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Mechanical experiments to identify homogeneous bodies

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ABSTRACT

All bodies are inhomogeneous at some scale but experience has shown that some of these bodies can be idealized as a homogeneous body. Here we examine which bodies can be idealized as a homogeneous body when they are subjected to a non-dissipative mechanical process. This is done by studying circumstances in which an inhomogeneous body admits pure stretch homogeneous deformations. Then, we devise experiments wherein these circumstances are prevented. If homogeneous deformation is observed in these devised experiments, the body could be modeled as a homogeneous body. We limit our analysis to a class of isotropic elastic bodies deforming from a stress free reference configuration whose Cauchy stress is explicitly related to left Cauchy–Green deformation tensor. It is further assumed that the constitutive relation is differentiable function of the position vector of material particles in the stress free reference configuration. Then, we find that a cuboid made of compressible and isotropic material could be modeled as a homogeneous body if it deforms homogeneously due to the application of the normal stresses on all of its six faces and the magnitude of the normal stresses on three orthogonal faces are different. A cuboid made of incompressible and isotropic material could be modeled as a homogeneous body, if it deforms homogeneously in two different biaxial experiments, such that the plane in which the forces are applied in the two biaxial experiments is mutually orthogonal.

1. Introduction

Use of composites, metallic alloys, concrete, polymers which are believed to be inhomogeneous is on the raise. Apart from these man made materials, many naturally occurring bodies like arteries, tendons, ligaments, valves are also thought to be inhomogeneous. Because of the need to understand the mechanical response of these bodies, there is an ever growing literature devoted to understand the issues in the deformation of these bodies under applied loads. This study examines which bodies can be idealized as a homogeneous body when they are subjected to a non-dissipative mechanical process.

According to Truesdell and Noll (1965), when one is interested in purely mechanical processes, two material points $P_1, P_2 \in \mathcal{B}$ are said to be materially uniform, if there exist two placers κ_1 and κ_2 such that the neighborhoods $N_{\mathbf{X}_1}$ of $\mathbf{X}_1 = \kappa_1(P_1)$ and $N_{\mathbf{X}_2}$ of $\mathbf{X}_2 = \kappa_2(P_2)$ are indistinguishable with respect to their mechanical response. A body is said to be homogeneous if all the material points are materially uniform with respect to a single placement. A body that is not homogeneous is said to be inhomogeneous. This study focuses on one class of inhomogeneous bodies for which, the Cauchy stress, $\boldsymbol{\sigma}$ depends explicitly on the deformation gradient, $\mathbf{F}$ and the position vector of the material particle identified in the stress free reference configuration, $\mathbf{X}$, i.e., $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}, \mathbf{X})$.

Many hold the opinion that the inhomogeneity of the type studied here could easily be decided by the body's response to electromagnetic radiation. They believe that if the body under investigation exhibits different responses in different regions as seen through, say, a microscope, it is inhomogeneous. However, different structures revealed under a microscope does not mean that the mathematical model of the body for its mechanical response should be different in these regions, if the mechanical properties and say, optical properties of the material are presumed[1] to be independent. Other reasons for the mathematical model for mechanical response could be different from that used for the response to electromagnetic radiation are explained below.

As inferred from its response to electromagnetic radiation all bodies are inhomogeneous at some scale. However, in case of bodies made of certain metals, say steel, having dimensions greater than a particular value seem to be robustly modeled using homogeneous models. Hence, it is believed that the homogeneous models are obtained through averaging the spatially varying

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[1] One cannot by knowing the refractive index, tell what the elastic properties of the material are.

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material parameters. These homogenization procedures are trying to replace a (spatially varying) function by a constant which cannot be robust unconditionally. In fact, for bodies undergoing large elastic deformations, Saravanan and Rajagopal (2003a,b, 2005, 2007) showed that the value of the constant material parameters in the homogeneous model so that the global load versus displacement relation is in agreement between the actual inhomogeneous body and its homogeneous counterpart, depends on the boundary value problem. Moreover, this constant material parameter varied by as much as 1800 percent with the boundary value problem used to determine these material parameters. This suggest that homogeneous models seem to work not because of homogenization but due to some other reason which could be that it is inherently homogeneous for its mechanical response under the investigated scenarios.

Taking the viewpoint that to answer questions of practical interest, such as, what is the maximum stress and displacement in a body subjected to some loading, the mathematical model for the body need not conform to the perceived reality that it is inhomogeneous, but can be an abstraction of the same. Akin to abstracting the earth as a point mass when one is interested in planetary motion, a rigid sphere when one is interested in studying eclipse we ask what would be a useful abstraction of a given body to capture some process that it is undergoing. Thus, in this point of view, an inhomogeneous model is required for a given body because some mechanical phenomena exhibited by this body can be captured only by abstracting it as an inhomogeneous body. In this spirit, the investigation here attempts on finding mechanical phenomena that requires a given body to be abstracted as an inhomogeneous body.

Towards this, in this article, we examine isotropic, inhomogeneous bodies, whose Cauchy stress depends explicitly on the left Cauchy–Green deformation tensor. It is assumed that the constitutive relation is a differentiable function of the position vector of material particles in the stress free reference configuration. On further assuming that this body undergoes a non-dissipative process from a stress free reference configuration, we examine scenarios when it would admit pure stretch homogeneous deformations when tested in the absence of any body forces. We find that a cuboid made of compressible and isotropic material can be considered to be homogeneous, if homogeneous[2] deformations are observed when the cuboid is subjected to normal stresses on all its six faces such that it does not result in a hydrostatic state of stress. A cuboid made of incompressible and isotropic material could be modeled as a homogeneous body, if homogeneous deformations are observed in two biaxial stretch experiments such that the plane in which the forces are applied is mutually orthogonal.

We emphasize that the above is a sufficient condition for abstracting a given body as a homogeneous body. On the other hand observing inhomogeneous deformations is only a necessary condition for the body to be inhomogeneous. Homogeneous body could also exhibit inhomogeneous deformations, because of the presence of body forces or non-uniform application of the boundary traction or due to the presence of inertial forces. Only on ruling out all these factors can the body be considered inhomogeneous. Thus, the proposed method seems to be a rationale way of deciding whether a given body can be idealized as homogeneous body or needs to be modeled as an inhomogeneous body.

Before proceeding further a few comments on the assumptions – isotropy and material functions being a differentiable function of the position vector – are necessary. First, we clarify that an inhomogeneous body can be made of isotropic constituents. The constitutive relation if for a point in the body and hence the material symmetry which restricts the form of this constitutive relation is also for a point. Inhomogeneity on the other hand is a statement about the form of the constitutive relation at different points. Since, point cannot have a structure there arises a conundrum as to the meaning of material symmetry. Thus, as even stated by Lekhnitskii (1981), the requirement that the symmetry of the constitutive relation be same as that of the material symmetry found based on the internal structure, is at best an assumption. Hence, it is advocated that one view material symmetry as a statement regarding the variation of the principal direction of the Cauchy stress with respect to the principal direction of the left Cauchy–Green deformation tensor. Paranjothi et al. (submitted for publication) presents experimental evidence and discusses practical difficulties associated with this view point. Consequently, a homogeneous body can be anisotropic and an inhomogeneous body can be made of isotropic constituents. This point that material symmetry and inhomogeneity are mutually exclusive cannot be overemphasized.

Next, the assumption that the material functions be differentiable function of the position vector needs discussion. Clearly, this assumption excludes bodies with voids, inclusions and the like. The results arrived here is applicable only for functionally graded materials. Relaxation of this assumption that the material functions be differentiable with respect to the position vector leads to mathematical complications and thereby obscuring the main thesis of this article that the idealization of a body as being homogeneous should be made based on the possibility of realizing homogeneous deformation field. Further, it is known (Varley and Cumberbatch, 1980; Ru et al., 2005) that a void or inclusion in a homogeneous matrix causes the deformation to be inhomogeneous when subjected to uniform far field loading. Therefore, it seems that scenarios when the deformation is homogeneous is more only for the case when the material functions are differentiable with respect to the position vector. However, a rigorous proof for the same is required and efforts are underway towards this. In Section 4, we briefly discuss how the result arrived at here could be used to study the case when the material functions are not differentiable with respect to position vector.

In the literature, it is prevalent to examine whether the body is subjected to homogeneous deformation. In fact, enormous care is taken to obtain homogeneous deformations, where possible. However, in most experiments only the surface deformation is measured. This surface measurement alone is not sufficient to determine if the realized deformation is homogeneous; deformation in the interior of the body also needs to be probed. On the other hand, if the surface deformation itself is non-uniform then the deformation is indeed inhomogeneous. There are reports of both the surface deformation being uniform (Rivlin and Saunders, 1951; Hariharaputhiran and Saravanan, 2010) and it being non-uniform (Kawamura et al., 1996; Lu et al., 2003; Paranjothi et al., 2011) in a pure stretch experiment showing the utility of the present approach to decide whether a given body can be approximated as homogeneous or otherwise. Further, X-ray computed tomography (Synolakis et al., 1996; Roux et al., 2008; You et al., 2009) as well as optical scanning tomography (Germaneau et al., 2007, 2008) techniques allows us to probe the deformation in the interior. As these techniques mature, the results in this paper would yield a practical tool for deciding when a body undergoing elastic deformations can be modeled as a homogeneous body.

One might think that the scale of observation would determine whether the deformation is homogeneous or not. This thinking stems from the observation that homogeneous deformation is seen in some bodies despite the fact that they are inhomogeneous at some length scale. However, mathematically a given deformation field would be either homogeneous or inhomogeneous with the

[2] If any straight line in the body deforms into another straight line the deformation is said to be homogeneous.

classification not dependent on the length scale of observation. Also, as shown in Pruša et al. (2013), the value of the deformation gradient estimated for an inhomogeneous deformation would be sensitive to how the measurement is made and that found for a homogeneous deformation would be insensitive to the measurement system. Of course, because of noise in the measurement no experimentally determined deformation would be strictly homogeneous. But it appears that for some bodies tested the deformation seems to be homogeneous on the surface beyond any doubt (Rivlin and Saunders, 1951; Hariharaputhiran and Saravanan, 2010). Thus, experimentally determining whether a deformation is homogeneous seems to be easy and it would be independent of the length scale of measurement.

Thus, as discussed above there are numerous reasons for a body to exhibit inhomogeneous deformations and despite this if it chooses to deform homogeneously then it could indeed be modeled as a homogeneous body. Further, this study shows that modeling a given body as inhomogeneous body is not a luxury but a necessity if the predicted and observed response of the body is to be in agreement.

2. Preliminaries

Let $\mathbf{X} \in \kappa_o(\mathcal{B})$ denote a typical particle belonging to the reference configuration of the body and let $\mathbf{x} \in \kappa_t(\mathcal{B})$ denote the position occupied by $\mathbf{X}$ at time t in the current configuration of the body. The motion of the body is defined through the mapping $\boldsymbol{\chi}$ that is one to one for each $t \in \mathcal{R}$, set of reals:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t). \tag{1}$$

Since, in this article we concern ourselves only with statics we call (1) the deformation field. Then, the deformation gradient, $\mathbf{F}$ is defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{2}$$

and the left and right Cauchy–Green deformation tensors, $\mathbf{B}$ and $\mathbf{C}$ respectively, are defined as:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^t, \quad \mathbf{C} = \mathbf{F}^t\mathbf{F}, \tag{3}$$

where the superscript t denotes the transpose operator. $\mathbf{C}$ being positive definite, we find that using the following set of invariants

$$J_1 = tr(\mathbf{C}), \quad J_2 = tr(\mathbf{C}^{-1}), \quad J_3 = \sqrt{det(\mathbf{C})} = det(\mathbf{F}), \tag{4}$$

minimizes the complexity of the ensuing analysis.

In this study, we restrict ourselves to inhomogeneous bodies made up of isotropic material undergoing elastic deformations from stress free reference configurations. In classical Cauchy elasticity (see Truesdell and Noll (1965)) for bodies made of isotropic, compressible materials, the Cauchy stress, $\boldsymbol{\sigma}$ and left Cauchy–Green deformation tensor, $\mathbf{B}$ are related through:

$$\boldsymbol{\sigma} = \alpha_0\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}, \tag{5}$$

where $\alpha_i = \hat{\alpha}_i(J_1, J_2, J_3, \mathbf{X})$ are the material response functions. It is assumed that α_i is a differentiable function of $\mathbf{X}$. Similarly, for bodies made of isotropic, incompressible materials, the Cauchy stress is given by:

$$\boldsymbol{\sigma} = -p\mathbf{1} + \beta_1\mathbf{B} + \beta_2\mathbf{B}^{-1}, \tag{6}$$

where $\beta_i = \hat{\beta}_i(J_1, J_2, \mathbf{X})$ are the material response functions such that β_i is a differentiable function of $\mathbf{X}$ and p is the Lagrange multiplier used to enforce the incompressibility constraint.

Since, the material is isotropic in the stress free reference configuration and the material response functions are written in terms of the invariants, it is such that

$$\alpha_i(\Lambda_1, \Lambda_2, \Lambda_3, \mathbf{X}) = \alpha_i(\Lambda_2, \Lambda_3, \Lambda_1, \mathbf{X}) = \alpha_i(\Lambda_3, \Lambda_1, \Lambda_2, \mathbf{X}), \tag{7}$$

$$\beta_i(\Lambda_1, \Lambda_2, \Lambda_3, \mathbf{X}) = \beta_i(\Lambda_2, \Lambda_3, \Lambda_1, \mathbf{X}) = \beta_i(\Lambda_3, \Lambda_1, \Lambda_2, \mathbf{X}), \tag{8}$$

where Λ_i's are the principal values of $\mathbf{C}$. We shall see that this property by which the principal values of $\mathbf{C}$ could be permuted is an important requirement to be checked, in the ensuing analysis. See for example, Ogden (1997) for the need of this requirement in bodies made of isotropic material.

Baker and Ericksen (1954) proposed that in a body made up of isotropic material, the greater principal stress should always occur in the direction of greater principal stretch, and hence, when $\Lambda_1 \neq \Lambda_2 \neq \Lambda_3$,

$$\alpha_1 - \frac{1}{\Lambda_1^2\Lambda_2^2}\alpha_2 > 0, \quad \text{for compressible materials,} \tag{9}$$

$$\beta_1 - \Lambda_3^2\beta_2 > 0, \quad \text{for incompressible materials,} \tag{10}$$

where $\alpha_i = \hat{\alpha}_i(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2, \frac{1}{\Lambda_1^2} + \frac{1}{\Lambda_2^2} + \frac{1}{\Lambda_3^2}, \Lambda_1\Lambda_2\Lambda_3, \mathbf{X}), \beta_i = \hat{\beta}_i(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2, \frac{1}{\Lambda_1^2} + \frac{1}{\Lambda_2^2} + \frac{1}{\Lambda_3^2}, \mathbf{X})$. Note in Eq. (10) $\Lambda_3 = 1/(\Lambda_1\Lambda_2)$ and that the Λ_i's in Eqs. (9) and (10) can be permuted.

On the other hand when $\Lambda_1 = \Lambda_2 = \Lambda \neq \Lambda_3$,

$$\alpha_1^e - \frac{1}{\Lambda^4}\alpha_2^e \geqslant 0, \quad \text{and} \quad \alpha_1^e - \frac{1}{\Lambda^2\Lambda_3^2}\alpha_2^e > 0, \quad \text{for compressible materials,} \tag{11}$$

$$\beta_1^e - \frac{1}{\Lambda^4}\beta_2^e \geqslant 0, \quad \text{and} \quad \beta_1^e - \Lambda^2\beta_2 > 0, \quad \text{for incompressible materials,} \tag{12}$$

where $\alpha_i^e = \hat{\alpha}_i(2\Lambda^2 + \Lambda_3^2, \frac{2}{\Lambda^2} + \frac{1}{\Lambda_3^2}, \Lambda^2\Lambda_3, \mathbf{X}), \beta_i^e = \hat{\beta}_i(2\Lambda^2 + \frac{1}{\Lambda^4}, \frac{2}{\Lambda^2} + \Lambda^4, \mathbf{X})$ Also, in Eqs. (9) and (11), all Λ_i's and Λ could (mathematically) take any positive value as do Λ_1 and Λ_2 in Eq. (10) and Λ in Eq. (12). The inequalities ((9)–(12) are called Baker–Ericksen inequalities. Please refer to Truesdell and Noll (1965) for a discussion on Baker–Ericksen inequalities and its usefulness to model common materials.

We shall neglect body forces and as we shall only consider static problems, the balance of linear momentum reduces to

$$div(\boldsymbol{\sigma}) = \mathbf{0}, \tag{13}$$

where the notation *div* stands for the divergence operator with respect to the current coordinates.

3. Feasibility of pure stretch homogeneous deformations

Let (X, Y, Z) denote the Cartesian coordinates of a material point in the reference configuration and (x, y, z) be the Cartesian coordinates of the same material point in the current configuration. Our interest is to find which inhomogeneous bodies admit deformations of the form

$$x = \lambda_x X, \quad y = \lambda_y Y, \quad z = \lambda_z Z, \tag{14}$$

where λ_i's are constants and denotes the stretch ratio along the i^{th} direction.

Now, the Cartesian components of the left Cauchy–Green deformation tensor in matrix representation for the assumed deformation (14) is,

$$\mathbf{B} = \begin{pmatrix} \lambda_x^2 & 0 & 0 \\ 0 & \lambda_y^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{pmatrix}. \tag{15}$$

Assuming that whatever boundary traction is required would be applied to realize this deformation, the only requirement for this deformation field to be feasible is that it satisfy the equilibrium equation (13). This can be ascertained only on specifying a constitutive relation. However, we note that for the assumed deformation (14) and the constitutive relation (5) and (6), the Cartesian matrix components of Cauchy stress in the body is given by

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}. \tag{16}$$

3.1. Compressible materials

First we explore the feasibility of these homogeneous deformations in inhomogeneous bodies made of compressible materials for which the Cauchy stress is given by (5). Substituting (15) in (5) and then substituting the ensuing result in (13) we obtain:

$$\left(\frac{\partial\alpha_0}{\partial X} + \lambda_x^2\frac{\partial\alpha_1}{\partial X} + \frac{1}{\lambda_x^2}\frac{\partial\alpha_2}{\partial X}\right)\frac{\partial X}{\partial x} = 0, \tag{17}$$

$$\left(\frac{\partial\alpha_0}{\partial Y} + \lambda_y^2\frac{\partial\alpha_1}{\partial Y} + \frac{1}{\lambda_y^2}\frac{\partial\alpha_2}{\partial Y}\right)\frac{\partial Y}{\partial y} = 0, \tag{18}$$

$$\left(\frac{\partial\alpha_0}{\partial Z} + \lambda_z^2\frac{\partial\alpha_1}{\partial Z} + \frac{1}{\lambda_z^2}\frac{\partial\alpha_2}{\partial Z}\right)\frac{\partial Z}{\partial z} = 0. \tag{19}$$

The above equations are obtained by assuming that the material response function, α_i's are differentiable with respect to $\mathbf{X}$. Then for the Eqs. (17)–(19) to hold it is required that:

$$\left(\frac{\partial\alpha_0}{\partial X} + \lambda_x^2\frac{\partial\alpha_1}{\partial X} + \frac{1}{\lambda_x^2}\frac{\partial\alpha_2}{\partial X}\right) = 0, \tag{20}$$

$$\left(\frac{\partial\alpha_0}{\partial Y} + \lambda_y^2\frac{\partial\alpha_1}{\partial Y} + \frac{1}{\lambda_y^2}\frac{\partial\alpha_2}{\partial Y}\right) = 0, \tag{21}$$

$$\left(\frac{\partial\alpha_0}{\partial Z} + \lambda_z^2\frac{\partial\alpha_1}{\partial Z} + \frac{1}{\lambda_z^2}\frac{\partial\alpha_2}{\partial Z}\right) = 0. \tag{22}$$

Here we assumed that all the three σ_i's are different from zero. If one or two of them are zero, as in biaxial or uniaxial state of stress, then only a subset of these equations need to hold. However, in case of biaxial or uniaxial state of stress, in addition to a subset of Eqs. (20)–(22), one or two algebraic equations also needs to hold. Hence, we consider uniaxial, biaxial and triaxial state of stress separately.

3.1.1. Triaxial state of stress

Here we assume that none of the σ_i = 0. Then, Eqs. (20)–(22) places restrictions on the spatial variation of the material response functions and/or the nature of the homogeneous deformations.

These Eqs. (20)–(22) could be integrated to obtain

$$\begin{aligned}\alpha_0 = {} & \frac{\lambda_x^2[\lambda_y^2+\lambda_z^2]F_1(Y,Z)}{\lambda_x^2[-\lambda_x^2+\lambda_y^2+\lambda_z^2]-\lambda_y^2\lambda_z^2} + \frac{\lambda_y^2[\lambda_x^2+\lambda_z^2]F_2(X,Z)}{\lambda_y^2[\lambda_x^2-\lambda_y^2+\lambda_z^2]-\lambda_x^2\lambda_z^2} \\ & + \frac{\lambda_z^2[\lambda_x^2+\lambda_y^2]F_3(X,Y)}{\lambda_z^2[\lambda_x^2+\lambda_y^2-\lambda_z^2]-\lambda_x^2\lambda_y^2},\end{aligned} \tag{23}$$

$$\begin{aligned}\alpha_1 = {} & -\frac{\lambda_x^2F_1(Y,Z)}{\lambda_x^2[-\lambda_x^2+\lambda_y^2+\lambda_z^2]-\lambda_y^2\lambda_z^2} - \frac{\lambda_y^2F_2(X,Z)}{\lambda_y^2[\lambda_x^2-\lambda_y^2+\lambda_z^2]-\lambda_x^2\lambda_z^2} \\ & - \frac{\lambda_z^2F_3(X,Y)}{\lambda_z^2[\lambda_x^2+\lambda_y^2-\lambda_z^2]-\lambda_x^2\lambda_y^2},\end{aligned} \tag{24}$$

$$\begin{aligned}\alpha_2 = {} & -\frac{\lambda_x^2\lambda_y^2\lambda_z^2F_1(Y,Z)}{\lambda_x^2[-\lambda_x^2+\lambda_y^2+\lambda_z^2]-\lambda_y^2\lambda_z^2} - \frac{\lambda_x^2\lambda_y^2\lambda_z^2F_2(X,Z)}{\lambda_y^2[\lambda_x^2-\lambda_y^2+\lambda_z^2]-\lambda_x^2\lambda_z^2} \\ & - \frac{\lambda_x^2\lambda_y^2\lambda_z^2F_3(X,Y)}{\lambda_z^2[\lambda_x^2+\lambda_y^2-\lambda_z^2]-\lambda_x^2\lambda_y^2},\end{aligned} \tag{25}$$

where $F_i(\cdot,\cdot)$ is a function of the respective spatial variables and has to be different from zero since it is assumed that the state of stress is triaxial. However, this general solution (23)–(25) is not possible for any choice of F_i's because the material response functions, α_i's do not meet the requirement (7) needed for the material to be isotropic in the stress free reference configuration.

Thus, the Eqs. (20)–(22) cannot be satisfied by appropriate choice of the material response functions alone. However, Eqs. (20)–(22) could be satisfied by placing restrictions on the material response functions and the nature of the homogeneous deformations.

Towards elucidating these restrictions, for this state of stress, we first integrate Eq. (20) with respect to X to obtain

$$\alpha_0 + \lambda_x^2\alpha_1 + \frac{1}{\lambda_x^2}\alpha_2 + F(Y,Z) = 0, \tag{26}$$

where $F(Y,Z)$ is yet to be determined function of Y and Z.

Differentiating (26) with respect to Y and subtracting the resulting expression from Eq. (21) we obtain:

$$(\lambda_y^2 - \lambda_x^2)\left(\frac{\partial\alpha_1}{\partial Y} - \frac{1}{\lambda_y^2\lambda_x^2}\frac{\partial\alpha_2}{\partial Y}\right) = \frac{\partial F}{\partial Y}. \tag{27}$$

While the right hand side of the Eq. (27) is a function of only Y and Z, the left hand side, in general, is a function of X, Y and Z. This contradiction can be reconciled only if any one of the following happens:

1. $\lambda_x = \lambda_y$ and $F(Y,Z) = \overline{F}(Z)$, where $\overline{F}(Z)$ is a yet to be determined function of Z.
2. $\alpha_i = \hat{\alpha}_i^+(J_1,J_2,J_3)\hat{h}(X,Y,Z)$ for $i = \{1,2\}$ with $\hat{\alpha}_i^+$ such that
$$\hat{\alpha}_1^+ - \frac{1}{\lambda_y^2\lambda_x^2}\hat{\alpha}_2^+ = 0 \tag{28}$$
and $F(Y,Z) = \overline{F}(Z)$, where $\overline{F}(Z)$ is a yet to be determined function of Z. Since, Eq. (28) has to hold for any arbitrary λ_x and λ_y, it violates the requirement (7), which needs to be met for isotropic materials. Hence, this is not a feasible solution.
3. α_1 and α_2 should be independent of X and
$$F(Y,Z) = (\lambda_y^2 - \lambda_x^2)\left(\alpha_1 - \frac{1}{\lambda_x^2\lambda_y^2}\alpha_2\right) + \overline{\overline{F}}(Z), \tag{29}$$
where $\overline{\overline{F}}(Z)$ is a yet to be determined function of Z.

Differentiating (26) with respect to Z and subtracting the resulting expression from Eq. (22) we obtain:

$$(\lambda_z^2 - \lambda_x^2)\left(\frac{\partial\alpha_1}{\partial Z} - \frac{1}{\lambda_z^2\lambda_x^2}\frac{\partial\alpha_2}{\partial Z}\right) = \frac{\partial F}{\partial Z}. \tag{30}$$

Substituting $F(Y,Z) = \overline{F}(Z)$, for the function $F(Y,Z)$ in (30), corresponding to the first case in the above solution, it is required that

$$(\lambda_z^2 - \lambda_x^2)\left(\frac{\partial\alpha_1}{\partial Z} - \frac{1}{\lambda_z^2\lambda_x^2}\frac{\partial\alpha_2}{\partial Z}\right) = \frac{d\overline{F}}{dZ}. \tag{31}$$

While the right hand side of the Eq. (31) is a function of only Z, the left hand side is a function of X, Y and Z in general. This contradiction can be reconciled only if one of the following happens:

1. $\lambda_x = \lambda_z$ and $\overline{F}(Z) = D_1$, where D_1 is a constant.
2. $\alpha_i = \hat{\alpha}_i^-(J_1,J_2,J_3)\hat{h}(X,Y,Z)$ for $i = \{1,2\}$ with $\hat{\alpha}_i^-$ such that
$$\hat{\alpha}_1^- - \frac{1}{\lambda_z^2\lambda_x^2}\hat{\alpha}_2^- = 0, \tag{32}$$
and $\overline{F}(Z) = D_1$, where D_1 is a constant. Since, Eq. (32) has to hold for any arbitrary λ_x and λ_z, it violates the requirement (7), which

needs to be met for isotropic materials. Hence, this is not a feasible solution.

3. α_1 and α_2 should be independent of X and Y and

$$\overline{F}(Z) = (\lambda_z^2 - \lambda_x^2)\left(\alpha_1 - \frac{1}{\lambda_x^2\lambda_z^2}\alpha_2\right) + D_1, \tag{33}$$

where D_1 is a constant.

Substituting (29) for the function $F(Y,Z)$ in (30) it is required that α_1 and α_2 should be independent of X and

$$(\lambda_z^2 - \lambda_y^2)\left(\frac{\partial\alpha_1}{\partial Z} - \frac{1}{\lambda_z^2\lambda_y^2}\frac{\partial\alpha_2}{\partial Z}\right) = \frac{d\overline{\overline{F}}}{dZ}. \tag{34}$$

As before, while the right hand side of the Eq. (34) is a function of only Z, the left hand side is a function of Y and Z in general. This contradiction can be reconciled only if one of the following happens:

1. $\lambda_y = \lambda_z$ and $\overline{\overline{F}}(Z) = D_2$, where D_2 is a constant.
2. $\alpha_i = \hat{\alpha}_i^*(J_1,J_2,J_3)\hat{h}(Y,Z)$ for $i = \{1,2\}$ with $\hat{\alpha}_i^*$ such that

$$\hat{\alpha}_1^* - \frac{1}{\lambda_y^2\lambda_x^2}\hat{\alpha}_2^* = 0, \tag{35}$$

and $\overline{\overline{F}}(Z) = D_2$, where D_2 is a constant. Since, Eq. (35) has to hold for any arbitrary λ_x and λ_y, it violates the requirement (7), which needs to be met for isotropic materials. Hence, this is not a feasible solution.

3. α_1 and α_2 should be independent of X and Y and

$$\overline{\overline{F}}(Z) = (\lambda_z^2 - \lambda_y^2)\left(\alpha_1 - \frac{1}{\lambda_y^2\lambda_z^2}\alpha_2\right) + D_2, \tag{36}$$

where D_2 is a constant.

Thus, Eqs. (20)–(22) admit the following feasible solutions:

Solution – 1: Corresponding to solution – 1 of Eqs. (27) and (31)

$$\lambda_x = \lambda_y = \lambda_z = \Lambda, \quad \text{and} \quad \alpha_0 = -\left[\Lambda^2\alpha_1 + \frac{1}{\Lambda^2}\alpha_2\right] - D_1, \tag{37}$$

with no restriction on how the material response functions, α_i depends on $\mathbf{X}$, however, $\alpha_i = \hat{\alpha}_i(3\Lambda^2, 3/\Lambda^2, \Lambda^3, \mathbf{X})$.

Solution – 2: Combining solution – 1 of Eq. (27) with the solution – 3 of Eq. (31) we obtain $\lambda_x = \lambda_y \neq \lambda_z$ and

$$\alpha_0 = -\left[\lambda_z^2\alpha_1 + \frac{1}{\lambda_z^2}\alpha_2\right] - D_1, \tag{38}$$

where $\alpha_i = \hat{\alpha}_i(J_1,J_2,J_3,Z)$. Here it is pertinent to note that if we began by integrating the y (or z) component of the equilibrium equation, instead of the material response function depending on Z, it could equivalently depend on X (or Y). Then, the stretch ratio in Eq. (38) should be replaced by stretch ratio in the appropriate direction. Hence, it is evident that this form (38) for the material response functions is not consistent with the requirement (7) arising due to the material being isotropic. Therefore, this is not a feasible solution.

Solution – 3: Combining solution – 3 of Eq. (27) and solution – 1 of Eq. (34) we obtain $\lambda_y = \lambda_z \neq \lambda_x$ and

$$\alpha_0 = -\left[\lambda_z^2\alpha_1 + \frac{1}{\lambda_z^2}\alpha_2\right] - D_2, \tag{39}$$

where $\alpha_i = \hat{\alpha}_i(J_1,J_2,J_3,Y,Z)$. The form for the material response function (39) is not consistent with the requirement (7) arising due to the material being isotropic. Therefore this is not a feasible solution.

Solution – 4: Combining solution – 3 of Eq. (27) and solution – 3 of Eq. (34) we obtain

$$\alpha_0 = -\left[\lambda_z^2\alpha_1 + \frac{1}{\lambda_z^2}\alpha_2\right] - D_2, \tag{40}$$

where $\alpha_i = \hat{\alpha}_i(J_1,J_2,J_3,Z)$. The form for the material response function (40) is not consistent with the requirement (7) arising due to the material being isotropic. Therefore this is also not a feasible solution.

Thus, only 1 solution is admissible. This feasible solution results in the state of stress being hydrostatic. Hence, it can be concluded from the above results that if homogeneous deformation is observed in a body made of isotropic, compressible material subjected to a triaxial state of stress, different from hydrostatic pressure, then the body is homogeneous.

3.1.2. Biaxial (plane) state of stress

Here we assume that $\sigma_z = 0$, without loss of generality. Consequently, (22) is trivially satisfied, by appropriate choice of the stretch ratios. Therefore for this state of stress, Eqs. (20) and (21) needs to be satisfied and we need to find a constant λ_z such that

$$\alpha_0 + \alpha_1\lambda_z^2 + \frac{\alpha_2}{\lambda_z^2} = 0. \tag{41}$$

In order to satisfy Eqs. (20) and (21) either the material response functions should be such that

$$\alpha_i = \tilde{\alpha}_i(J_1,J_2,J_3,Z), \tag{42}$$

or following an analysis similar to that presented for triaxial stress state, we find that there exists two more sets of solution.

Set 1:

$$\lambda_x = \lambda_y = \lambda, \quad \text{and} \quad \alpha_0 + \lambda^2\alpha_1 + \frac{1}{\lambda^2}\alpha_2 - \widetilde{F}(Z) = 0, \tag{43}$$

where $\alpha_i = \hat{\alpha}_i(J_1,J_2,J_3,\mathbf{X})$ and $\widetilde{F}(Z)$ is a function that depends only on Z.

Set 2:

$$\alpha_1 = \frac{\alpha_2}{\lambda_x^2\lambda_y^2}, \quad \text{and} \quad \alpha_0 = -\left[\frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2}\right]\alpha_2 - \widetilde{F}^*(Z) = 0, \tag{44}$$

where $\alpha_2 = \hat{\alpha}_2(J_1,J_2,J_3,\mathbf{X})$ and $\widetilde{F}^*(Z)$ is a function that depends only on Z. However, it should be noted that this form for the material response functions violates the requirement (7) which is needed for the material to be isotropic. Hence, this solution is not feasible.

Thus, only two out of the three solutions that mathematically satisfies Eqs. (20), (21) and (41) are physically admissible.

Solution 1: The only form of the material response functions that satisfy (42) and result in a constant λ_z satisfying (41) is,

$$\alpha_i = \hat{f}_i(J_1,J_2,J_3)\hat{g}(Z). \tag{45}$$

Solution 2: Now, we explore for what forms of the material response function, does there exist admissible solutions for Eqs. (41) and (43). Towards this, substituting (43) in (41) we find that a constant λ_z should satisfy,

$$\widetilde{F}(Z) + [\lambda_z^2 - \lambda^2]\left[\alpha_1 - \frac{\alpha_2}{\lambda_z^2\lambda^2}\right] = 0. \tag{46}$$

It then follows that (46) admit three classes of solutions. In the first class of solution we assume $\widetilde{F}(Z) = 0$ and require that $\lambda_z = \lambda$. Then the Cauchy stress, $\boldsymbol{\sigma} = \mathbf{0}$, which is not of interest. For the second class of solution too we assume $\widetilde{F}(Z) = 0$ but now we need to find a constant λ_z such that it satisfies

$$\alpha_1 - \frac{\alpha_2}{\lambda_z^2\lambda^2} = 0. \tag{47}$$

It can immediately be seen that only if the material response functions $\alpha_i = \hat{f}_i(J_1, J_2, J_3)\hat{h}(X, Y, Z)$, we can find a constant λ_z satisfying (47). Even then, only if the material response functions violate the Baker–Ericksen inequality (11) we would be able to find a λ_z satisfying (47). For the final class of solution, we assume $\widetilde{F}(Z) \neq 0$. Then, for a constant λ_z to satisfy (46) $\alpha_i = \hat{f}_i(J_1, J_2, J_3)\widetilde{F}(Z)$.

Hence, we find that if an inhomogeneous body made of compressible and isotropic material that obeys Baker–Ericksen inequality is subjected to biaxial state of stress, then it would exhibit homogeneous deformation only if the material response functions vary spatially along the direction of the normal to the plane in which the biaxial stress is applied.

However, inhomogeneous body made of compressible and isotropic material that do not obey Baker–Ericksen condition when subject to biaxial state of stress exhibits homogeneous deformations for any arbitrary dependance of the material response functions on $\mathbf{X}$. But in this case, the material response functions dependance on the kinematical quantity is constrained by Eq. (47).

Stating the above differently, if a body made of isotropic, compressible material which obeys Baker–Ericksen condition exhibits homogeneous deformation in two biaxial stress experiments such that the plane in which the stresses are applied is orthogonal in the two biaxial tests then the body is homogeneous. If the body is made of a material that do not obey Baker–Ericksen condition then observing homogeneous deformations in any number of biaxial tests would not help us conclude it is homogeneous.

3.1.3. Uniaxial state of stress

Without loss of generality, here we assume that $\sigma_y = \sigma_z = 0$. This implies that Eqs. (21) and (22) are satisfied trivially, by appropriate choice of the stretch ratios. Hence, Eq. (20) needs to be satisfied and the equations

$$\alpha_0 + \alpha_1\lambda_y^2 + \frac{\alpha_2}{\lambda_y^2} = 0, \tag{48}$$

$$\alpha_0 + \alpha_1\lambda_z^2 + \frac{\alpha_2}{\lambda_z^2} = 0, \tag{49}$$

should hold for some constant λ_y and λ_z.

For Eq. (20) to hold, the material response function should be such that

$$\alpha_i = \overline{\alpha}_i(J_1, J_2, J_3, Y, Z), \tag{50}$$

or

$$\alpha_0 = F^u(Y, Z) - \lambda_x^2\alpha_1 - \frac{1}{\lambda_x^2}\alpha_2, \tag{51}$$

with no restriction on how α_i's depend on $\mathbf{X}$.

To examine whether Eqs. (48) and (49) has solutions with constant λ_y and λ_z when the material response function are of the form (50) or (51), we rewrite these equations to obtain,

$$\left[\lambda_y^2 - \lambda_z^2\right]\left[\alpha_1 - \frac{1}{\lambda_y^2\lambda_z^2}\alpha_2\right] = 0. \tag{52}$$

Then, immediately it transpires that there are two solutions to (52) and (49).

Solution 1: The first solution that we study is one for which,

$$\lambda_y = \lambda_z = \Lambda, \quad \text{and} \quad \alpha_0^u + \alpha_1^u\Lambda^2 + \frac{\alpha_2^u}{\Lambda^2} = 0, \tag{53}$$

where $\alpha_i^u = \hat{\alpha}_i(\lambda_x^2 + 2\Lambda^2, 1/\lambda_x^2 + 2/\Lambda^2, \lambda_x\Lambda^2, \mathbf{X})$, and Λ is to be found by solving the nonlinear Eq. (53b) given a λ_x.

If and only if $\hat{\alpha}_i(J_1, J_2, J_3, \mathbf{X}) = f_i(J_1, J_2, J_3)g(Y, Z)$, there exist a constant Λ that satisfies (50) and (53b).

Substituting (51) in (53b) we require

$$F^u(Y, Z) + [\Lambda^2 - \lambda_x^2]\left[\alpha_1^u - \frac{1}{\lambda_x^2\Lambda^2}\alpha_2^u\right] = 0, \tag{54}$$

to hold for constant Λ and λ_x. Eq. (54) could be satisfied in three different ways. The first solution that we consider is one in which $F^u(Y, Z) = 0$ and $\Lambda = \lambda_x$. This is a trivial solution since for this case the Cauchy stress is zero tensor. The second solution corresponds to the case where

$$F^u(Y, Z) = 0, \quad \text{and} \quad \alpha_1^u - \frac{1}{\lambda_x^2\Lambda^2}\alpha_2^u = 0. \tag{55}$$

Clearly, for (55b) to hold Baker–Ericksen inequality (11) has to be violated. Further, for a constant λ_x and Λ to satisfy (55b), $\hat{\alpha}_i(J_1, J_2, J_3, \mathbf{X}) = f_i^+(J_1, J_2, J_3)\ g^+(X, Y, Z)$. The third solution corresponds to the case when $F^u(Y, Z) \neq 0$. In this case, the form of α_i's that satisfies (54) is $\hat{\alpha}_i(J_1, J_2, J_3, \mathbf{X}) = f_i^-(J_1, J_2, J_3)F^u(Y, Z)$.

Solution 2: The second solution that satisfies (52) happens only if Baker–Ericksen conditions (9) are violated. For this case, we need to find λ_y and λ_z by solving the two nonlinear equations

$$\alpha_1 - \frac{1}{\lambda_y^2\lambda_z^2}\alpha_2 = 0, \quad \text{and} \quad \alpha_0 + \alpha_1\lambda_z^2 + \frac{\alpha_2}{\lambda_z^2} = 0, \tag{56}$$

given a λ_x. Here again, if and only if $\hat{\alpha}_i(J_1, J_2, J_3, \mathbf{X}) = f_i(J_1, J_2, J_3)g(Y, Z)$, there exist a constant λ_y and λ_z that satisfies (50) and (56).

Substituting (51) in (56) we obtain

$$\alpha_1 - \frac{1}{\lambda_y^2\lambda_z^2}\alpha_2 = 0, \quad \text{and} \quad F^u(Y, Z) + [\lambda_z^2 - \lambda_x^2]\left[\alpha_1 + \frac{\alpha_2}{\lambda_z^2\lambda_x^2}\right] = 0. \tag{57}$$

Thus, in order to find a constant λ_y and λ_z given a λ_x that satisfy (57), $\hat{\alpha}_i(J_1, J_2, J_3, \mathbf{X}) = f_i^*(J_1, J_2, J_3)F^u(Y, Z)$.

Thus, we find that an inhomogeneous body made up of compressible, isotropic material subjected to uniaxial state of stress would deform homogeneously if the material response functions do not vary along the direction of the applied uniaxial stress and they are of some special form (50) or (51). Stating this differently, if a compressible isotropic body exhibits homogeneous deformation in three different uniaxial stretch experiments, such that direction of the uniaxial stress in the three experiments is mutually orthogonal, then the body is homogeneous.

3.2. Incompressible materials

Next, we explore the feasibility of these homogeneous deformations in inhomogeneous and incompressible bodies for which the Cauchy stress is given by (6). Substituting (15) in (6) and the resulting Cauchy stress in (13) we obtain:

$$\left(-\frac{\partial p}{\partial X} + \lambda_x^2\frac{\partial\beta_1}{\partial X} + \frac{1}{\lambda_x^2}\frac{\partial\beta_2}{\partial X}\right)\frac{\partial X}{\partial x} = 0, \tag{58}$$

$$\left(-\frac{\partial p}{\partial Y} + \lambda_y^2\frac{\partial\beta_1}{\partial Y} + \frac{1}{\lambda_y^2}\frac{\partial\beta_2}{\partial Y}\right)\frac{\partial Y}{\partial y} = 0, \tag{59}$$

$$\left(-\frac{\partial p}{\partial Z} + \lambda_z^2\frac{\partial\beta_1}{\partial Z} + \frac{1}{\lambda_z^2}\frac{\partial\beta_2}{\partial Z}\right)\frac{\partial Z}{\partial z} = 0. \tag{60}$$

The above equations has to hold, since the material response functions, β_i's are differentiable with respect to $\mathbf{X}$. Noticing the similarity between Eqs. (58)–(60) with Eqs. (17)–(19) we follow the same procedure as that adopted for compressible materials to obtain the following results. Therefore, we specialize to various stress states.

3.2.1. Triaxial state of stress

When $\sigma_i \neq 0$, Eqs. (58)–(60) has to be satisfied. Solving, which we obtain that

$$p = \left[\lambda_z^2 \beta_1 + \frac{1}{\lambda_z^2}\beta_2\right] + D_3, \quad (61)$$

with the material response functions being independent of X and Y; i.e., $\beta_i = \hat{\beta}_i(J_1, J_2, Z)$ where D_3 is a constant. Here it is pertinent to note that if instead of the material response function depending on Z, it could equivalently depend on X or Y, in which case the stretch ratio in Eq. (61) should be replaced by stretch ratio in the appropriate direction.

Thus, inhomogeneous, incompressible, isotropic bodies subjected to triaxial state of stress could deform homogeneously when the material response functions depends on only one spatial coordinate. Moreover, this homogeneous deformation results in the following Cauchy stress field,

$$\sigma_{xx} = (\lambda_x^2 - \lambda_z^2)\left[\beta_1 - \frac{1}{\lambda_x^2\lambda_z^2}\beta_2\right] - D_3, \quad (62)$$

$$\sigma_{yy} = (\lambda_y^2 - \lambda_z^2)\left[\beta_1 - \frac{1}{\lambda_y^2\lambda_z^2}\beta_2\right] - D_3, \quad (63)$$

$$\sigma_{zz} = -D_3, \quad (64)$$

when the material response functions β_i's are independent of X and Y and $\lambda_z = 1/(\lambda_x\lambda_y)$, in order to satisfy the incompressibility requirement.

Another solution to Eqs. (58)–(60) is: $\lambda_x = \lambda_y \neq \lambda_z$ and $\beta_1 = \beta_2/(\lambda_x^2\lambda_z^2)$ and $p = \beta_2(1/\lambda_x^2 + 1/\lambda_z^2) + D_4$, which on enforcing the incompressibility criterion reduces to requiring,

$$\lambda_x = \lambda_y = \frac{1}{\sqrt{\lambda}}, \quad \lambda_z = \lambda, \quad \text{and} \quad \beta_1 = \frac{\beta_2}{\lambda}, \quad \text{and} \quad p = \beta_2\left[\lambda + \frac{1}{\lambda^2}\right] + D_4 \quad (65)$$

with $\beta_2 = \hat{\beta}_2(\lambda^2 + 2/\lambda, 2\lambda + 1/\lambda)\hat{h}(X, Y, Z)$. The Cauchy stress corresponding to this solution is hydrostatic. However, material response functions satisfying (65) violate the Baker–Ericksen inequality (12).

We also note that, solution corresponding to (37) results in the following: the incompressibility requirement, $J_3 = \Lambda^3 = 1$ necessitates that $\Lambda = 1$ and hence now, $p = \beta_1 + \beta_2 - D_5$ where $\beta_i = \hat{\beta}_i(3, 3, \mathbf{X})$ and D_5 is a constant. This implies that the body has not been deformed and hence a trivial solution that need not be explored further.

Hence, we conclude that inhomogeneous, incompressible, isotropic bodies subjected to hydrostatic state of stress could deform homogeneously provided the material response functions violate the Baker–Ericksen inequalities but satisfy (65). Also, these bodies could deform homogeneously when the material functions only vary along a particular direction and the applied traction on a surface whose normal coincides with the direction along which the material function varies is uniform but nonuniform in other planes.

Stating the above result differently, the given incompressible, isotropic cuboid is homogeneous, if the homogeneous deformation is observed on application of a uniform triaxial state of stress, different from hydrostatic stress.

3.2.2. Biaxial (plane) state of stress

Assuming that $\sigma_z = 0$, now (58) and (59) needs to be satisfied and we should be able to find a constant λ_z such that

$$-p + \lambda_z^2\beta_1 + \frac{1}{\lambda_z^2}\beta_2 = 0. \quad (66)$$

We find that if p is such that,

$$p = \lambda_z^2\beta_1 + \frac{1}{\lambda_z^2}\beta_2, \quad (67)$$

then Eq. (66) is satisfied. Then, in order to satisfy (58) and (59) either

$$\beta_i = \overline{\beta}_i(J_1, J_2, Z), \quad (68)$$

or β_i's and/or λ_i's should be such that

$$(\lambda_x^2 - \lambda_z^2)\left(\frac{\partial\beta_1}{\partial X} - \frac{1}{\lambda_x^2\lambda_z^2}\frac{\partial\beta_2}{\partial X}\right) = 0, \quad \text{and} \quad (\lambda_y^2 - \lambda_z^2)\left(\frac{\partial\beta_1}{\partial Y} - \frac{1}{\lambda_y^2\lambda_z^2}\frac{\partial\beta_2}{\partial Y}\right) = 0, \quad (69)$$

holds. There are four possible solutions to (69).

Set 1: One set of solutions requires $\lambda_x = \lambda_y = \lambda_z$. However, since the material being studied here is incompressible, $\lambda_x\lambda_y\lambda_z = 1$ which implies that $\lambda_x = \lambda_y = \lambda_z = 1$, that is the body is not deformed and hence this class of solutions is not of interest here.

Set 2: In the second set of solutions, we assume that $\lambda_x \neq \lambda_y \neq \lambda_z$. Therefore for Eq. (69) to hold we require that

$$\frac{\partial\beta_1}{\partial X} - \frac{1}{\lambda_x^2\lambda_z^2}\frac{\partial\beta_2}{\partial X} = 0, \quad \text{and} \quad \frac{\partial\beta_1}{\partial Y} - \frac{1}{\lambda_y^2\lambda_z^2}\frac{\partial\beta_2}{\partial Y} = 0. \quad (70)$$

Integrating the above equations we obtain

$$\beta_1 - \frac{1}{\lambda_x^2\lambda_z^2}\beta_2 = F(Y), \quad \text{and} \quad \beta_2 = F(Y)\frac{\lambda_x^2\lambda_y^2\lambda_z^2}{(\lambda_y^2 - \lambda_x^2)}, \quad (71)$$

where $F(Y)$ is a yet to be determined function of Y. The expression for β_i's in Eq. (71) violates the requirement (8) needed when the material being modeled is isotropic. Hence, solutions of this kind is not possible.

Set 3: To obtain the third set of solutions we require that $\lambda_x = \lambda_z \neq \lambda_y$ and that

$$\frac{\partial\beta_1}{\partial Y} - \frac{1}{\lambda_y^2\lambda_z^2}\frac{\partial\beta_2}{\partial Y} = 0. \quad (72)$$

However, since $\lambda_x = \lambda_z$ and $\sigma_z = 0, \sigma_x = 0$. Therefore the state of stress is uniaxial and hence not of interest here.

Set 4: Similarly, for the last set of solution wherein $\lambda_x \neq \lambda_y = \lambda_z$ and

$$\frac{\partial\beta_1}{\partial X} - \frac{1}{\lambda_x^2\lambda_z^2}\frac{\partial\beta_2}{\partial X} = 0, \quad (73)$$

also results in an uniaxial state of stress and hence of not interest here.

Instead of beginning with the assumption that $\sigma_z = 0$, we could have assumed $\sigma_x = 0$ or $\sigma_y = 0$. Then, we can by a similar analysis, as above, conclude that the material response functions depend only on X or Y respectively.

Summarizing the results in this subsection, an isotropic, incompressible, inhomogeneous body subjected to biaxial state of stress would exhibit homogeneous deformations only if the material response functions vary along the coordinate direction in which no stress is applied. Stating this result differently, if one observes homogeneous deformation in two biaxial stretch experiments such that the plane along which the body is stretched in the two tests are mutually orthogonal, then the given isotropic, incompressible body can be considered as a homogeneous body.

3.2.3. Uniaxial state of stress

Without loss of generality, let us assume that $\sigma_y = \sigma_z = 0$. Then, we should be able to find constant λ_y and λ_z such that equations

$$-p+\lambda_y^2\beta_1+\frac{1}{\lambda_y^2}\beta_2=0,\quad \text{and}\quad -p+\lambda_z^2\beta_1+\frac{1}{\lambda_z^2}\beta_2=0, \tag{74}$$

hold. Also, Eq. (58) has to hold.

Rewriting Eq. (74) as

$$p=\lambda_z^2\beta_1+\frac{1}{\lambda_z^2}\beta_2,\quad \text{and}\quad \left(\lambda_y^2-\lambda_z^2\right)\left(\beta_1-\frac{1}{\lambda_y^2\lambda_z^2}\beta_2\right)=0, \tag{75}$$

we find that either $\lambda_y=\lambda_z$ or $\lambda_y\neq\lambda_z$ and

$$\beta_1-\frac{1}{\lambda_y^2\lambda_z^2}\beta_2=0. \tag{76}$$

Immediately we observe that for Eq. (76) to hold, Baker–Ericksen inequality (10) has to be violated and for constant λ_y and λ_z to satisfy (76), $\beta_i=\hat{\beta}_i(J_1,J_2)\,\hat{g}(X,Y,Z)$.

Next, we find the restriction on the material response functions so that (58) holds for each of the above two classes of solutions. When Lagrange multiplier is given by Eq. (75a) and $\lambda_y=\lambda_z$, (58) reduces to requiring

$$(\lambda_x^2-\lambda_z^2)\left(\frac{\partial\beta_1}{\partial X}-\frac{1}{\lambda_x^2\lambda_z^2}\frac{\partial\beta_2}{\partial X}\right)=0 \tag{77}$$

Eq. (77) admits three classes of solutions.

First, we explore a class of solution, for which, $\lambda_x=\lambda_z$. This solution coupled with the incompressibility constraint implies that $\lambda_x=\lambda_y=\lambda_z=1$. This means that the body is not deformed. Consequently, this solution is not of interest.

Next, two classes correspond to the case where

$$\frac{\partial\beta_1}{\partial X}-\frac{1}{\lambda_x^2\lambda_z^2}\frac{\partial\beta_2}{\partial X}=0. \tag{78}$$

This would happen when $\beta_i=\hat{\beta}_i(J_1,J_2,Y,Z)$ or when $\beta_i=\hat{\beta}_i^+(J_1,J_2)\,\overline{g}(X,Y,Z)$ with $\hat{\beta}_i^+$'s such that

$$\hat{\beta}_1^+-\frac{1}{\lambda_x^2\lambda_z^2}\hat{\beta}_2^+=0. \tag{79}$$

Clearly, for material response functions that satisfy (79) the Baker–Ericksen condition (12) would be violated.

Now, we find the restriction that (58) places when Lagrange multiplier is given by Eq. (75a), $\lambda_y\neq\lambda_z$ and material response function satisfy (76). As before, substituting (75a) in (58), it reduces to requiring

$$(\lambda_x^2-\lambda_z^2)\left(\frac{\partial\beta_1}{\partial X}-\frac{1}{\lambda_x^2\lambda_z^2}\frac{\partial\beta_2}{\partial X}\right)=0 \tag{80}$$

Recognizing the similarity between Eqs. (77) and (80), we conclude that (80) also has three solutions.

The first case that we study is one for which, $\lambda_x=\lambda_z\neq\lambda_y$ and we should be able to find a constant λ_y and λ_x such that (76) holds. This would happen when $\beta_i=\hat{\beta}_i^+(J_1,J_2)\,\overline{g}(X,Y,Z)$ and $\hat{\beta}_i^+(J_1,J_2)$ is such that

$$\hat{\beta}_1^+-\frac{1}{\lambda_y^2\lambda_z^2}\hat{\beta}_2^+=0. \tag{81}$$

However, by virtue of $\sigma_z=0$ and $\lambda_x=\lambda_z, \sigma_x=0$, contrary to the assumption. Hence, solution of this kind is not feasible.

The second class of solutions to (80) is when $\beta_i=\hat{\beta}_i(J_1,J_2,Y,Z)$. However, for a constant λ_y and λ_z to satisfy (76), $\beta_i=\hat{\beta}_i^+(J_1,J_2)g^+(Y,Z)$ and $\hat{\beta}_i^+(J_1,J_2)$ is such that (81) holds.

The final class of solutions to (80) is when $\lambda_x\neq\lambda_y\neq\lambda_z$ and $\hat{\beta}_i^-(J_1,J_2)\,\overline{g}(X,Y,Z)$ where $\hat{\beta}_i^-$ are such that

$$\hat{\beta}_1^--\frac{1}{\lambda_y^2\lambda_z^2}\hat{\beta}_2^-=0,\quad \text{and}\quad \hat{\beta}_1^--\frac{1}{\lambda_x^2\lambda_z^2}\hat{\beta}_2^-=0, \tag{82}$$

holds. However, now for (82) to hold $\lambda_x=\lambda_y$ which is in contradiction with our initial requirement that $\lambda_x\neq\lambda_y$. Hence, this class of solutions also is not feasible.

Thus, isotropic, incompressible, inhomogeneous bodies admit homogeneous deformations when the material response functions do not vary along the direction of the applied uniaxial stress or the material response functions are such that they do not satisfy the Baker–Ericksen condition and are of the form $\beta_i=\hat{\beta}_i^+(J_1,J_2)\overline{g}(X,Y,Z)$.

Stating the above result differently, if an isotropic, incompressible body satisfies the Baker–Ericksen inequalities and admits homogeneous deformations when subjected to three different uniaxial stress experiments such that the direction in which the stress is applied in these experiments are mutually orthogonal, then the body can be idealized as a homogeneous body. On the other hand if the Baker–Ericksen inequalities are violated then one cannot conclude that the given body is homogeneous by observing homogeneous deformation in any number of uniaxial experiments.

Hence we conclude that for incompressible bodies, if homogeneous deformation is observed in two different biaxial experiments performed such that the applied biaxial displacements are in the $x\&y$ and $y\&z$ (or $x\&z$) directions, then the incompressible, isotropic body could be idealized to be homogeneous when it is being subjected to a non-dissipative process.

4. Discussion

By virtue of the assumption that the material response functions are differentiable functions with respect to the position vector of the material particles in the stress free reference configuration, important classes of inhomogeneous bodies, like bodies with inclusion, voids, layered composites are not considered in the above analysis. However, if we assume that over a small region, of size ϵ, on either size of the interface of the two materials, the material parameters vary smoothly from one value to that of the other, the above conclusions would still be true. Even then, we have to show that the limit wherein $\epsilon\to 0$ exist for the above result to be applicable for bodies with inclusion, layered composites, etc. This work is in progress and the results of the same communicated at a later date.

Similarly, it should also be stated that the status of the present results for materials undergoing small deformations and obeying Hooke's law needs to be examined as this is not a special case, but a limiting case of the framework used here.

5. Conclusion

In this article, we show that if a body, say a cuboid, made of a compressible material can be homogeneously stretched by applying loads on three orthogonal faces such that it does not result in a hydrostatic state of stress, then the body can be abstracted as a homogeneous body. In case the cuboid is made of an incompressible material, if we can homogeneously stretch it in two different biaxial experiments, such that the plane in which the forces are applied in the two biaxial experiments is mutually orthogonal then the cuboid can be approximated as homogeneous. While a rigorous proof of the above statements has been provided in the case of functionally graded materials, the same for the case of bodies with inclusions, voids or layered composites is at best sketchy.

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