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Multiaxial fatigue – An overview

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Abstract. Many engineering components are subjected to multiaxial type of fatigue loading and the fatigue life relation based on uniaxial testing needs modifications before it could be used for multiaxial condition. In this paper a brief review of the different approaches based on stress and strain is presented for the correlation of bulk parameters with the fatigue life and the merits and limitations are discussed.

Keywords. Multiaxial fatigue; stress-strain parameters; in-phase out-of-phase fatigue; high temperature fatigue; life assessment.

1. Introduction

Multiaxial fatigue is the major consideration in the design of many of the structural components such as automobile parts, gas turbine components, power reactors etc. The parts will be subjected to varying loads of different amplitudes and frequencies in two or three directions depending on the thickness. This will give rise to biaxial or multiaxial fatigue and the assessment of fatigue life is an important design requirement for reliable operation and safety.

Most of the fatigue data are obtained based on tests performed under uniaxial loading condition. However, engineering components in general, are subjected to complex loading involving bending, torsion and axial forces. The design of these components is carried out based on fatigue properties under uniaxial loading conditions. It will be in the form of endurance limit or the low cycle strain range fatigue life relation. However, these data can serve only as an input information for the life prediction of components subjected to biaxial or multiaxial fatigue. The safe life assessment has to be carried out through suitable methods based on the deformation mechanisms that will be operative during the fatigue process. An excellent review of the state-of-the-art up to 1980 has been presented by Garud (1981). Many symposium proceedings have also been brought out in the last decade highlighting the developments made in the area of multiaxial fatigue (Miller & Brown 1985; Brown & Miller 1989; Kussmaul *et al* 1991).

A list of symbols is given at the end of the paper

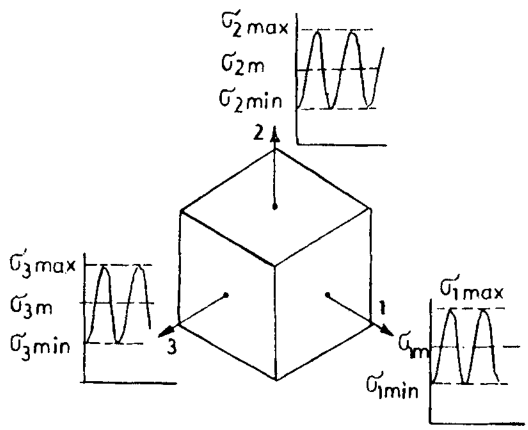

Figure 1. An element subjected to triaxial fatigue stressing.

2. Life prediction models

2.1 *State of stress and strain*

Normally components are designed based on high cycle fatigue data where the stress levels are below the yield strength of the material and general yielding of the material is not present. The design approach is based on the state of stress acting on the component. A typical element subjected to triaxial fatigue loading is shown in figure 1 and Mohr's circle for state of stress is shown in figure 2. Three principal stresses in general, σ_1, σ_2, and σ_3, are considered for evaluating the equivalent stress that is taken for life calculations. The stress that governs the deformation could be the maximum principal stress, σ_1, or the maximum shear stress, τ_{max}, given by the relation

$$\tau_{max} = (\sigma_1 - \sigma_3)/2. \tag{1}$$

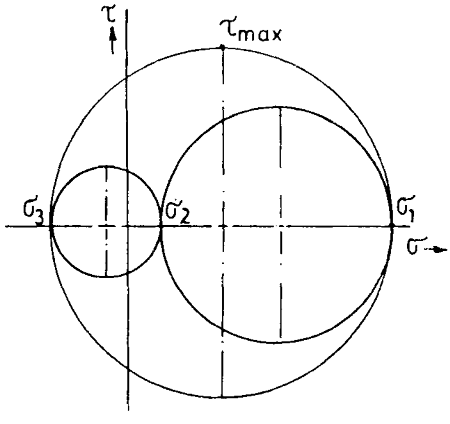

Figure 2. Mohr's circle for triaxial state of stress.

Octahedral shear stress, τ_{oct}, is also considered as a controlling parameter for deformation and is given by

$$\tau_{oct} = (1/3)[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}. \quad (2)$$

The corresponding strains, i.e., maximum shear strain and the octahedral shear strain are given by

$$\gamma_{max} = \varepsilon_1 - \varepsilon_2, \quad (3)$$

and

$$\gamma_{oct} = (2/3)[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]^{1/2}. \quad (4)$$

In terms of equivalent normal stress and strain we have

$$\tau_{oct} = 2^{1/2}/3\bar{\sigma}, \quad (5)$$

and

$$\gamma_{oct} = 2^{1/2}\bar{\varepsilon}. \quad (6)$$

In the following, stress- and strain-based approaches for multiaxial fatigue are considered for the calculation of the equivalent parameters.

2.2 *Stress based approach*

Consider a machine member subjected to a triaxial pulsating load along the three principal axes. The stresses vary from a minimum to a maximum with a mean value. For simplicity it is assumed that the stresses are in-phase and the principal axes are fixed. Under such a condition, Sines (1955) proposed a relation in the form

$$[(\sigma_{a1} - \sigma_{a2})^2 + (\sigma_{a2} - \sigma_{a3})^2 + (\sigma_{a3} - \sigma_{a1})^2]^{1/2} + [A(\sigma_{m1} + \sigma_{m2} + \sigma_{m3}] = 2^{1/2}\sigma_n, \quad (7)$$

where σ_{ai} is the alternating principal stress, σ_{mi} the mean stress, A a constant, σ_n the uniaxial fully reversed fatigue stress that is expected to give the same life on smooth specimens as the multiaxial state.

The above relation for biaxial loading will be reduced to

$$(\sigma_{a1}^2 - \sigma_{a1}\sigma_{a2} + \sigma_{a2}^2)^{1/2} + A(\sigma_{m1} + \sigma_{m2})/2^{1/2} = \sigma_n. \quad (8)$$

This relation gives a family of ellipses with their centres decided by the mean stresses σ_{m1} and σ_{m2}, as shown in figure 3. Thus, lines of constant octahedral shear stress are ellipses in such a plane. The different ellipses give the areas of safe alternating stresses on a stress plane for biaxial stresses. For any biaxial in-phase alternating stress that corresponds to the fatigue criterion, successive instantaneous states of stress are represented by points on a diameter of such an ellipse. The centre of the ellipse is the mean stress. Ellipses that are located on a line of constant octahedral normal mean stress 45° downward to the right, are of equal size. Increasing the mean stress decreases the size of the ellipse. The approach assumes that (i) there is no gross yielding, and (ii) the alternating stresses can be represented by principal alternating stresses along fixed principal axes.

Sines (1955) showed a good correlation with experimental results in high cycle fatigue. Static torsion has been found to have less effect than static bending on alternating fatigue strength when gross yielding is avoided.

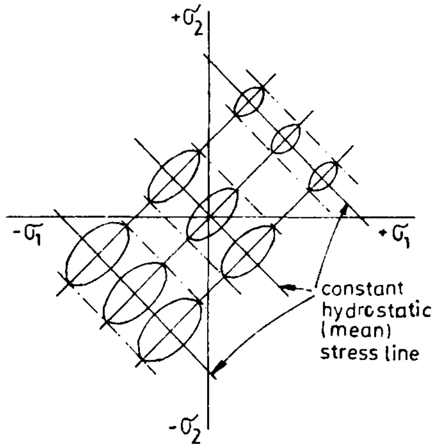

Figure 3. Areas of safe alternating stresses on a biaxial stress-plane.

Findley (1957, 1959) introduced the influence of normal stress acting on the maximum shear stress plane and has given the fatigue life relation as

$$\tau_a + k\sigma_{n\,\max} = f(N_f) \tag{9}$$

where k is a constant. Thus the allowable alternating shear stress, τ_a, for a given fatigue life decreases with an increase in the maximum normal stress $\sigma_{n\,\max}$ on the plane of the critical alternating shear stress.

Combining the approaches due to Sines and Findley, an equation for the fatigue equivalent stress has been proposed in the form,

$$\begin{aligned}&[(\sigma_{a1}-\sigma_{a2})^2+(\sigma_{a2}-\sigma_{a3})^2+(\sigma_{a3}-\sigma_{a1})^2]^{0\cdot5}\\&+A[\sigma_{m1}+\sigma_{m2}+\sigma_{m3}]+B[\sigma_{a1}+\sigma_{a2}+\sigma_{a3}]=C,\end{aligned} \tag{10}$$

where σ_{ai} are the principal alternating stress ranges, and σ_{mi} the principal static stresses.

Kakuno & Kawada (1979) were able to show good correlation with experimental results obtained from bending and torsion fatigue testing. The above criterion implies that there will be fatigue failure under hydrostatic alternating stresses i.e., $\sigma_{a1} = \sigma_{a2} = \sigma_{a3}$. The constant B which determines the contribution of the hydrostatic alternating component could be positive or negative indicating the deterimental or beneficial effects. However, for cases where there is phase lag, the effect is not clearly brought out in the equation.

McDiarmid (1985, 1987, 1991, 1993) introduced a modification in the stress approach and gave the modified equation in the form

$$\tau_a(1 + (\sigma_{n\,\max}/2\sigma_{\mathrm{ult}})) = F(N_f), \tag{11}$$

where σ_{ult} is the tensile strength. The function F depends on whether the cracking grows along the surface (mode III) or grows into the surface (mode II). The first case occurs in torsion loading and the second one under biaxial tension loading. The theory has been extended for the cases of out-of-phase biaxial stresses and has been shown that the out-of-phase stresses produce shorter fatigue lives than equal in-phase stresses. Typical

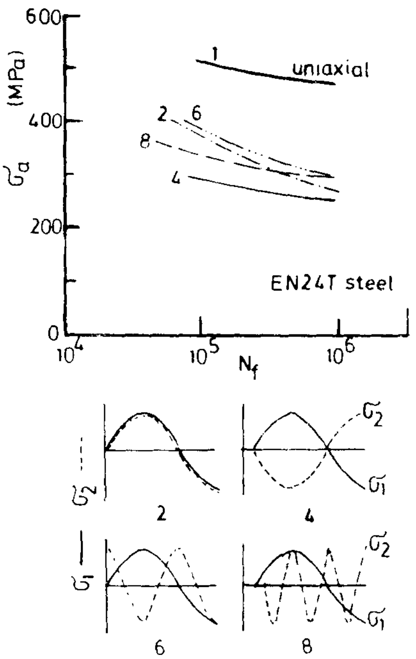

Figure 4. Effect of phase lag and different frequency on S–N curve of EN 24 steel.

experimental results for both out-of-phase and different frequencies are shown in figure 4 for the material EN 24 steel. S–N curve (case 1) gives for uniaxial condition. In biaxial condition, when σ_1 and σ_2 are both in-phase (case 2), the fatigue strength is reduced as compared to uniaxial loading. When both of them are completely out-of-phase (case 4), the strength is still lower. Similarly, it can be seen that different frequencies for σ_1 and σ_2 also (cases 6 and 8) reduce the fatigue strength as compared to the uniaxial condition.

A relation of the type

$$\tau_{\max} + k\sigma_{n\,\max} = CN_f^{-\alpha} \tag{12}$$

has been used to compute the life.

For out-of-phase angle of 90°

$$\tau + k\delta_n = \tau \cos wt + k\sigma \sin wt \tag{13}$$

For maximum value of $(\tau + k\sigma_n)$ we have

$$\tan wt = k\sigma/\tau = k/\lambda_\sigma, \tag{14}$$

where $\lambda_\sigma = \tau_a/\sigma_a$.

Thus the maximum value of $(\tau + k\sigma_n)$ can be found for different values of $\lambda_\sigma (= \tau_a/\sigma_a)$.

The analysis can also be extended to cases of different frequencies. The experimental results on 1045 steel and the applicability of the model are shown in figure 5. In the out-of-phase loading the damage is likely to be more severe. Though, in the out-of-phase cases, the maximum normal stresses do not occur simultaneously with maximum shear stresses in any plane, the analysis used here maximises the $(\tau + k\sigma_n)$ critical parameter for each particular loading condition.

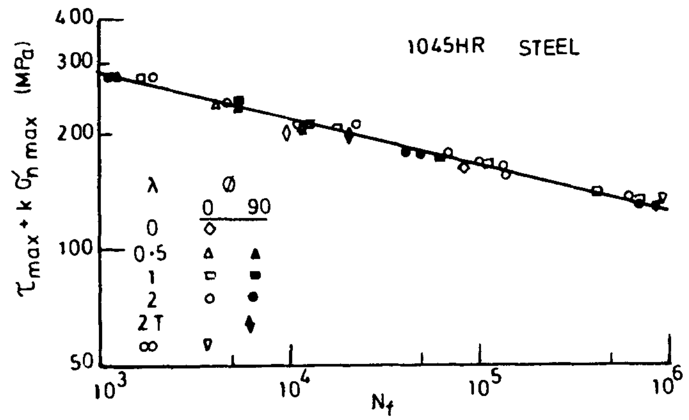

Figure 5. Relation between the equivalent stress due to McDiarmid and fatigue life of 1045 steel.

In the stress based approach two criteria are normally used – one based on maximum shear stress amplitude (Tresca) to correlate the life and incorporated in the ASME Boiler and Pressure Vessel Code Section III – and the other based on octahedral shear stress amplitude (von Mises) incorporated in Code Case N-47-12 of Section III.

Both these methods compute an equivalent fully reversed uniaxial stress amplitude from an applied multiaxial state of stress. Life estimation is made by using the relation

$$\sigma_{eq} = \sigma'_f(N_f)^{-\alpha}. \tag{15}$$

Notches are quite common in components subjected to multiaxial fatigue, such as crank shafts, connecting rods, transmission shafts etc., and hence stress concentration factors should also be taken into consideration.

In cases of shafts or machine members subjected to both bending and torsion the amplitude of the maximum shearing stress used as equivalent stress amplitude in the above equation, is calculated using the Tresca criterion as

$$\tau_{max} = \sigma/2[1 + K^2 + (1 + 2K^2\cos(2\phi) + K^4)^{0\cdot 5}]^{1/2}, \tag{16}$$

where σ is the notch bend stress amplitude including the stress concentration factor Kb in bending, τ is the notch torsional shearing stress amplitude including the stress concentration factor Kt in torsion, $K = 2\tau/\sigma$ and ϕ is the phase angle between applied bending and torsional loading.

In the second approach, based on octahedral shear stress amplitude, the equivalent stress amplitude, τ_{eq}, is given by

$$\tau_{eq} = (\sigma/2)[1 + 3/4K^2 + (1 + (3/2)K^2\cos(2\phi) + (9/16)K^4)^{0\cdot 5}]^{0\cdot 5}. \tag{17}$$

The effects of stress concentration in bending and torsion are to be included in both the approaches. In some cases the fatigue notch factor K_f could be assumed to be unity. However, this assumption is likely to lead to over estimation of fatigue life, specially in high strength materials. In high cycle fatigue, out-of-phase loading appears to be more damaging than in-phase loading for the same applied load levels.

2.3 *Strain based approach*

Fatigue lives under multiaxial loading may differ significantly from those observed under equivalent uniaxial loading conditions due to additional damage and deformation mechanisms that are activated under first condition. Many investigations have been conducted over the last two decades on thin-walled tubular specimens to characterize the fatigue behaviour of several engineering alloys under combined axial and torsional loading conditions. These investigations involved both in-phase and out-of-phase strain-controlled axial–torsional fatigue tests. The materials investigated included aluminium alloys, low alloy steels, stainless steels, and nickel base super alloys (Socie & Shield 1984; Fatemi & Stephens 1989; Jayaraman & Ditmars 1989; Nitta *et al* 1989; Sreeramesh & Bonacuse 1993).

Many life prediction models have been proposed to estimate fatigue life under combined cyclic axial–torsional loading conditions (Haverd & Topper 1971; Miller & Brown 1984; Fatemi & Kurath 1988; Fatemi & Socie 1988; Socie 1989). Some of these models include parameters based on the von Mises equivalent strain range, the principal stresses or strains and on a combination of maximum shear strain and the normal stress and/or normal strain acting on the maximum shear plane. Most of these life prediction models have been verified with room temperature axial–torsional data.

The local strain based approach has become an important tool for prediction of fatigue life. Many attempts have been made to identify the damage parameters capable of predicting multiaxial fatigue. The important among them are

(a) maximum principal strain amplitude;
(b) maximum shear strain amplitude;
(c) von Mises effective strain amplitude;
(d) Brown–Miller approach;
(e) Lohr–Ellison approach;
(f) other strain-based approaches;
(g) energy-based approach.

2.3a *Maximum principal strain approach*: In the maximum principal strain approach the principal strain $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are determined by an appropriate transformation of the measured or applied strains ε_{ij}. For correlating multiaxial fatigue tests, the range of maximum principal strain on the plane that experiences the maximum principal strain range is considered the dominant parameter to describe damage. The strain life equation is written in the form

$$\Delta\varepsilon_1 = A(N_f)^{-\alpha} + B(N_f)^{-\beta}, \qquad (18)$$

where, $A = \sigma'_f/E$ and $B = \varepsilon'_f$.

2.3b *Maximum shear strain approach*: In the case of maximum shear strain range approach the shear strain range $\Delta\gamma_{max}$ is given by

$$\Delta\gamma_{max} = (\varepsilon_1 - \varepsilon_3)_{max}. \qquad (19)$$

This can be related to the fatigue life N_f as

$$\Delta\gamma_{max} = A(N_f)^{-\alpha} + B(N_f)^{-\beta}. \qquad (20)$$

In the case of a shaft with a notch subjected to bending and torsion the relation for shear strain range, $\Delta\gamma$ can be written as

$$\Delta\gamma_{max} = [\Delta\varepsilon_x^2(1+v)^2 + \Delta\gamma_{xy}^2]^{0\cdot 5}, \quad \text{for in-phase loading,} \tag{21}$$

and

$$\Delta\gamma_{max} = \text{greater of} \begin{bmatrix} \Delta\gamma_{xy} \\ \Delta\varepsilon_x(1+v) \end{bmatrix}, \quad \text{for } 90^\circ \text{ out-of-phase loading,} \tag{22}$$

$\Delta\gamma_{max}$ is the range of notch maximum shear strain, $\Delta\varepsilon_x$ is stabilized notch bending strain range, and $\Delta\gamma_{xy}$ is stabilized notch torsional shear strain range.

The equivalent strain range is found from

$$\Delta\bar{\varepsilon} = \Delta\gamma_{max}/(1+v).$$

The elastic and plastic values of Poisson's ratio have been taken as 0·29 and 0·5.

2.3c *The von Mises equivalent strain range model*: The von Mises effective strain may be thought of as the root mean square of the maximum principal shear strain normalised to axial loading. It is also called the octahedral shear strain.

This model gives the multiaxial equivalent strain range in the form

$$\Delta\bar{\varepsilon} = [1/(2^{1/2}(1+v))]/[(\Delta\varepsilon_1 - \Delta\varepsilon_2)^2 + (\Delta\varepsilon_2 - \Delta\varepsilon_3)^2 + (\Delta\varepsilon_3 - \Delta\varepsilon_1)^2]^{1/2}, \tag{23}$$

where

$$\Delta\bar{\varepsilon} = \text{von Mises effective strain amplitude.}$$

In the case of combined bending and torsion the relation can be written as

$$\Delta\varepsilon_1 \text{ or } \Delta\varepsilon_2 = (1/2)(\Delta\varepsilon_x + \Delta\varepsilon_y) \pm [(\Delta\varepsilon_x - \Delta\varepsilon_y)^2 + \Delta\gamma_{xy}]^{0\cdot 5} \tag{24}$$

$$\Delta\varepsilon_3 = [-v/(1-v)](\Delta\varepsilon_1 + \Delta\varepsilon_2), \tag{25}$$

where $\Delta\varepsilon_y$ is the circumferential strain range.

The effective strain is used as an equivalent uniaxial strain amplitude to predict fatigue life. The variable Poisson's ratio, v, is defined as

$$v = (v_{el}\varepsilon_{el} + v_p\varepsilon_p)/\varepsilon_t \tag{26}$$

where ε_{el}, ε_p, ε_t are elastic, plastic and total strains respectively and are used in cases of proportional loading.

2.3d *The Brown–Miller approach*: Figure 6 shows the Mohr's circle based on maximum principal strain in triaxial loading situation. In fatigue fracture two strain parameters are important, that is, (i) the size of the largest circle created in a cycle $(\varepsilon_1 - \varepsilon_3)$, and (ii) the location of this circle in strain space, i.e., $(\varepsilon_1 + \varepsilon_3)/2$. Thus it can be seen that equivalent stress or strain calculation on the basis of Tresca or von Mises criterion may not be sufficient to calculate the fatigue life.

Brown & Miller (1973) (B–M) postulated that failure under multiaxial fatigue is governed by a functional relation between the maximum range of shear strain and the range of normal strain acting on the plane where the former occurs. Thus

$$(\varepsilon_1 - \varepsilon_3)/2 = f[\varepsilon_1 + \varepsilon_3]/2 \tag{27}$$

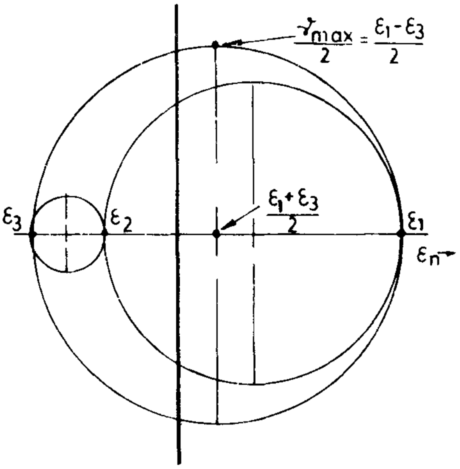

Figure 6. Mohr's circle based on maximum principal strains.

The approach proposed is for fixed principal strain axes and in-phase straining. The function f is not defined and varies with life.

The B–M approach considers that, in addition to the amplitude of the maximum shear strain γ_{max}, the amplitude of the normal strain ε_n acting perpendicular to the plane of maximum shear strain amplitude. Thus we have

$$(\gamma_{max}/g)^i + (\varepsilon_n/h)^i = 1. \tag{28}$$

Contours of constant life on γ_{max} versus ε_n called Lamda planes are obtained. The constants g, h, and i were found to vary with life. However, it was suggested that a safe and simple design criterion could be established with $i = 2$, so that

$$(\gamma_{max}/g)^2 + (\varepsilon_n/h)^2 = 1 \tag{29}$$

A typical Lamda plane and the constant life contours on the γ_{max} and $(\varepsilon_1 + \varepsilon_3)/2$ axes are shown in figure 7. The figure also shows how stage I and stage II cracks grow in relation to the surface. There are two cases (A) and (B). Stage I growth plane is inclined at 45° to the principal direction while stage II is perpendicular. In case (A) type, cracks are contained in the surface layers. This state occurs in torsion testing and the growth of these cracks is by mode II along the surface and mode III in depth. In case (B) type, the crack growth situation is more dangerous. The cracks grow from the surface to inside the material. The stage I crack is driven inside by a mode II mechanism, but the mode I plays an increasing role as crack enters stage II. As can be seen in figure 7, the same life time of 500 cycles occurs when the plastic shear strain is 0·325 in a plane strain test and 0·6 in a pure torsion test. ε_n is zero in both cases. This is because in the former case a dangerous case (B) type crack is generated while in the latter case of torsion test, a case (A) type crack propagates along the surface.

Extending this approach Kandil *et al* (1982) showed that the life, N_f, can be related to the strain criterion in the form

$$\Delta\gamma_{max} + S\varepsilon_n = A_1(\sigma_f'/E)(N_f)^{-\alpha} + A_2\varepsilon_f'(N_f)^{-\beta}, \tag{30}$$

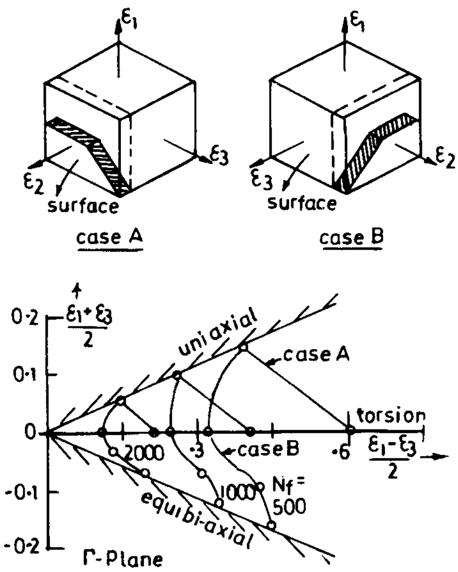

Figure 7. Representation of constant life contours on the Lamda plane. Two types of crack growth (A) along the surface and (B) inside the material are also shown.

where S = material parameter = unity,

$$A_1 = 2(1 + v_e) + S(1 - v_e) = 3{\cdot}3,$$

$$A_2 = 2(1 + v_p) + S(1 - v_p) = 3{\cdot}5.$$

2.3e *The Lohr–Ellison approach*: Taking into consideration the shear strain range that makes the crack propagate from the surface to inside the material, Lohr & Ellison (1980) proposed a critical plane theory, leading to the formulation of the relation

$$\Delta\gamma^* + k\varepsilon_n^* = C. \tag{31}$$

$\Delta\gamma^*$ is the maximum shear strain acting on a plane which intersects the surface at 45° and will control fatigue life in the low cycle regime. $\Delta\varepsilon_n^*$ is the direct strain normal to this plane and may have a modifying influence.

When the minimum principal strain is primarily normal to the surface the $\Delta\gamma^*$ is given by

$$\Delta\gamma^*/2 = \Delta(\varepsilon_1 - \varepsilon_2)_{max}/2 \tag{32}$$

If the second principal strain is most nearly normal to the surface, the value is given by

$$\Delta\gamma^*/2 = \Delta(\varepsilon_1 - \varepsilon_2). \tag{33}$$

The normal strain to the plane of shear is given by

$$\Delta\varepsilon_n/2 = [\varepsilon_1 + \varepsilon_3(\text{or } \varepsilon_2)]/4. \tag{34}$$

The third (or the second) principal strain is normal to the surface.

A life relationship based on the parameter can be developed as

$$\Delta\gamma^* + k\varepsilon_n^* = 2{\cdot}88(\sigma_f'/E)(2N_f)^{-\alpha} + 3{\cdot}2\varepsilon_f'(2N_f)^{-\beta}, \tag{35}$$

where $k = 0{\cdot}4$. The constants depend on the value of k as in the previous example.

The above relation proposed by Lohr & Ellison (1980) is similar to the theory proposed by Kandil *et al* (1982) which yields the relation

$$\Delta\gamma_p + S\varepsilon_p = F(N_f). \tag{36}$$

Thus both the theories with suitable values for the constants k and S should fit the data equally well.

2.3f *Other strain based approaches*: Walker (1970) proposed an effective strain, ε_{eff} parameter for biaxial fatigue loading as

$$\varepsilon_{eff} = (\Delta\gamma_0)^m(\varepsilon_T)^{1-m}, \tag{37}$$

where

$$\Delta\gamma_0 = 2/3[\Delta\gamma_x^2 + (\Delta\gamma_x - \Delta\gamma_y)^2 + \Delta\gamma_y^2]^{1/2}, \tag{38}$$

$$\varepsilon_T = \langle\varepsilon_x\rangle + \langle\varepsilon_y\rangle + \langle\varepsilon_z\rangle. \tag{39}$$

$\Delta\gamma_i$ is the principal shearing strain ranges, ε_i is the principal normal strains that give the maximum value of ε_T, $\langle\varepsilon\rangle = \varepsilon$, if $\varepsilon > 0$, are zero otherwise.

The $\Delta\gamma_0$ parameter, referred to as the octahedral shearing strain range, is considered the main parameter coupled with the total strain ε_T.

A total strain criterion in the form

$$\varepsilon_T = (2/3)^{1/2}[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2]_{max}^{1/2} \tag{40}$$

was proposed by Zamrik (1972).

ε_i are the three instantaneous principal strains. ε_T is the maximum value of the right hand side. This approach is similar to von Mises criterion. The total strain ε_T was related to N_f in the form,

$$\varepsilon_T = A(N_f)^{-\alpha}, \tag{41}$$

and was found to give good correlation for both in-phase and out-of-phase combined torsion and push-pull straining of tubular specimens.

2.3g *Energy based approach*: Plastic work approach uses uniaxial stress–strain data to compute the hysteresis energy per cycle, resulting from bending and torsional straining, in-phase and out-of-phase. The plastic work per cycle is defined for this loading as

$$W_p = \int \sigma_x d\varepsilon_x^p + \tau_{xy} d\gamma_{xy}^p, \tag{42}$$

where σ_x and τ_{xy} are the axial and torsional components of the stress tensor. $d\varepsilon_x^p$ and $d\gamma_{xy}^p$ are the corresponding infinitesimal load increment. It is assumed that

$$W_p = AN_f^{-\alpha} \tag{43}$$

where the constants A and α are fitted from uniaxial strain life data.

3. Data analysis and discussion

Figure 8(a–e) show the correlation of experimental data with prediction based on (a) principal strain theory, (b) von Mises criterion, (c) maximum shear strain theory, (d)

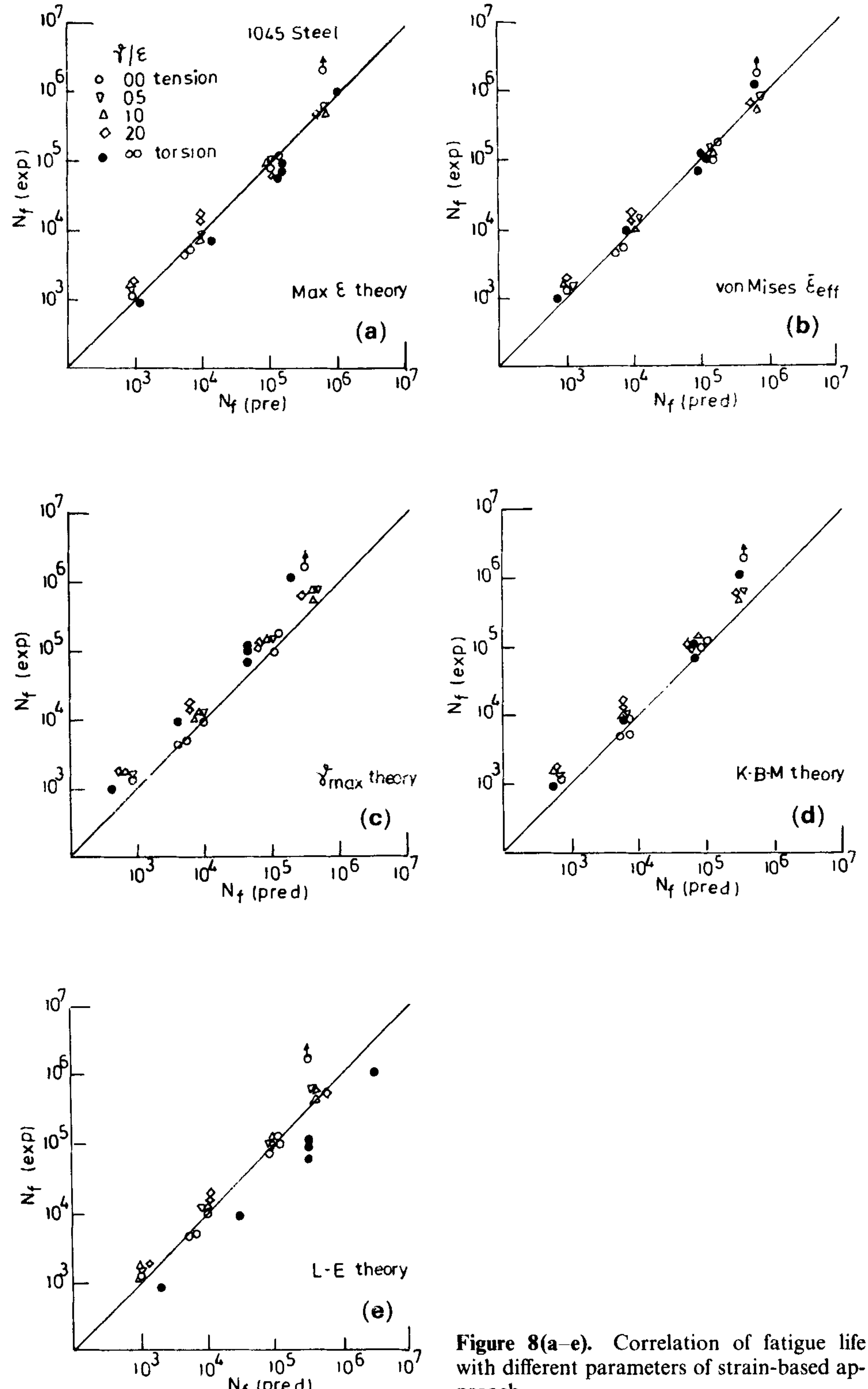

Figure 8(a–e). Correlation of fatigue life with different parameters of strain-based approach.

Kandil, Brown and Miller theory and (e) Lohr and Ellison approach. The material tested was 1045 steel and results are for smooth thin walled tubular specimen (Fash *et al* 1985). It can be seen that the maximum strain theory and von Mises criterion give good correlation with evenly distributed scatter. In the case of maximum shear strain approach, the prediction is conservative at low as well as high cycles regions. So is the case with the Kandil–Brown–Miller approach also. In the case of the Lohr–Ellison approach the scatter is wide and the torsion results consistently fall out with the prediction which is always non-conservative. Fash *et al* (1985) also found that the aforementioned approaches offer considerably less correlation for notched shaft specimens.

A comparison of the degree of correlation found using a number of failure criterion as applied to automobile steel is shown in figure 9 (McDiarmid 1993). The octahedral shear strain approach as also the approaches due to Brown & Miller (1973) and McDiarmid (1993) modified shear stress approach appear to give fairly good correlation within a factor of ± 2. Other approaches give large scatter. The octahedral shear strain criterion has the disadvantage of not being able to account for the effect of mean stress or strain. The other two criteria can account for mean stress/strain within their normal stress/strain terms. Thus the modified shear stress based on the critical plane criterion of failure, i.e.,

$$\tau^*_{\max} + k\sigma^*_{n\,\max} = \text{constant} \tag{44}$$

could reasonably be accurate in correlating both the in-phase and 90° out-of-phase tension–torsion SAE experimental data.

SAE experimental programme (Tipton 1989) has shown that von Mises criterion provided reasonably good estimates of finite life under in-phase loading. In the case of out-of-phase loading non-conservative estimates are obtained. It was also observed by Tipton that all the methods gave somewhat more conservative predictions for pure torsion or loadings with higher ratios of torsion to bending. Further, all prediction methods were conservative at long life and somewhat non-conservative at lower cycles, particularly for bending or loadings with larger ratios of bending to torsion.

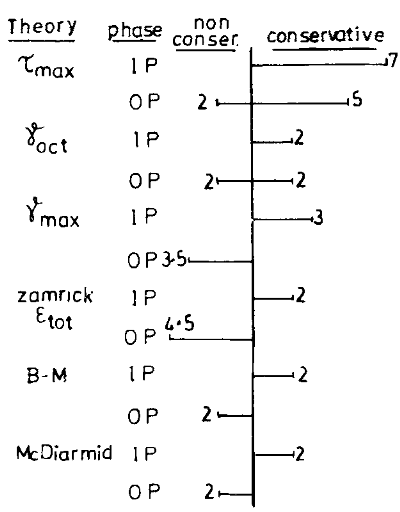

Figure 9. Degree of correlation (scatter width) based on different parametric approaches.

Multiaxial fatigue behaviour of notched specimens will depend on stress concentration and stress gradient, as in uniaxial fatigue, and also on the specific type of notch geometry and its influence on cracking behaviour.

Correlations made by conventional strain based methods (Tresca and von Mises) for in-phase tests are fairly sensitive to the Poisson's ratio assumed (i.e., the degree of plasticity). The use of a variable Poisson's ratio with the von Mises approach eliminates this sensitivity. However, the use of a variable Poisson's ratio is invalid for out-of-phase loading.

The ASME Boiler and Pressure Vessel Elevated Temperature Code Sec III uses the von Mises correlation for non-proportional loading with Poisson's ratio as 0·5 (the fully plastic value). However, predictions made by this approach are generally worse (more non-conservative) than predictions made by the same procedure with Poisson's ratio = 0·29 (the fully elastic value). All out-of-phase predictions using this approach are non-conservative, regardless of Poisson's ratio.

The Lohr–Ellison method makes life predictions similar to those of the Brown and Miller approach for bending dominated loading. However, for tests where torsional loading dominates, predictions are nonconservative. The application of this approach to out-of-phase loading is not clearly possible since cracks which are deemed harmless by this approach, could grow detrimentally due to rotating principal stresses.

In the case of out-of-phase biaxial fatigue, constant amplitude stresses of different frequencies produce non-constant amplitude shear stress variations (Miller *et al* 1967). For these conditions the normal stress acting on the maximum shear stress plane is also of non-constant amplitude. The analysis of McDiarmid (1985) concludes that the maximum shear stress criterion using damaging shear stress amplitude appears to be a reasonable basis for fatigue life evaluation under out-of-phase biaxial stresses of different frequencies.

In general out-of-phase stresses could be more damaging than in-phase stresses of the same magnitude, though opposite trends are also observed in certain materials. The total strain approach due to Zamrik (1972) can be applied to out-of-phase loading by maximizing the function ε_T. However, the total strain expression reduces to von Mises criterion when the Poisson's ratio is 0·5. Kanazawa *et al* (1977, 1979) showed that out-of-phase loading in 1% Cr–Mo–V steel is more damaging than in-phase loading in the low cycle regime. Further, they observed that both the Tresca and von Mises type criteria are not conservative for out-of-phase multiaxial loading. They proposed that the shear strain range and the normal strain amplitude on the maximum shear plane should govern out-of-phase multiaxial fatigue, but offered no functional relationship.

When ε_1, ε_2, and ε_3 rotate during every cycle in out-of-phase loading, a number of cracking systems will start operating and endurance will be reduced because of the large value of ε_n that can be obtained. However, in some cases, the life time may be extended due to crack growth on one plane being obstructed by other cracks on other planes. Taking a simple bi-axial loading, $\sigma_1 = \sigma_2$ and $\sigma_3 =$ zero, three distinct cracking systems can operate in the material. Depending on the phase angle $\emptyset$ one or more cracks will grow thereby either increasing or decreasing the endurance life. Figure 10 shows the effect of phase angle $\emptyset$ on the endurance cycles in such a simple system of loading (Miller & Brown 1984).

The plastic work life predictions are comparable to those made by the von Mises and the Brown and Miller approaches for in-phase and are reasonably good for

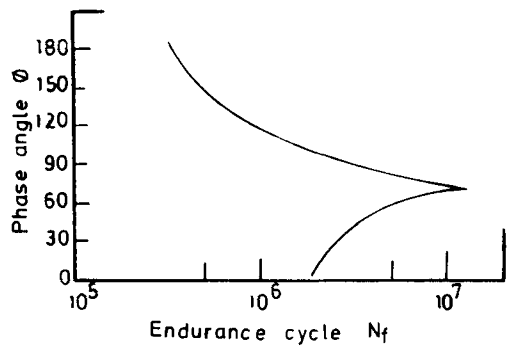

Figure 10. Effect of phase angle on the endurance life.

out-of-phase loadings. However, the main difficulty with this approach is that when plastic strains are small, slight variations in their computation can lead to large differences in predicted life. Further, numerical difficulties may be encountered in incremental plasticity routines specially in 90° out-of-phase loading.

Several new equivalent strain criteria have been proposed in recent times, as other criteria like Tresca, octahedral shear strain etc., do not give good predictions, even for in-phase loading. The inadequacy of the octahedral shear strain approach is clearly shown in the case of EN 15 R steel for in-phase tension torsion test results as shown in figure 11a. If $R(\varepsilon)$ is the ratio of γ_{oct} to $\gamma_{oct\ torsion}$, i.e.,

$$R(\varepsilon) = \gamma_{oct}/\gamma_{oct\ torsion}. \tag{45}$$

it should be equal to unity if octahedral shear strain criterion is valid. However, the values of $R(\varepsilon_{pp})$ (push-pull loading) deviate significantly from unity indicating the inadequacy of the γ_{oct} criterion.

Realising the above inadequacy of von Mises criterion, Troshchenko *et al* (1993) proposed a modified octahedral shear strain approach. The octahedral shear stress component γ_{oct*} is taken in the form,

$$\gamma_{oct*} = 0{\cdot}816[(4/3)(1+v^*)^2\varepsilon_z^2 + \gamma_{z\theta}^2]^{1/2}, \tag{46}$$

for a material subjected to axial and torsional loading.

v^* is the Poisson's ratio corresponding to the achieved level of strains. The modified $\delta_{oct(modified)}$ is a function of state of strain and N_f. Thus

$$\gamma_{oct(modified)} = 4\psi(\varepsilon, N_f)\gamma_{oct*} \tag{47}$$

The function $\psi(\varepsilon, N_f)$ can be evaluated for different phase angles. The correlation between the modified octahedral shear strain and the fatigue life is shown in figure 11b for En 15 R steel. The agreement appears to be good for both in-phase and out-of-phase loading, though separate endurance curves for reversed tension–compression and for torsion are required for the computation. The approach which takes into account the non-proportionality of the deformation process and the dependence of the equivalent strain parameter on life time, is somewhat complicated and involves detailed calculation.

Sreeramesh & Bonacuse (1993) have analysed the behaviour of Haynes 188 alloy in in-phase and out-of-phase axial-torsional fatigue at 760°C. The correlation based on the von

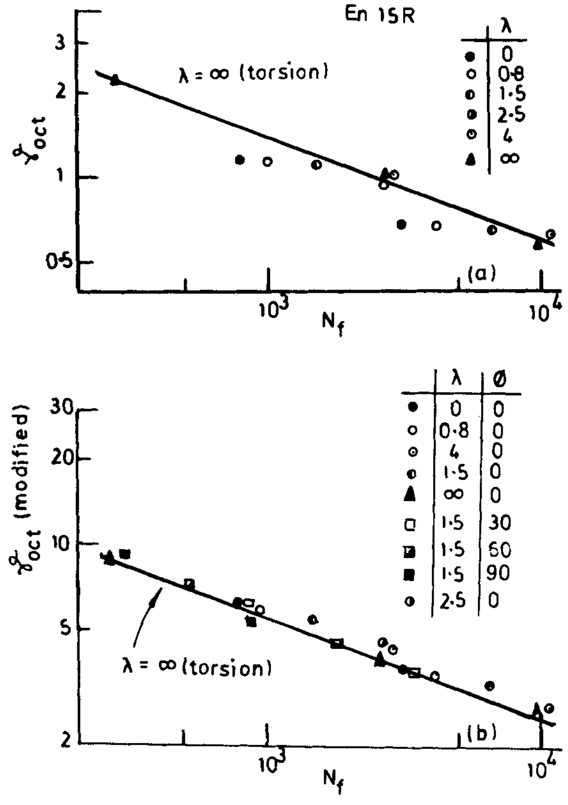

Figure 11. (**a**) Relation between octahedral shear strain and fatigue life for different phase angles. (**b**) Correlation of fatigue life with the modified octahedral shear.

Mises criterion, Fatemi–Socie–Kurath (F–S–K) (1988) and Smith–Watson–Topper (S–W–T) (1970) approaches are shown in figures 12a, b and c respectively.

The von Mises equivalent strain range $\Delta\varepsilon_{eq}$ is given by

$$\Delta\varepsilon_{eq} = \{[\Delta\varepsilon_x - \Delta\varepsilon_y)^2 + (\Delta\varepsilon_y - \Delta\varepsilon_z)^2 + (\Delta\varepsilon_z - \Delta\varepsilon_x^2)] + (3/2)[\Delta\gamma_{xy}^2 + \Delta\gamma_{yz}^2 + \Delta\gamma_{zx}^2]\}^{1/2}/[2^{1/2}(1 + v_{eff})],$$

with

$$v_{eff} = [(\Delta\varepsilon_e v_e + \Delta\varepsilon_p v_p)/\Delta\varepsilon_t]. \tag{48}$$

The von Mises equivalent strain range for in-phase tests has been computed with the maximum axial strain as the reference because in an in-phase test the axial and engineering shear strains reach their respective maxima simultaneously. However, in an out-of-phase test the axial and the engineering shear strains reach their respective maxima at separate points in a cycle. Therefore for each out-of-phase axial torsion test the equivalent strain range is computed twice: once with the maximum axial strain as the reference and a second time with the maximum shear strain as the reference. The larger of the two values is used for fatigue life prediction.

The F–S–K parameter is given as

$$\gamma_{max}[1 + k(\sigma_n^{max}/\sigma_{yield})] = F(N_f) \tag{49}$$

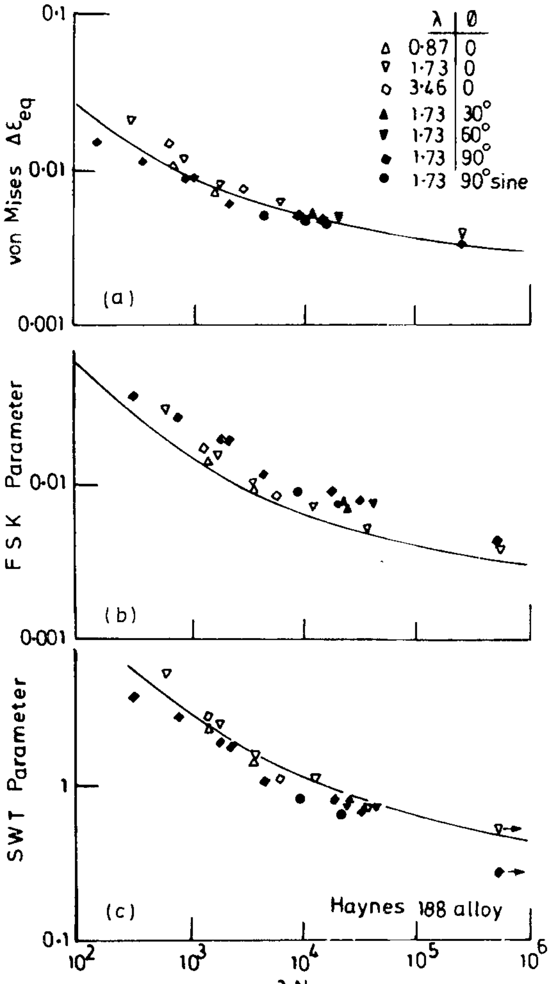

Figure 12(a–c). Correlation of elevated temperature fatigue life of Haynes 188 alloy with different parameters.

The F–S–K model has been successfully used to predict room temperature in- and out-of-phase axial torsional fatigue lives for materials that exhibit a shear mode of failure.

The S–W–T parameter is given by

$$\Delta\varepsilon_1 \sigma_{1\max} = \sigma'_f \varepsilon'_f (2N_f)^{-\beta} + (\sigma_f'^2/E)(2N_f)^{-\alpha}. \tag{50}$$

In the S–W–T model the magnitude and the plane of the maximum principal strain amplitude $\Delta\varepsilon_1$ are determined first for a given cyclic of axial torsional loading. Then $\Delta\varepsilon_1$ is multiplied by the maximum normal stress σ_1 that occurs on that plane. The constants in (50) are obtained from uniaxial fatigue properties. The S–W–T parameter is successful in predicting the room temperature fatigue life under in- and out-of-phase axial torsional loading specially in materials where cracks initiate and propagate perpendicular to the maximum principal strain direction.

The description of the results by von Mises criterion appears to be the best among the three approaches considered. The F–S–K parameter is very conservative and under-predicts, whereas the scatter in the S–W–T approach is large and the prediction is somewhat non-conservative upto a factor of 4.

In high temperature multiaxial fatigue, damage accumulation is accelerated by the presence of mean component of stress and strain. The effect of time-dependent creep should also be taken into account in such cases. Further, ratchetting phenomenon will take place which will increase the strain accumulation. Hence the analysis becomes more complicated compared to ambient temperature multiaxial fatigue.

4. Concluding remarks

Important criteria and parameters for multiaxial fatigue are reviewed in this presentation. The Tresca and von Mises criteria appear to be inadequate and hence approaches which take into account both shear stress/strain and normal stress/strain acting on critical planes as the controlling parameters have been developed. However, they are also not able to predict the fatigue life under different conditions of loading. Many times the predictions are either conservative or non-conservative and the scatter is more than by a ± 2 factor.

The stress-based approaches can be applied to high cycle fatigue where gross yielding is not encountered. In low cycle fatigue, strain-based criteria will be better suited. Energy-based approach shows promise in predicting the life. However, energy is a scalar quantity and fatigue crack growth is controlled by stress/strain on certain critical planes which is a vector quantity. Further accurate computing of the plastic work where the plastic strain range is small will be difficult and will lead to erroneous results.

The stress–strain path in multiaxial loading condition is more important in the fatigue failure process (Wu & Yang 1987) and hence the deformation mode under multiaxial loading must also be considered for crack nucleation and its subsequent growth.

Thus the prediction methodology of life in multiaxial fatigue is rather very much involved and as their development stands to-day the approaches made are not complete in all respects.

List of symbols

A, B, C	constants;
E	Young's modulus;
F	function;
K	ratio shear stress/normal stress;
N	number of cycles to failure;
υ	Poisson's ratio;
α, β	exponents;
γ	shear strain;
Δ	range;
$\Delta\sigma$	stress range $= 2\sigma a$;
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	principal strains;
λ	biaxial strain ratio, γ/ε;
σ	stress;

$\rho_1, \sigma_2, \sigma_3$	principal stresses;
σ_n	normal stress;
σ_{eq}	equivalent stress;
σ_f	fatigue strength coefficient;
σ_m	mean stress;
σ_a	alternating component of the stress;
τ	shear stress;
ϕ	phase angle.

Suffixes

1, 2, 3	principal directions.

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