$ and $A^{\text{box}\,2}$ in order to obtain row n of the matrix A. However, from Eq. (A.2) we see that it is sufficient to determine only the first $[n/2]$ elements of row n as that completely determines row n of $\widehat{F}$. However, this reduction is accompanied by the need to evaluate $A^{\text{box}\,2}$ which is yet another computation. Hence, we preferred to work with the matrix F. However, one should be open to using the matrix $\widehat{F}$ if one has an algorithm to directly compute it. Then, the additional effort to compute $A^{\text{box}\,2}$ might be worth it.

A.1. The box transform

Define the following generating function for the A-matrix

$$A(q,t)=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}a_{m+r+1,r}\frac{q^m t^r}{r!}, \tag{A.3}$$

along with similar definitions for $A^{\text{box}\,2}(q,t)$ and $\widehat{F}(q,t)$. Then, Eq. (A.2) implies that the generating functions have a simple relation. One has

$$A(q,t)=A^{\text{box}\,2}(q,t)\times\widehat{F}(q,t).$$

It is due to this property that we refer to Eq. (A.3) as the *box transform*. Similarly, one defines

$$C(q,t)=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}c_{m,r}\frac{q^m t^r}{r!},$$

along with similar definitions for $C^{\text{box}\,2}(q,t)$ and $\widehat{C}(q,t)$. Again, one has

$$C(q,t)=C^{\text{box}\,2}(q,t)\times\widehat{C}(q,t).$$

There is an obvious extension to our considerations by replacing the symmetric box of size two by one of size b. Again, relations of the kind that we considered between FD's that fit in the box and those that don't appear. For instance, one has

$$A(q,t)=A^{\text{box}\,b}(q,t)\times\widehat{F}(q,t),$$

where $\widehat{A}(q,t)$ is the generating function of FD's that don't fit into a box of size b and do not have reducible parts that fit into the box.

References

[1] A.O.L. Atkin, P. Bratley, I.G. Macdonald, J.K.S. McKay, Some computations for m-dimensional partitions, Proc. Cambridge Philos. Soc. 63 (1967) 1097–1100.
[2] G.E. Andrews, The Theory of Partitions, Cambridge Math. Lib., Cambridge University Press, Cambridge, 1998, reprint of the 1976 original.
[3] P.A. MacMahon, Combinatory Analysis, vols. I, II (bound in one Volume), Dover Phoenix Editions, Dover Publications, Inc., Mineola, NY, 2004, reprint of An Introduction to Combinatory Analysis, 1920 and Combinatory Analysis, vols. I, II, 1915, 1916.
[4] F.Y. Wu, G. Rollet, H.Y. Huang, J.M. Maillard, C.-K. Hu, C.-N. Chen, Directed compact lattice animals, restricted partitions of an integer, and the infinite-state Potts model, Phys. Rev. Lett. 76 (1996) 173–176.
[5] D.P. Bhatia, M.A. Prasad, D. Arora, Asymptotic results for the number of multidimensional partitions of an integer and directed compact lattice animals, J. Phys. A 30 (7) (1997) 2281–2285.
[6] S. Balakrishnan, S. Govindarajan, N.S. Prabhakar, On the asymptotics of higher-dimensional partitions, J. Phys. A 45 (2012) 055001, arXiv:1105.6231 [cond-mat.stat-mech].
[7] P. Bratley, J.K.S. McKay, Algorithm 313: Multi-dimensional partition generator, Commun. ACM (1967) 1.
[8] D.E. Knuth, A note on solid partitions, Math. Comp. 24 (1970) 955–961.
[9] The on-line encyclopedia of integer sequences, published electronically at http://oeis.org, 2012.
[10] S.B. Ekhad, The number of m-dimensional partitions of eleven and twelve, published electronically at http://www.math.rutgers.edu/~zeilberg/pj.html.
[11] Wikipedia, Spanning tree – Wikipedia, the free encyclopedia, published online at http://en.wikipedia.org/wiki/Spanning_tree, accessed 19 March 2012.
[12] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009.

[13] I. Pak, Lectures on bijections at IPAM, available at http://www.math.ucla.edu/~pak/lectures/lectures-IPAM.htm, accessed 9 March 2009.
[14] S. Govindarajan, Notes on higher-dimensional partitions, arXiv:1203.4419 [math.CO].
[15] N. Destainville, R. Mosseri, F. Bailly, Configurational entropy of codimension-one tilings and directed membranes, J. Stat. Phys. 87 (1997) 697–754.