Applied Mathematics Letters 24 (2011) 1369–1373

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Multiple integral equations arising in the theory of water waves

S.R. Manam

Department of Mathematics, Indian Institute of Technology Madras, Chennai, 600036, India

ARTICLE INFO

Article history:
Received 7 April 2010
Received in revised form 8 March 2011
Accepted 10 March 2011

Keywords:
Singular Integral equation
Logarithmic singularity
Trigonometric kernels
Surface tension
Flexural gravity waves

ABSTRACT

A quick method of solution for multiple integral equations which are defined over a partition consisting of three intervals of the positive axis and whose kernel is the combination of trigonometric functions has been explained. The solution procedure can be extended to deal with similar integral equations defined over any finite partition of the positive axis. To represent the solution uniquely, certain solvability criteria are obtained in terms of the forcing functions involved. Limiting cases of dual integral equations over two disjoint intervals are discussed.

1. Introduction

Solving boundary value problems with mixed boundary conditions in different branches of mathematical physics is of long-standing interest to scientists and engineers and the problem is often reduced to that of solving dual integral equations (see [1,2]). Chakrabarti et al. [3,4] studied linear water wave scattering by vertical barriers by reducing the corresponding boundary value problem to dual integral equations with a trigonometric kernel. The behavior of one of the integrals of these dual integral equations at the point where the boundary condition changes plays a crucial role in determining their solution.

The motivation for handling these kinds of integral equations is manyfold in the context of the capillary–gravity or structural gravity wave scattering in deep water. In fact, while performing the weakly nonlinear analysis on the time-harmonic capillary–gravity or flexural gravity wave scattering in deep water, with the aid of the small wave slope parameter, a leading order solution is required to deal with the higher order problems. In two-dimensional wave scattering by rigid vertical barriers, this leading order boundary value problem is peculiar in the sense that the solution satisfies the Laplace equation with higher order boundary conditions. Rhodes-Robinson (see [5–7]) handled a variety of these problems in the context of water wave problems under the surface tension effect.

Evans [8] first studied the effect of surface tension on the surface water waves scattered by partial vertical barriers via a complex variable technique. Later, Rhodes-Robinson (see [9]) obtained results for the same problem by extending William's [10] so called reduction method. However, these kinds of boundary value problems can be reduced to multiple integral equations with trigonometric kernels. These integral equations may be defined over many disjoint intervals for a physical situation of structural water waves involving vertical barriers with a single gap or many gaps in it and are handled here to obtain their solution.

The multiple integral equations under consideration are

$$\frac{2}{\pi}\int_0^\infty A(\xi)\left[\sum_{k=0}^{n} c_k \frac{\partial^{2k+1}}{\partial y^{2k+1}} + c_0\right]\sin\xi y d\xi = f(y), \quad y \in L = (a,b), 0 < a < b$$

E-mail address: manam@iitm.ac.in.

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.03.012

$$\frac{2}{\pi}\int_0^\infty \xi A(\xi)\left[\sum_{k=0}^{n} c_k \frac{\partial^{2k+1}}{\partial y^{2k+1}} + c_0\right]\sin\xi y\mathrm{d}\xi = g(y), \quad y\in[0,\infty)\setminus L, \tag{1.1}$$

where

$$g(y) = \begin{cases} g_1(y), & 0\le y<a \\ g_2(y), & b\le y<\infty, \end{cases}$$

and $c_i, i = 0, 1, 2, \ldots, n$, are real or complex constants. These integral equations can be uniquely solvable when the functions g and f are suitably differentiable. They arise in the capillary–gravity or flexural gravity wave scattering in deep water by partial vertical wave-makers or barriers or a vertical barrier with a gap in it (see [8,9,11]). In this context, L represents the complement of the vertical wave-maker or barrier position along the positive axis.

In the present note, an attempt has been made to solve the above multiple integral equations (1.1) completely by converting them into a set of logarithmic singular integral equations. With the aid of the bounded solutions of these singular integral equations, a possible unique solution for the multiple integral equations is obtained under certain restrictions or conditions on the forcing functions f and g.

2. The method of solution

For solving the multiple integral equations (1.1) with $L = (a, b)$, they can be equivalently written as a set of differential equations

$$\mathcal{T}\left[\frac{2}{\pi}\int_0^\infty A(\xi)\sin\xi y\mathrm{d}\xi\right] = f(y), \quad y\in L$$
$$\mathcal{T}\left[\frac{2}{\pi}\int_0^\infty \xi A(\xi)\sin\xi y\mathrm{d}\xi\right] = g(y), \quad y\in[0,\infty)\setminus L,$$

where $\mathcal{T} = \sum_{k=0}^{n} c_k \frac{\partial^{2k+1}}{\partial y^{2k+1}} + c_0$. Upon solving the above $(2n + 1)$th-order ordinary differential equations, they are transformed into a new set of multiple integral equations

$$\frac{2}{\pi}\int_0^\infty A(\xi)\sin\xi y\mathrm{d}\xi = \sum_{k=0}^{2n} D_k e^{\lambda_k y} + \mathcal{T}^{-1}[f(y)] \equiv h_1(y), \quad a<y<b \tag{2.1}$$

$$\frac{2}{\pi}\int_0^\infty \xi A(\xi)\sin\xi y\,\mathrm{d}\xi = \begin{cases} \sum_{k=0}^{2n} E_k e^{\lambda_k y} + \mathcal{T}^{-1}[g_1(y)] \equiv h_2(y), & 0<y<a, \\ \sum_{k=0}^{2n} F_k e^{\lambda_k y} + \mathcal{T}^{-1}[g_2(y)] \equiv h_3(y), & b<y<\infty, \end{cases} \tag{2.2}$$

where $\mathcal{T}^{-1}[f(y)]$, $\mathcal{T}^{-1}[g_1(y)]$, $\mathcal{T}^{-1}[g_2(y)]$ are the particular integrals with respect to the differential operator $\mathcal{T}$, λ_k, $k = 0, 1, 2, 3, \ldots, 2n$, are the roots of the polynomial equation $\sum_{k=0}^{n} c_k x^{2k+1} + c_0 = 0$ and the constants D_k, E_k, F_k are unknowns to be determined.

In the context of deep water waves, the dispersion relation is a polynomial equation $\sum_{k=0}^{n} c_k x^{2k+1} + c_0 = 0$, for some specified constants c_k $(k = 0, 1, 2, \ldots, n)$. The constants are so defined that the dispersion relation has a unique positive real root which accounts for a progressive wave mode in a homogeneous fluid. Physically, complex roots represent either dissipative progressive wave modes or evanescent wave modes. The degree of the dispersion relation is the same as the order of the boundary condition at the undisturbed interface between the floating structure and water surface. The degrees are 3 and 5 for the cases of capillary or membrane-coupled gravity waves and flexural gravity waves respectively.

Making $y \to 0$, we have from Eq. (2.2) that

$$h_2^{(2i)}(y) = 0, \quad i = 0, 1, 2, \ldots, n, \text{ i.e.,} \qquad \sum_{k=0}^{2n} E_k\lambda_k^{2i} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}[\mathcal{T}^{-1}(g_1(y))l]\Big|_{y=0} = 0, \quad i = 0, 1, 2, \ldots, n, \tag{2.3}$$

where the superscript in parentheses hereafter denotes the order of differentiation.

The multiple integral equations (2.1) and (2.2) can be differentiated up to $2i$ times, $i = 0, 1, 2, \ldots, n$, and the resulting sets of multiple integral equations are given by

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty (-1)^i \xi^{2i} A(\xi)\sin\xi y\mathrm{d}\xi &= \sum_{k=0}^{2n} D_k\lambda_k^{2i} e^{\lambda_k y} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}\mathcal{T}^{-1}[f(y)] \\ &\equiv h_1^{(2i)}(y), \quad a<y<b, \text{ for } i = 0, 1, 2, \ldots, n.\end{aligned} \tag{2.4}$$

$$\frac{2}{\pi}\int_0^\infty (-1)^i \xi^{2i+1} A(\xi)\sin\xi y\,\mathrm{d}\xi$$
$$= \begin{cases} \sum_{k=0}^{2n} E_k \lambda_k^{2i} \mathrm{e}^{\lambda_k y} + \dfrac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}\mathcal{T}^{-1}[g_1(y)] \equiv h_2^{(2i)}(y), & 0 < y < a, \\ \sum_{k=0}^{2n} F_k \lambda_k^{2i} \mathrm{e}^{\lambda_k y} + \dfrac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}\mathcal{T}^{-1}[g_2(y)] \equiv h_3^{(2i)}(y), & b < y < \infty, \end{cases} \quad \text{for } i = 0, 1, 2, \ldots, n. \tag{2.5}$$

Now, on defining

$$P_{2i+1}(y) = \frac{2}{\pi}(-1)^i \int_0^\infty \xi^{2i+1} A(\xi)\sin\xi y \mathrm{d}\xi, \quad a < y < b, i = 0, 1, 2, \ldots, n, \tag{2.6}$$

it becomes clear that $P_{2i+1}(y) = P_1^{(2i)}(y), i = 0, 1, 2, \ldots, n$.

Application of the Fourier sine transform on Eqs. (2.5) and (2.6) gives

$$\xi^{2i+1} A(\xi) = \int_0^\infty Q_{2i+1}(y)\sin\xi y \mathrm{d}y, \tag{2.7}$$

where

$$Q_{2i+1}(y) = \begin{cases} (-1)^i P_1^{(2i)}(y), & a < y < b \\ (-1)^i h_2^{(2i)}(y), & 0 < y < a \\ (-1)^i h_3^{(2i)}(y), & b < y < \infty, \end{cases} \quad i = 0, 1, 2, \ldots, n.$$

It is clear from the relation (2.7) that the unknown function $A(\xi)$ can be represented in $n+1$ different ways. Hence certain compatible conditions must be satisfied for $A(\xi)$ to be the same and they are determined as

$$h_2^{(2i)}(a) = h_3^{(2i)}(b), \quad i = 0, 1, 2, \ldots, n-1,$$

i.e.,

$$\sum_{k=0}^{2n} E_k \lambda_k^{2i} \mathrm{e}^{\lambda_k a} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}[\mathcal{T}^{-1}[g_1(y)]]\bigg|_{y=a} = \sum_{k=0}^{2n} F_k \lambda_k^{2i} \mathrm{e}^{\lambda_k b} + \frac{\mathrm{d}^{2i}}{\mathrm{d}y^{2i}}[\mathcal{T}^{-1}[g_2(y)]]\bigg|_{y=b}, \quad \text{for } i = 0, 1, 2, \ldots, n-1, \tag{2.8}$$

$$\frac{\mathrm{d}^{2i+1}P_1}{\mathrm{d}y^{2i+1}}\bigg|_{y=a} = (-1)^i \frac{\mathrm{d}^{2i+1}h_2}{\mathrm{d}y^{2i+1}}\bigg|_{y=a}, \quad i = 0, 1, 2, \ldots, n-1, \tag{2.9}$$

and

$$\frac{\mathrm{d}^{2i+1}P_1}{\mathrm{d}y^{2i+1}}\bigg|_{y=b} = (-1)^i \frac{\mathrm{d}^{2i+1}h_3}{\mathrm{d}y^{2i+1}}\bigg|_{y=b}, \quad i = 0, 1, 2, \ldots, n-1. \tag{2.10}$$

At this stage it may be remarked here that, even after converting the multiple integral equations (2.4) and (2.5) into a logarithmic singular integral equation, the bounded solution will not be in a suitable form for computing its derivatives for use in the relations (2.9) and (2.10). Hence, these relations are required to be in a suitable form to represent in terms of the unknown constants involved.

The restriction of the bounded solution $P_1^{(2i)}(u)$ is because of its natural utility in applications and the nature of the bounded solution will be clear from what follows.

Substituting $\xi^{2i}A(\xi)$, $i = 0, 1, 2, \ldots, n$, from Eq. (2.7) into the relation (2.4) yields a set of singular integral equations

$$\frac{1}{\pi}\int_a^b P_1^{(2i)}(u)\log\left|\frac{u+t}{u-t}\right|\mathrm{d}u = -\int_0^a h_2^{(2i)}(u)\log\left|\frac{u+t}{u-t}\right|\mathrm{d}u - \int_b^\infty h_3^{(2i)}(u)\log\left|\frac{u+t}{u-t}\right|\mathrm{d}u + h_1^{(2i)(y)}$$
$$\equiv R_i(t), \quad i = 0, 1, 2, \ldots, n, \tag{2.11}$$

where we have utilized the following relation (see [12, Eq. 3.741(1)]):

$$\int_0^\infty \frac{\sin\xi y\sin\xi t}{\xi}\mathrm{d}\xi = -\frac{1}{2}\log\left|\frac{y-t}{y+t}\right|, \quad \text{for } y, t \in (0, \infty).$$

The bounded solutions of the logarithmic singular integral equations (see [13]), described by the relation (2.11), are given by

$$P_1^{(2i)}(u) = \frac{2}{\pi}\sqrt{(u^2-a^2)(b^2-u^2)}\int_a^b \frac{tR_i^{(1)}(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}(u^2-t^2)}\mathrm{d}t, \quad a < u < b, \tag{2.12}$$

provided that

$$\int_a^b \frac{tR_i^{(1)}(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\mathrm{d}t = 0, \quad \text{for } i = 0, 1, 2, \ldots, n \tag{2.13}$$

and

$$\frac{(a\pi - J_1)}{J_2}\int_a^b \frac{R_i(t)\mathrm{d}t}{\sqrt{(t^2-a^2)(b^2-t^2)}} + \int_a^b \sqrt{\frac{t^2-a^2}{b^2-t^2}}R_i(t)\mathrm{d}t + \int_a^b t\sqrt{\frac{t^2-a^2}{b^2-t^2}}R_i^{(1)}(t)\mathrm{d}t = 0$$
$$\text{for } i = 0, 1, 2, \ldots, n, \tag{2.14}$$

where

$$J_1 = \int_a^b \sqrt{\frac{t^2-a^2}{b^2-t^2}} \log\left|\frac{a+t}{a-t}\right| \mathrm{d}t \quad \text{and} \quad J_2 = \int_a^b \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} \log\left|\frac{a+t}{a-t}\right| \mathrm{d}t.$$

The bounded property of the solutions is that $P_1^{(2i)}(u) = 0$, $i = 0, 1, 2, \ldots, n$, at the end points a and b.

The conditions (2.9) and (2.10) have to be modified into an alternative form. This can be achieved in many ways by using the solution (2.12). By multiplying by u^j on both sides of the relation (2.12) and integrating from a to b, one may obtain an integral relation involving the left hand sides of (2.9) and (2.10) for each positive integer j. The required quantities $\left.\frac{\mathrm{d}^{2i+1}P_1}{\mathrm{d}y^{2i+1}}\right|_{y=a}$ and $\left.\frac{\mathrm{d}^{2i+1}P_1}{\mathrm{d}y^{2i+1}}\right|_{y=a}$ on the left hand sides of (2.9) and (2.10) can be obtained from any two such integral relations. In the process of obtaining the modified conditions as shown below, j is chosen to be an odd integer for the sake of obtaining simplified integral relations.

First, by multiplying by u^3 on the left hand side of the relation (2.12) and integrating from a to b, it may be derived that

$$\int_a^b u^3 P_1^{(2i)}(u)\mathrm{d}u = b^3 \left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=b} - a^3 \left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=a} + 6\int_a^b uP_1^{(2i-2)}(u)\mathrm{d}u, \quad \text{for } i = 1, 2, 3, \ldots, n. \tag{2.15}$$

Then by making use of the definite integrals

$$\int_a^b \frac{u^3\sqrt{(u^2-a^2)(b^2-u^2)}}{u^2-t^2}\mathrm{d}u = -\frac{\pi}{2}\left[t^4 - \frac{1}{2}(a^2+b^2)t^2 - \frac{1}{8}(a-b)^2(a+b)^2\right]$$
$$\int_a^b \frac{u\sqrt{(u^2-a^2)(b^2-u^2)}}{u^2-t^2}\mathrm{d}u = -\frac{\pi}{2}\left[t^2 - \frac{1}{2}(a^2+b^2)\right],$$

which are evaluated by the standard contour integration technique, it may be derived from the relations (2.12) and (2.15) that

$$\int_a^b \frac{\left[t^5 - \frac{1}{2}(a^2+b^2)t^3 - \frac{1}{8}(a-b)^2(a+b)^2 t\right]R_i^{(1)}(t) - 6\left[t^3 - \frac{1}{2}(a^2+b^2)t\right]R_{i-1}^{(1)}(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\mathrm{d}t$$
$$+ b^3 \left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=b} - a^3 \left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=a} = 0, \quad \text{for } i = 1, 2, 3, \ldots, n. \tag{2.16}$$

Secondly, by multiplying the relation (2.12) by u and integrating from a to b, it can be obtained that

$$b\left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=b} - a\left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=a} + \int_a^b \frac{\left[t^3 - \frac{1}{2}(a^2+b^2)t\right]R_i^{(1)}(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\mathrm{d}t = 0, \quad i = 1, 2, \ldots, n. \tag{2.17}$$

Now, the determination of $\left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=a}$ and $\left.\frac{\mathrm{d}^{2i-1}P_1}{\mathrm{d}y^{2i-1}}\right|_{y=b}$ from the relations (2.16) and (2.17) modifies the conditions (2.9) and (2.10) as

$$(-1)^i\int_a^b \frac{\left[t^3 - \frac{1}{2}(a^2+b^2)t\right]R_i^{(1)}(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\mathrm{d}t + ah_2^{(2i-1)}(a) - b\,h_3^{(2i-1)}(b) = 0, \quad i = 1, 2, \ldots, n \tag{2.18}$$

and

$$(-1)^i\int_a^b \frac{S(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\mathrm{d}t - a(b^2-a^2)h_2^{(2i-1)}(a) = 0, \quad i = 1, 2, \ldots, n \tag{2.19}$$

where $S(t) = \left[t^5 - \frac{1}{2}(a^2+3b^2)t^3 - \frac{1}{8}(a^4 - b^2(6a^2+b^2))t\right]R_i^{(1)}(t) - 6\left[t^3 - \frac{1}{2}(a^2+b^2)t\right]R_{i-1}^{(1)}(t)$.

Thus, the relations (2.3), (2.8), (2.13), (2.14), (2.18) and (2.19) together will determine the $6n + 3$ unknown parameters that appear in Eqs. (2.1) and (2.2) completely. Thus, we conclude here that the multiple integral equations (1.1) have a solution (possibly unique) provided a set of solvability criteria are satisfied by the forcing functions f and g.

It may be remarked that in Eq. (2.5), accommodating infinity along the y axis, one can equate certain constants, whose coefficients $e^{\lambda_k y}$ with the root λ_k have a positive real part, to zero. These extra conditions are naturally utilized in the physical problems of practical interest. Also, it may be worth here discussing the limiting cases of $a \to 0$ or $b \to \infty$ and comparing with the corresponding dual integral equations defined over two disjoint intervals. In the former case, the dual integral equations are of a special type having an isolated point at $y = 0$ in the definition, while in the latter case, the integral equations are reduced to those defined over two disjoint intervals and the formal conditions of solvability exactly match those of [14]. However, it is not straightforward to obtain the limit of the modified solvability criteria involving the functions $h_k^{(2i+1)}(y)$, $k = 1, 2, 3$, as given in the relations (2.18) and (2.19).

For multiple integral equations over disjoint intervals greater than what was considered, the reduced weakly singular integral equations are defined over the same number of intervals. Their bounded solution provides a set of solvability criteria (see [15]). In fact, one may get a number $2l$ of solvability conditions for the multiple integral equations defined over the positive real axis, where l is the number of finite intervals of type (L_1, L_2) with $L_1 > 0$.

3. Conclusions

A quick and elementary method of solution has been worked out for some special multiple integral equations defined over disjoint intervals with trigonometric functions as the kernel. These equations arise in connection with the scattering of time-harmonic capillary or membrane-coupled gravity waves or flexural gravity waves in deep water by partial vertical barriers or a vertical barrier with single or many gaps in it. The multiple integral equations which are worked out here govern the physical problem of deep water wave scattering by a complete barrier with a single gap in it. When the barrier has many gaps, governing multiple integral equations are defined over many disjoint intervals along the depth. In this case, they can be reduced to a weakly singular integral equation over that many disjoint intervals and upon utilizing the solution and solvability criteria as given in [15], one can work out the unique solution for these integral equations.

Acknowledgement

The author thanks the unanimous referee for his or her valuable comments which helped in revising the paper.

References

[1] I.N. Sneddon, The Use of Integral Transforms, McGraw-Hill, New York, 1972.
[2] Debnath Lokenath, Integral Transforms and their Applications, CRC Press, Boca Raton, USA, 1995.
[3] A. Chakrabarti, B.N. Mandal, Solution of some dual integral equations, ZAMM Z. Angew. Math. Mech. 78 (1998) 141–144.
[4] A. Chakrabarti, S. Banerjea, T. Sahoo, A unified approach to problems of scattering of surface water waves by vertical barriers, J. Aust. Math. Soc. Ser. B 39 (1997) 93–103.
[5] P.F. Rhodes-Robinson, On the forced three-dimensional surface waves in a channel in the presence of surface tension, Proc. Cambridge Philos. Soc. 75 (1974) 405–426.
[6] P.F. Rhodes-Robinson, On surface waves in the presence of immersed vertical boundaries I, Quart. J. Mech. Appl. Math. 32 (1979) 109–124.
[7] P.F. Rhodes-Robinson, Note on the reflexion of water waves at a wall in the presence of surface tension, Math. Proc. Cambridge Philos. Soc. 92 (1982) 369–373.
[8] D.V. Evans, The influence of surface tension on the reflection of water waves by a plane vertical barrier, Proc. Cambridge Philos. Soc. 64 (1968) 795–810.
[9] P.F. Rhodes-Robinson, The effect of surface tension on the progressive waves due to incomplete vertical wave-makers in water of infinite depth, Proc. R. Soc. Lond. Ser. A 435 (1991) 293–319.
[10] W.E. Williams, Note on the scattering of water waves by a vertical barrier, Proc. Cambridge Philos. Soc. 62 (1966) 507–509.
[11] S.R. Manam, Scattering of membrane coupled gravity waves by partial vertical barriers, ANZIAM J. 51 (2009) 241–260.
[12] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, London, 1980.
[13] A. Chakrabarti, S.R. Manam, Solution of the logarithmic singular integral equation, Appl. Math. Lett. 16 (2003) 369–373.
[14] S.R. Manam, On the solution of dual integral equation, Appl. Math. Lett. 20 (2007) 391–395.
[15] S.R. Manam, Solution of logarithmic singular integral equation over multiple intervals, Appl. Math. Lett. 16 (2003) 1031–1037.