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Nonnegative generalized inverses and certain subclasses of singular Q-matrices

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ABSTRACT

The notion of Q-matrices is quite well understood in the theory of linear complementarity problems. In this article, the author considers three variations of Q-matrices, typically applicable for singular matrices. The main result presents a relationship of these notions (for a Z-matrix) with the nonnegativity of the Moore–Penrose inverse of the matrix concerned.

1. Introduction

A real square matrix A is called a Z-matrix if all the off-diagonal entries of A are nonpositive. It follows that a Z-matrix A can be written as $A = sI - B$, where $B \geqslant 0$. Here and in the rest of the article, we use the notation $C \geqslant 0$ for a real matrix C to denote that all its entries are nonnegative. A similar definition holds for row and column vectors. A Z-matrix A is called an M-matrix if, in the above decomposition, we also have $s \geqslant \rho(B)$, where $\rho(.)$ denotes the spectral radius. An M-matrix A is invertible if and only if $s > \rho(B)$ which holds if and only if $A^{-1} \geqslant 0$. For details we refer to Chapter

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6 of [3], where more than 50 equivalent conditions for a Z-matrix to be an invertible M-matrix are given.

As usual, let $\mathbb{R}^{m\times n}$ denote the space of all real matrices of order $m\times n$ and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. Let $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ be given. The linear complementarity problem denoted by $LCP(A, q)$ is the problem of determining if there is $x \in \mathbb{R}^n$ such that $x \geqslant 0, y = Ax + q \geqslant 0$ and $\langle x, y\rangle = x^T y = 0$. In this connection, A is called a Q-matrix, if $LCP(A, q)$ has a solution for all $q \in \mathbb{R}^n$. If A is a Q-matrix, we also say that A has the Q-property. Another well known property forms the P-matrix class, which we define next. $A \in \mathbb{R}^{n\times n}$ is called a P-matrix if all the principal minors of A are positive. A generalization of the P-matrix notion is P_0. A is called a P_0-matrix if all the principal minors of A are nonnegative. These classes of matrices have been widely studied in the theory and applications of linear complementarity problems [5]. We also point out to the recent survey article [4]. Let us just point out a well known result that if A is a Z-matrix, then A is a P-matrix if and only if A is a Q-matrix (with the additional property that $LCP(A, q)$ has a unique solution for all q) [3].

Let us set the stage for the main result of this article, by recalling a theorem that has motivated it. Gowda and Tao [7] proved a result characterizing the Q-property of linear transformations over Euclidean Jordan algebras with a proper cone having the Z-property. For our purposes we only consider the following particular case:

Theorem 1.1 [7, Theorem 6]. *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following conditions are equivalent:*

(a) A has the Q-property.
(b) A^{-1} exists and $A^{-1} \geqslant 0$.
(c) There exists $d > 0$ such that $Ad > 0$.
(d) A^T has the Q-property.
(e) $(A^T)^{-1}$ exists and $(A^T)^{-1} \geqslant 0$.
(f) There exists $u > 0$ such that $A^T u > 0$.

Let us consider the condition given in the statement (b). The variations of Q-matrices studied in this article stem from the natural consideration of replacing this condition by the statement: $A^\dagger \geqslant 0$, where $A^\dagger$ denotes the Moore–Penrose (generalized) inverse of A (which always exists, for any A, singular, even rectangular). Recall that the Moore–Penrose inverse $A^\dagger$ of $A \in \mathbb{R}^{m\times n}$ is the unique $X \in \mathbb{R}^{n\times m}$ which satisfies the following Penrose equations: $AXA = A$; $XAX = X$; $(AX)^T = AX$ and $(XA)^T = XA$, where the superscript T denotes the operation of transposition. We shall also discuss the notion of the group (generalized) inverse $A^\#$ for a real matrix A of order $n \times n$, which is defined to be the unique $X \in \mathbb{R}^{n\times n}$ (if such an X exists) satisfying the equations: $AXA = A$; $XAX = X$ and $XA = AX$. Thus, we shall also consider the statement: $A^\# \geqslant 0$.

In trying to obtain necessary and/or sufficient conditions mimicking Theorem 1.1, our efforts naturally lead to (apparently) three distinct classes of singular Q-matrices. Using these definitions, we present our main result (Theorem 2.14). The version for the group inverse is presented in Theorem 2.18.

In the rest of the introduction, we summarize certain important properties of the two classes of generalized inverses mentioned above, and include two preliminary results.

For $A \in \mathbb{R}^{m\times n}$, the symbols $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the range space and the null space of A, respectively. It is well known that the Moore–Penrose $A^\dagger$, defined as above, exists for all matrices $A \in \mathbb{R}^{m\times n}$ whereas the group inverse $A^\#$ need not exist. A well known equivalent condition for its existence is the condition $\mathcal{R}(A) = \mathcal{R}(A^2)$ which in turn, is equivalent to $\mathcal{N}(A) = \mathcal{N}(A^2)$. Another necessary and sufficient condition is that $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are complementary subspaces of $\mathbb{R}^n$. A square matrix A is called range symmetric (or EP, by some authors) if it satisfies $\mathcal{R}(A) = \mathcal{R}(A^T)$. A rather well known result for a range symmetric matrix A is that the Moore–Penrose inverse and the group inverse coincide. In particular, if A is range symmetric, then $AA^\dagger = A^\dagger A$. We shall be using the following formulas repeatedly in the proofs: $\mathcal{R}(A^T) = \mathcal{R}(A^\dagger)$; $\mathcal{N}(A^T) = \mathcal{N}(A^\dagger)$; $\mathcal{R}(A) = \mathcal{R}(A^\#)$; $\mathcal{N}(A) = \mathcal{N}(A^\#)$; If $x \in \mathcal{R}(A^T)$ then $x = A^\dagger Ax$ and if $x \in \mathcal{R}(A)$ then $x = A^\# Ax$. We refer the reader to the book [1] for proofs of the statements mentioned above.

The following result is frequently used in the proofs:

Lemma 1.2. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $Ax = b$ has a solution if and only if $AA^\dagger b = b$. In that case, the general solution is given by $x = A^\dagger b + z$ for some $z \in \mathcal{N}(A)$.*

We conclude this introductory section with the next result which states that if the Moore–Penrose inverse of a singular Z-matrix (which is also a P_0-matrix), is nonnegative, then the matrix is permutationally similar to the direct sum of an invertible M-matrix and a zero matrix of an appropriate order.

Theorem 1.3 [8, Theorem 3.9]**.** *Let A be a singular nonzero Z-matrix which is also a P_0-matrix. Then the following statements are equivalent:*

(i) $A^\dagger \geqslant 0$.

(ii) *There exists a permutation matrix S such that $SAS^T = \begin{pmatrix} T & O \\ O & O \end{pmatrix}$ where T is a nonsingular Z-matrix.*

(iii) *rank A=rank A^2 and $A^\# \geqslant 0$.*
Furthermore, if one of above condition holds, then $A^\dagger = A^\#$.

Remarks 1.4. It is important for our purposes to observe that in the statement (ii), the matrix T is a P-matrix. This will be used later (see the proof of the implication $(b') \implies (d)$ in Theorem 2.14).

2. Variations of Q-property

In this section, first we propose three variations of the Q-property, viz., A has the pseudo Q-property (Definition 2.1), A is a presumably rank deficient Q-matrix of type I (Definition 2.7) and type II (Definition 2.11). Each notion is followed by two examples, one illustrating that a certain matrix satisfies the specified property and another one not belonging to the class of matrices with that property. We then proceed to prove the main result of this article, viz., Theorem 2.14 for the Moore–Penrose inverse version and present a similar result in Theorem 2.18 for the case of the group inverse.

Definition 2.1. Let $A \in \mathbb{R}^{n\times n}$. A is said to have the pseudo Q-property if for every $q \in \mathcal{R}(A)$, $LCP(A, q)$ has a solution in $\mathcal{R}(A^T)$. More precisely, A has the pseudo Q-property if for every $q \in \mathcal{R}(A)$, there exists $x \in \mathcal{R}(A^T)$ such that $x \geqslant 0,\ y = Ax - q \geqslant 0$ and $\langle x, y\rangle = 0$.

Remarks 2.2. While the pseudo Q-property is neither implied by nor implies the Q-property, it is clear that for an invertible matrix, these notions are equivalent. In the next result, we present a class of Q-matrices that are also pseudo Q-matrices.

Theorem 2.3. *Let $A \in \mathbb{R}^{n\times n}$ be a range symmetric Q-matrix such that $AA^\dagger \geqslant 0$. Then A is a pseudo Q-matrix.*

Proof. Let $q \in \mathcal{R}(A)$. Since A is a Q-matrix, there exists $x \in \mathbb{R}^n$ such that $x \geqslant 0$, $y = Ax - q \geqslant 0$ and $\langle x, y\rangle = 0$. Set $w = AA^\dagger x = A^\dagger Ax$. Then $w \geqslant 0$ and $w \in \mathcal{R}(A^T)$. Also $Aw = AA^\dagger Ax = Ax$ so that $Aw - q = y \geqslant 0$. Now, since $q \in \mathcal{R}(A)$, it follows that $y \in \mathcal{R}(A)$ so that $y = AA^\dagger y = A^\dagger Ay$. Thus, we have $\langle w, y\rangle = \langle A^\dagger Ax, y\rangle = \langle x, A^\dagger Ay\rangle = \langle x, y\rangle = 0$. This shows that A is a pseudo Q-matrix. □

Let $C = \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix}$, where F and G are square matrices. It can be shown that C is not necessarily a Q-matrix, even if F and G are Q-matrices. Worse, even when G is the zero matrix, C need not be a

Q-matrix. However, the latter is a pseudo Q-matrix, if F is invertible, as we show next (see also the proof of $(b') \implies (d)$ in Theorem 2.14).

Theorem 2.4. *Let $C = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, where $F \in \mathbb{R}^{n\times n}$ and the zero blocks are such that C is a square matrix. If F is an invertible Q-matrix, then C is a pseudo Q-matrix.*

Proof. Let $q = (q^1, q^2)^T \in \mathcal{R}(C)$. Then $q^2 = 0$. Since F is a Q-matrix, there exists $x^1 \geqslant 0$ such that $y = Fx^1 - q^1 \geqslant 0$ and $\langle x^1, y^1\rangle = 0$. Set $x = (x^1, 0)$ and $y = Cx - q$. Then $x \geqslant 0$, $x \in \mathcal{R}(C^T)$ (since F is invertible) and $y = (Fx^1 - q^1, 0)^T \geqslant 0$. Also $\langle x, y\rangle = \langle x^1, y^1\rangle = 0$. Hence C is a pseudo Q-matrix. $\square$

Example 2.5. Let $A = (E, 0)$, where E is the $2n \times n$ matrix all of whose entries are 1, and 0 denotes the $2n\times n$ zero matrix. Let us show that A is not a Q-matrix but is a pseudo Q-matrix. First, let $q^* = (e, -e)^T$, where e denotes the row vector with n coordinates all of which are 1. Let $x = (u, v)^T$, with $u, v \in \mathbb{R}^n$, be a solution of $LCP(A, q^*)$. We then have $u \geqslant 0$ and $v \geqslant 0$. Also, $Ax + q^* = ((\alpha + 1)e, (\alpha - 1)e)^T$, where $\alpha = \sum_{i=1}^n u_i$. So, the inequality $0 \leqslant Ax+q^*$ implies that $\alpha \geqslant 1$. The condition $\langle x, Ax+q^*\rangle = 0$, reduces to $(1 + \alpha)\alpha + (-1 + \alpha)\beta = 0$, where $\beta = \sum_{i=1}^n v_i \geqslant 0$. This gives $\alpha = 0$, a contradiction. Hence A is not a Q-matrix.

On the other hand, if $q \in \mathcal{R}(A)$, then $q = Ez$ for some $z \in \mathbb{R}^n$ so that $q = \gamma e^{2n}$ for some real number γ. If $x = (u, v)^T \in \mathcal{R}(A^T)$, then $v = 0$. Also, $Ax + q = (\alpha + \gamma)e^{2n}$, where $\alpha = \sum_{i=1}^n u_i$. The requirements for x to be a solution now get transformed to the inequalities $u \geqslant 0$, $\alpha + \gamma \geqslant 0$ and $(\alpha + \gamma)\alpha = 0$. If $\gamma < 0$, then we choose $u = -\frac{\gamma}{n}e$ so that $u \geqslant 0$ and $\alpha + \gamma = 0$. If $\gamma \geqslant 0$, then $u = 0$ is a choice so that we have a solution in either case. Hence A is a pseudo Q-matrix.

Example 2.6. Let $B = \begin{pmatrix} E & 0 \\ -E & 0 \end{pmatrix}$, where E is the $n \times n$ matrix all of whose entries are 1 and the zeros denote zero blocks of order $n \times n$. Let $q^* = (e, -e)^T$, where e is as above. Let $x = (u, v)^T$ with $u, v \in \mathbb{R}^n$. Then the inequality $0 \leqslant Bx + q^* = (Eu + e, -Eu - e)^T$ implies that $Eu = -e$ so that $u \ngeqslant 0$. Thus B is not a Q-matrix. Since $q^* \in \mathcal{R}(B)$, it follows that B is also not a pseudo Q-matrix.

Definition 2.7. Let $A \in \mathbb{R}^{n\times n}$ with $r = rank(A)$. Then A is called a presumably rank deficient Q-matrix of type I, if there exists a permutation matrix S such that $LCP(SAS^T, q)$ has a solution for all $q = (q^1, q^2)^T$ whenever $q^2 \in \mathbb{R}^{n-r}$ satisfies $q^2 \geqslant 0$. Here, S may be the $n \times n$ identity matrix.

Remarks 2.8. First, observe that if A is a Q-matrix, then A is a presumably rank deficient Q-matrix of type I. This follows from the fact that if A is a Q-matrix then for any permutation matrix S, the matrix SAS^T is also a Q-matrix [6]. It is also clear that, for the same reason as above, a presumably rank deficient Q-matrix A of type I which is also invertible, has the Q-property. Given $A \in \mathbb{R}^{n\times n}$, to verify that A is a presumably rank deficient Q-matrix of type I, only in the worst case, one may have to consider all the $n!$ matrices of the form SAS^T. On the other hand, to verify that A is not in this class, in all cases, one will have to study all the $n!$ matrices of the form SAS^T. Due to this complexity, in the next two examples we restrict our attention to 2×2 matrices.

Example 2.9. Let A be as given in Example 2.5 (with $n = 1$). Let us emphasize once again that A is not a Q-matrix. We show that A is a presumably rank deficient Q-matrix of type I. Observe that $rank(A) = 1$. We show that $LCP(A, q)$ has a solution for all $q = (q_1, q_2)^T$, where $q_2 \geqslant 0$. If $q_1 \geqslant 0$, then we choose $x = 0$ and if $q_1 < 0$, then we choose $x_1 = -q_1$ and $x_2 = 0$. It can be verified that the vector $x = (x_1, x_2)^T$, defined as above is a solution for the given $LCP(A, q)$.

Example 2.10. Let B be as given in Example 2.6 (with $n = 1$). We show that B is not a presumably rank deficient Q-matrix of type I. Observe that $rank(B) = 1$. Let $q^* = (-1, 0)^T$. We show that

$LCP(SBS^T, q^*)$ does not have a solution when $S = I$ or $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In the first case, the inequality $0 \leqslant Bx + q^* = (x_1 - 1, -x_1)^T$ is clearly infeasible. In the second case, set $C = SBS^T = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$. Then $0 \leqslant Cx + q^* = (-x_2 - 1, x_2)^T$ yields $x_2 \leqslant -1$, so that $x \ngeqslant 0$.

Definition 2.11. Let $A \in \mathbb{R}^{n \times n}$. Then A is called a presumably rank deficient Q-matrix of type II, if for some $0 \neq d \geqslant 0$, the problems $LCP(A^T, 0)$ and $LCP(A^T, d)$ have precisely one solution in $\mathcal{R}(A)$, namely the zero solution.

Example 2.12. Let $A = (E, 0)$ be as given in Example 2.5. We show that A is a presumably rank deficient Q-matrix of type II. We must show that $LCP(A^T, d)$ has only $x = 0$ as a solution in $\mathcal{R}(A)$ for some (in fact, for all) $d \geqslant 0$.

First, let x be a solution of $LCP(A^T, d)$ in $\mathcal{R}(A)$. Then $x = \gamma e^*$ for some $\gamma \in \mathbb{R}$, where $e^* = (e, e) \in \mathbb{R}^{2n}$, with e being defined as before. Let $d = (r, s)^T \geqslant 0$ be arbitrary. Then $A^T x + d = (\gamma e + r, \gamma e + s)^T$. Thus, $r, s \geqslant 0$ together with the condition that $0 = \langle x, A^T x + d\rangle$ gives rise to the equation $2n\gamma^2(n + \sum_{i=1}^n r_i) + 2n\gamma \sum_{i=1}^n s_i = 0$, where each term (in particular, each factor) is nonnegative. Since the second factor of the first term is positive, we conclude that $\gamma = 0$. Hence $LCP(A^T, d)$ has only $x = 0$ as a solution in $\mathcal{R}(A)$, as was required to prove.

Example 2.13. Rather interestingly, B defined as in Example 2.6, which has been shown to be not a pseudo Q-matrix nor a presumably rank deficient Q-matrix of type I, turns out to be a presumably rank deficient Q-matrix of type II. We omit the details.

We are now in a position to prove the main result of this article. Inspired by Theorem 1.1, among other things, we show that every Z-matrix A possessing the pseudo Q-property must be a presumably rank deficient Q-matrix of type II. Also, if A is a Z-matrix having the pseudo Q-property and is a P_0-matrix, then A is a presumably rank deficient Q-matrix of type I.

Theorem 2.14. *Let $A \in \mathbb{R}^{n \times n}$ be a non-zero Z-matrix. Consider the following statements:*

(*a*) *A has the pseudo Q-property.*
(*b*) *$A^\dagger \geqslant 0$.*
(*c*) *There exists $0 \neq q \geqslant 0$ such that $Aq \in \mathbb{R}^n_+ + \mathcal{N}(A^T)$.*
(*d*) *A is a presumably rank deficient Q-matrix of type I.*
(*e*) *A is a presumably rank deficient Q-matrix of type II.*

Then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (e)$. Consider
(b') A is a P_0-matrix and $A^\dagger \geqslant 0$.
Then $(b') \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$: Let $p \geqslant 0$ and $r = A^\dagger p$. We must show that $r \geqslant 0$. By Lemma 1.2, we have $p = Ar + z$, for some $z \in \mathcal{N}(A^T)$, so that $q := p - z = Ar \in \mathcal{R}(A)$. Since A has the pseudo Q-property, there exists $x \in \mathcal{R}(A^T)$ such that

$$x \geqslant 0, \quad y = Ax - q \geqslant 0 \quad \text{and} \quad \langle x, y\rangle = 0.$$

We have $\langle q, y\rangle = \langle p, y\rangle - \langle z, y\rangle = \langle p, y\rangle$, since $y = Ax - q = A(x - r) \in \mathcal{R}(A)$ and $z \in \mathcal{N}(A^T)$. Since A is a Z-matrix, we then have

$$0 \geqslant \langle Ax, y\rangle = \langle y + q, y\rangle = \| y \|^2 + \langle p, y\rangle.$$

Since both the terms on the extreme right are nonnegative, it follows that $y = 0$ and so $q = Ax$. Thus $r = A^\dagger p = A^\dagger(q + z) = A^\dagger(Ax + z) = x + A^\dagger z = x \geqslant 0$, where we have used the fact that $A^\dagger z = 0$, since $\mathcal{N}(A^\dagger) = \mathcal{N}(A^T)$.

$(b) \Rightarrow (c)$: Let $p \geqslant 0$ and $q = A^\dagger p$. Then $q \geqslant 0$. Also, $p = Aq + z$ for some $z \in \mathcal{N}(A^T)$, by Lemma 1.2. Thus $Aq = p - z \in \mathbb{R}^n_+ + \mathcal{N}(A^T)$. Let us observe that, if necessary, we could choose p such that all its coordinates are positive, so that $0 \neq q \geqslant 0$ holds (since $0 \neq A^\dagger \geqslant 0$).

$(c) \Rightarrow (e)$: Since (c) holds, there exists $q^* \geqslant 0$ such that $Aq^* \in \mathbb{R}^n_+ + \mathcal{N}(A^T)$. We show that $LCP(A^T, 0)$ and $LCP(A^T, q^*)$ have only zero as a solution in $\mathcal{R}(A)$. Let $u \in \mathcal{R}(A)$ be any solution of $LCP(A^T, tq^*),\ t = 0$ or 1. Then $u \geqslant 0$, $v = A^T u + tq^* \geqslant 0$ and $\langle u, v\rangle = 0$. Since A is a Z-matrix, we have $0 \geqslant \langle Av, u\rangle = \langle v, A^T u\rangle = \langle A^T u + tq^*, A^T u\rangle = \| A^T u \|^2 + t\langle q^*, A^T u\rangle = \| A^T u \|^2 + t\langle Aq^*, u\rangle$. Since $Aq^* \in \mathbb{R}^n_+ + \mathcal{N}(A^T)$, $u \geqslant 0$ and $u \in \mathcal{R}(A)$ it follows that $\langle Aq^*, u\rangle \geqslant 0$. It now follows that $A^T u = 0$. This implies that $u = 0$. Thus A is a presumably rank deficient Q-matrix of type II.

$(b') \Rightarrow (d)$: Let $A^\dagger \geqslant 0$. Since A is a Z-matrix which is also a P_0-matrix, by Theorem 1.3, there exists a permutation matrix S such that $B = SAS^T = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$, where T is an invertible M-matrix, that is a P-matrix. Let $q = (q^1, q^2)^T$ with $q^2 \in \mathbb{R}^{n-r},\ q^2 \geqslant 0$. We have $r = rank(A) = rank(T)$. Since T is a P-matrix and $q^1 \in \mathbb{R}^r$, there exists $x^1 \in \mathbb{R}^r$ such that

$$x^1 \geqslant 0, \quad y^1 = Tx^1 + q^1 \geqslant 0 \quad \text{and} \quad \langle x^1, y^1\rangle = 0.$$

Set $x = (x^1, 0)^T$ and $y = Bx + q$. Then $x \geqslant 0$ and $y = (Tx^1 + q^1, q^2)^T \geqslant 0$. Also $\langle x, y\rangle = 0$. This shows that A is a presumably rank deficient Q-matrix of type I. $\square$

Remarks 2.15. Let B be as defined in Example 2.10. Then B is not a presumably rank deficient Q-matrix of type I. Note that B is a Z-matrix and is also a P_0-matrix. It can be shown that the Moore–Penrose inverse $B^\dagger = \frac{1}{2}B^T \not\geqslant 0$. Hence, the assumption that $A^\dagger \geqslant 0$ is indispensable in the implication $(b') \Rightarrow (d)$.

Remarks 2.16. Let us give an alternative proof for the implication $(a) \Rightarrow (b)$ in Theorem 2.14. A well known result of Berman and Plemmons [2] (a proof is provided below) states that condition (b) is equivalent to the statement:

$$Au \in \mathbb{R}^n_+ + \mathcal{N}(A^T), \qquad u \in \mathcal{R}(A^T) \Rightarrow u \geqslant 0.$$

We show that the statement above holds. Let $q = Au \in \mathbb{R}^n_+ + \mathcal{N}(A^T)$ and $u \in \mathcal{R}(A^T)$. By (a), since $q \in \mathcal{R}(A)$, there exists $p \in \mathcal{R}(A^T)$ such that

$$p \geqslant 0, \quad r = Ap - q \geqslant 0 \quad \text{and} \quad \langle p, r\rangle = 0.$$

Since A is a Z-matrix, we have

$$0 \geqslant \langle Ap, r\rangle = \langle r + q, r\rangle = \| r \|^2 + \langle q, r\rangle.$$

Let $q = q^1 + q^2$ with $q^1 \geqslant 0$ and $q^2 \in \mathcal{N}(A^T)$. Then $\langle q^2, r\rangle = 0$, since $r \in \mathcal{R}(A)$ and $\langle q, r\rangle = \langle q^1, r\rangle \geqslant 0$, since $r \geqslant 0$. Thus both the terms on the right in the inequality above are nonnegative. Hence, $r = 0$ and so $Ap = q = Au$ so that $p - u \in \mathcal{N}(A)$. Also, $p - u \in \mathcal{R}(A^T)$ and so $u = p \geqslant 0$.

Next, for the sake of completeness and ready reference, we prove the result of Berman and Plemmons, mentioned as above.

Theorem 2.17. *Let $A \in \mathbb{R}^{m\times n}$. Then $A^\dagger \geqslant 0$ if and only if $Au \in \mathbb{R}^m_+ + \mathcal{N}(A^T),\ u \in \mathcal{R}(A^T) \Rightarrow u \geqslant 0$.*

Proof. Let $A^\dagger \geqslant 0$. Let $Au = v + w,\ v \in \mathbb{R}^m_+,\ w \in \mathcal{N}(A^T)$ and $u \in \mathcal{R}(A^T)$. Then $u = A^\dagger Au = A^\dagger v \geqslant 0$, proving one way implication. Conversely, suppose that $Au \in \mathbb{R}^m_+ + \mathcal{N}(A^T),\ u \in \mathcal{R}(A^T) \Rightarrow u \geqslant 0$.

Let $v \geqslant 0$ and $x = A^{\dagger}v$. We must show that $x \geqslant 0$. We have $v = Ax + y$, $y \in \mathcal{N}(A^T)$. Thus $Ax = v - y \in \mathbb{R}^m_+ + \mathcal{N}(A^T)$. Also, by definition, $x \in \mathcal{R}(A^T)$. Hence it follows that $x \geqslant 0$, completing the proof. $\square$

We conclude this article with an adaptation of Theorem 2.14 for the group inverse.

Theorem 2.18. *Let $A \in \mathbb{R}^{n\times n}$ be a non-zero Z-matrix. Consider the following statements:*

(a) For every $q \in \mathcal{R}(A)$, the problem LCP(A, q) has a solution in $\mathcal{R}(A)$.
(b) Suppose that $A^{\#}$ exists. Then $A^{\#} \geqslant 0$.
(c) There exists $q \geqslant 0$ such that $A^Tq \in \mathbb{R}^n_+ + \mathcal{N}(A)$.
(d) A is a presumably rank deficient Q-matrix of type I.

Then $(a) \Rightarrow (b) \Rightarrow (c)$. Suppose now that $A^{\#}$ exists. Consider the statement:
(b') A is a P_0-matrix and $A^{\#} \geqslant 0$.
Then the implication $(b') \Rightarrow (d)$ holds.
Suppose that, in addition, A is range symmetric. Consider the statement:
(e) For some $0 \neq d \geqslant 0$, the problems LCP$(A, 0)$ and LCP(A, d) have precisely one solution in $\mathcal{R}(A)$, namely the zero solution.
We then have the implication: $(c) \Rightarrow (e)$.

Proof. The proofs for the implications $(a) \Rightarrow (b) \Rightarrow (c)$ are similar to those in Theorem 2.14.

$(b') \Rightarrow (d)$: If $A^{\#}$ exists, and $A^{\#} \geqslant 0$, then by Theorem 1.3, we have $A^{\#} = A^{\dagger}$ and so the proof follows from the corresponding implication in Theorem 2.14.

$(c) \Rightarrow (e)$: Let A be range symmetric, viz., $\mathcal{R}(A) = \mathcal{R}(A^T)$. Then $\mathcal{N}(A^T) = \mathcal{N}(A)$. From (c), we then have: There exists $q^* \geqslant 0$ such that $A^Tq^* \in \mathbb{R}^n_+ + \mathcal{N}(A)$, since $\mathcal{N}(A^T) = \mathcal{N}(A)$. We claim that the only solution of $LCP(A, tq^*)$ for $t = 0$ or 1, in $\mathcal{R}(A)$ is the zero solution. Let $u \in \mathcal{R}(A)$ be any solution. Then $u \geqslant 0$, $v = Au + tq^* \geqslant 0$ and $\langle u, v\rangle = 0$. So, as before, we have $0 \geqslant \langle Au, v\rangle = \| Au \|^2 + t\langle u, A^Tq^*\rangle$. Since $A^Tq^* \in \mathbb{R}^n_+ + \mathcal{N}(A)$, $u \in \mathcal{R}(A)$ and $\mathcal{R}(A) = \mathcal{N}(A^T)^{\perp} = \mathcal{N}(A)^{\perp}$ it follows that $\langle u, A^Tq^*\rangle \geqslant 0$. Thus $Au = 0$ and so $u = 0$, as we set out to prove. $\square$

3. Conclusions

Let us reiterate the fact that the main result (Theorem 2.14) was motivated by a characterization theorem in [7] for Q-matrices in the class of Z-matrices. The framework considered in [7] was symmetric cones in Euclidean Jordan algebras. However, we have considered variations of Q-property pertaining only to classical linear complementarity theory. The reason that we have narrowed our attention in this article is due to the fact that presently we are not aware of an analogue of Theorem 1.3 (which is the main tool in the proof of the implication $(b') \Longrightarrow (d)$ in Theorem 2.14) for singular Z-transformations even for the space of real symmetric matrices. Let us conclude by pointing out that such an extension will be interesting in its own right.

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