Magnetohydrodynamic shock wave formation: Effect of area and density variation

R. I. Sujith

Citation: Physics of Plasmas (1994-present) **12**, 052116 (2005); doi: 10.1063/1.1901693
View online: http://dx.doi.org/10.1063/1.1901693
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/12/5?ver=pdfcov
Published by the AIP Publishing

PHYSICS OF PLASMAS 12, 052116 (2005)

Magnetohydrodynamic shock wave formation: Effect of area and density variation

R. I. Sujith[a)]
Department of Aerospace Engineering, Indian Institute of Technology Madras, Chennai 600036, India

(Received 9 September 2004; accepted 14 March 2005; published online 11 May 2005)

The nonlinear steepening of finite amplitude magnetohydrodynamic (MHD) waves propagating perpendicular to the magnetic field is investigated. The nonlinear evolution of a planar fast magnetosonic wave in a homentropic flow field is understood well through simple waves. However, in situations where the wave is moving through a variable area duct or when the flow field is nonhomentropic, the concept of simple waves cannot be used. In the present paper, the quasi-one-dimensional MHD equations that include the effect of area variation and density gradients are solved using the wave front expansion technique. The analysis is performed for a perfectly conducting fluid and also for a weakly conducting fluid. Closed form solutions are obtained for the nonlinear evolution of the slope of the wave front in the limits of infinitely large and small conductivity. A general criterion for a compression wave to steepen into a shock is obtained. An analytical expression for the location of shock formation is derived. The effect of area variation and density gradient on shock formation is studied and examples highlighting the same are presented. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1901693]

I. INTRODUCTION

The nonlinear steepening of finite amplitude magnetohydrodynamic (MHD) waves propagating perpendicular to the magnetic field (fast magnetosonic waves) is investigated in this paper. The profile of a finite amplitude fast magnetosonic wave gets distorted and tends to steepen as a consequence of nonlinearity. The compressive part of the wave pulse travels faster than the expansive part and hence wave crests tend to catch up with wave troughs. This evolution will lead to a fast reshaping of the initially smooth velocity and magnetic field profiles with subsequent breakdown at the leading part of the wave resulting in shock formation.[1] This phenomena is of great interest in many areas such as the study of solar flares[1,2] and other astrophysical phenomena,[3] discharges between electrodes in laser cavities, applications relating to thermonuclear fusion, MHD generators to name a few.

A number of studies have been performed to investigate shock formation in MHD flows. Vrsnak and Lulic[1,2] studied the formation and evolution of a large amplitude fast magnetosonic waves in a perfectly conducting low β plasma, in the context of the solar corona.[1,2] Using an analysis based on simple waves, they derived explicit expressions for the time and the distance needed for the transformation of the perturbation's leading edge into a shock wave. Stefenlo *et al.*[4] developed an exact analytical solution of the nonlinear MHD equations for planar disturbances having a certain initial profile traveling across the external magnetic field in a cold plasma. Ödblom[5] derived analytical solutions for the nonlinear evolution of planar MHD waves propagating either across or along the magnetic field in warm, isothermal, and adiabatic plasma using simple waves. Bharadwaj[6] studied the formation of shock waves in reactive MHD flows in a cylindrical geometry. Orta *et al.*[7] used numerical simulations to follow the shock formation. However, all these studies were performed in uniform medium. Except for Ref. 6 which was performed in cylindrical coordinates, all other studies dealt with plane waves in homogeneous medium in constant area ducts.

The nonlinear steepening of a planar wave in a homentropic flow is understood well through simple waves.[1,2,4,5] The characteristics of a simple wave have constant slopes and the Riemann invariants are constant along these characteristics. However, in the case of variable area ducts or stratified media, the Reimann invariants of the system are not constant along the characteristics. As a result, shock formation time and distance are underestimated or overestimated (depending on the direction of propagation in relation to the density or area gradient) if these effects are neglected. MHD flows in variable area ducts has application in lasers and other terrestrial applications. Wave propagation through stratified media has applications in astrophysical applications. In the present paper, the analysis is performed taking into account the effects of area variation and density gradients.

The steepening of acoustic waves (non-MHD) in the presence of area and entropy gradients has been studied by many authors. The effect of spherical geometry was discussed by Appert *et al.*[8] in the context of nucleation of liquids. Lin and Szeri[9] investigated the steepening of acoustic waves in the presence of entropy gradients, in the context of sonoluminescence of bubbles. Tyagi and Sujith[10] investigated the steepening of acoustic waves in variable area ducts in the presence of entropy gradients. They[9,10] used the wave front expansion technique to obtain an evolution equation for the slope of the wave front. In the present paper, the analysis is further generalized to describe the case of nonlinear dis-

[a)]Telephone: +91-44-22578166, +91-44-22578152. FAX: +91-44-22570545. Electronic mail: sujith@iitm.ac.in

tortion of finite amplitude fast magnetosonic waves in the presence of area and density gradients.

The rest of the paper is organized as follows: In Sec. II, the method of wave front expansion is used to determine the time evolution of a wave front of a fast magnetosonic wave in an infinitely conducting medium. The effect of variable area ducts and stratified medium are presented in Sec. III. In Sec. IV the steepening of the wave in a weakly conducting fluid is discussed.

II. SHOCK FORMATION IN A MEDIUM WITH INFINITE CONDUCTIVITY

A. Governing equations

In the system considered here, the disturbance is propagating along the x direction, perpendicular to the magnetic field which is along the z direction ($\vec{B}=B\hat{e}_z$). Assuming an inviscid and non-heat-conducting, isentropic gas, the quasi-one-dimensional equations of motion for a gas with infinitely large conductivity (magnetic Reynolds number much larger than one) can be written as follows.[11–13]

For continuity,

$$\frac{\partial \rho}{\partial t}+\frac{1}{A}\frac{\partial}{\partial x}(\rho A u)=0, \tag{1}$$

where, A is the area of cross section.

For Faraday's law,

$$\frac{\partial \vec{B}}{\partial t}+(\vec{u}\cdot\vec{\nabla})\vec{B}+(\vec{\nabla}\cdot\vec{u})\vec{B}=(\vec{B}\cdot\vec{\nabla})\vec{u}. \tag{2a}$$

Applying this to the quasi-one-dimensional system considered here, the right-hand side will be zero and Eq. (2a) reduces to

$$\frac{\partial B}{\partial t}+\frac{1}{A}\frac{\partial}{\partial x}(BAu)=0. \tag{2b}$$

For momentum,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x}+\frac{1}{\mu}B\frac{\partial B}{\partial x}=0. \tag{3}$$

For energy,

$$(p_t+up_x)-a^2(\rho_t+u\rho_x)=0, \tag{4}$$

where, $a=\sqrt{\partial p/\partial\rho|_{s=\text{const}}}$ is the isentropic speed of sound.

The momentum equation can be rewritten as

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p^*}{\partial x}=0. \tag{5}$$

Here, $p^*=p+(B^2/2\mu)$ is the sum of the hydrodynamic and magnetic pressures. Also, Eqs. (1) and (2b) show that in the one-dimensional (1D) situation, when the frozen-in condition is satisfied, B/ρ is a constant.[1]

Combining Eqs. (2b) and (4) gives

$$\frac{\partial p^*}{\partial t}+u\frac{\partial p^*}{\partial x}+\frac{\rho a^{*2}}{A}\frac{\partial(Au)}{\partial x}=0, \tag{6}$$

where the fast magnetosonic velocity a^* is defined by

$$a^{*2}=a^2+\alpha^2, \tag{7a}$$

where

$$\alpha=\sqrt{\frac{B^2}{\rho\mu}} \tag{7b}$$

is the Alfven speed.

These equations form a hyperbolic system, and along with the equation of state, they completely describe the flow. The undisturbed medium is assumed to be at rest. To study the propagation and distortion of a wave in the medium, these equations are manipulated and written along their characteristics in the (t,x) plane,

$$\frac{d^+u}{dt}+\frac{1}{a^*\rho}\frac{d^+p^*}{dt}+\frac{ua^*}{A}\frac{dA}{dx}=0 \quad \text{on } C_+\text{:}\frac{d^+x}{dt}=u+a^*, \tag{8}$$

$$\frac{d^-u}{dt}-\frac{1}{a^*\rho}\frac{d^-p^*}{dt}-\frac{ua^*}{A}\frac{dA}{dx}=0 \quad \text{on } C_-\text{:}\frac{d^-x}{dt}=u-a^*, \tag{9}$$

$$\frac{ds}{dt}=0 \quad \text{on } C_0\text{:}\frac{dx}{dt}=u, \tag{10}$$

where s is the entropy, which can be expressed in terms of the pressure and density as $p/\rho^\gamma=\exp[s/c_\nu]$ where c_ν is the specific heat at constant volume

$$\frac{d}{dt}\left(\frac{B}{\rho}\right)=0 \quad \text{on } C_0\text{:}\frac{dx}{dt}=u. \tag{11}$$

In the above set of equations d^+/dt, d^-/dt, and d/dt are the derivatives taken along the C_+, C_-, and C_0 characteristics, respectively.

In the present problem, a finite amplitude wave with compact support having a discontinuity in its first derivative at the wave front is considered. It can be shown that the leading edge of the wave propagates along the characteristics C_+ and C_- with velocities a^* and $-a^*$, respectively.[12] Since the undisturbed medium is at rest, C_0 characteristics are absent at the wave front. The rate of steepening of the leading edges is followed using the technique of wave front expansion. A shock forms when the slope at the leading edge becomes infinity. The disadvantage of the present method is that it neglects the possibility of formation of shock in the middle of the wave. It however illustrates the effect of area variation and density variation on the nonlinear distortion of the wave. Furthermore, the method yields a closed form solution which is of valued significance.

B. Wave front expansion

In the neighborhood of the wave front, a new coordinate system, $\xi=x-X(t)$ is defined, where $X(t)$ is the position of the wavefront. Physically, ξ represents the distance measured from the wave front. Therefore, (i) $\xi=0\Rightarrow x=X(t)$, describes the motion of the wavefront; (ii) $\xi>0$ is the region of the undisturbed quiescent medium into which the wave propagates; (iii) $\xi<0$ is the region behind the wave front where the flow is unsteady. The motion of the leading edge of the right running wave is governed by the following equation:

$$\dot{X}(t) = a_0^*[X(t)], \tag{12}$$

where "$\dot{}$" indicates time derivative, and the subscript "0" indicates the value of the flow variable in the undisturbed medium. Henceforth, the analysis is performed for the right running wave. The left running wave can be analyzed in a similar fashion. As the wave propagates, a flow variable λ can be expanded in terms of its derivatives at the wave front as follows:

$$\lambda(\xi,t) = \lambda_0[X(t)] + \xi\lambda_1(t) + \frac{\xi^2}{2}\lambda_2(t) + \dots \quad \text{for } \xi < 0, \tag{13a}$$

$$\lambda(\xi,t) = \lambda_0[X(t)] + \xi\lambda_0'[X(t)] + \frac{\xi^2}{2}\lambda_0''[X(t)] + \dots \quad \text{for } \xi > 0, \tag{13b}$$

where λ indicates p^*, ρ, or u and $\lambda_1, \lambda_2, \dots$ denote the corresponding spatial derivatives behind the wave front. Since $\xi > 0$ is an undisturbed region, $u_0[X(t)]=0$. Further, to clarify, since a discontinuity in the first and the higher derivatives are present at the wave front,

$$\lambda_1(t) \neq \lambda_0'[X(t)], \quad \lambda_2(t) \neq \lambda_0''[X(t)], \dots, \tag{14}$$

where "$'$" indicates spatial derivatives. Left-hand side of the inequalities in Eq. (14) are derivatives behind the wave front (disturbed region) and right-hand side are derivatives in front of the wave front (undisturbed region). However, for the cross-sectional area $A_0(x)$, all the derivatives are known on both sides of the wave front, i.e., $A_1(t)=A_0'[X(t)]$, $A_2(t)=A_0''[X(t)]$ and so on. If $a^*=a^*(p^*,\rho)$, then,[14]

$$a^*(p^*,\rho) = a_0^*[X(t)] + \xi p_1^*\left(\frac{\partial a_0^*}{\partial p^*}\right)_\rho + \xi\rho_1\left(\frac{\partial a_0^*}{\partial \rho}\right)_{p^*} + \dots. \tag{15}$$

Furthermore, the derivatives with respect to t can be obtained using:[14]

$$\left[\frac{\partial}{\partial t}\right]_x := \left[\frac{\partial}{\partial t}\right]_\xi + \left[\frac{\partial \xi}{\partial t}\right]_x \left[\frac{\partial}{\partial \xi}\right] = \frac{\partial}{\partial t} - a_0^*[X(t)]\frac{\partial}{\partial \xi}. \tag{16}$$

The terms in Eqs. (1), (5), and (6) are expanded using Eqs. (13a), (15), and (16). Comparing the coefficients of ξ^0 and ξ^1, one finds ξ^0 terms,

$$a_0^*(\rho_0' - \rho_1) + \rho_0 u_1 = 0, \tag{17}$$

$$p_1^* = \rho_0 u_1 a_0^*, \tag{18}$$

$$(p_0^{*\prime} - p_1^*) - a_0^{*2}(\rho_0' - \rho_1) = 0; \tag{19}$$

ξ^1 terms,

$$(\rho_0' - \rho_1)a_0^* A_1 + A_0(\dot{\rho}_1 - \rho_2 a_0^*) + \rho_0 A_0 u_2 + 2\rho_0 A_1 u_1 + 2\rho_1 u_1 A_0 = 0, \tag{20}$$

$$-\rho_1 u_1 a_0^* + \rho_0(\dot{u}_1 - u_2 a_0^*) + \rho_0 u_1^2 + p_2^* = 0, \tag{21}$$

$$\dot{p}_1^* - a_0^* p_2^* + p_1^* u_1 = 2a_0^*\left(p_1^*\frac{\partial a^*}{\partial p^*} + \rho_1\frac{\partial a^*}{\partial \rho}\right)(\rho_0' - \rho_1)a_0^* + a_0^{*2}(\dot{\rho}_1 - \rho_2 a_0^* + \rho_1 u_1). \tag{22}$$

The matrix formed by the coefficients of first derivatives in Eqs. (17)–(19) is singular. Hence, the first sets of equations reduce to $p_0^{*\prime}=0$, which is as expected in a quiescent medium.

Similarly, coefficients of the second derivatives, Eqs. (20)–(22), form a singular matrix. Hence on elimination of second derivatives and changing the thermodynamic state variables from (ρ,p) to (ρ,s), Eqs. (20)–(22) yields, after some manipulation:

$$\frac{du_1}{dt} + \left[\frac{3}{2}a_0^{*\prime} + \frac{1}{2}\frac{\rho_0'}{\rho_0}a_0^* + \frac{a_0}{2}\frac{A_0'}{A_0}\right]u_1 + \Gamma_0 u_1^2 = 0. \tag{23}$$

$\Gamma = 1/a^*\,\partial(\rho a^*)/\partial\rho|_s$ is the nonlinearity parameter.[15] The value of Γ can be evaluated, knowing the equation of state.

To trace the evolution of the wave front as a function of its propagation distance, a change of variable, t to $y=X(t)$ is effected. Here, y denotes the position of the wave front. As a result, Eq. (23) reduces to

$$\frac{du_1}{dy} + \left[\frac{3}{2}\frac{a_0^{*\prime}}{a_0^*} + \frac{1}{2}\frac{\rho_0'}{\rho_0} + \frac{1}{2}\frac{A_0'}{A_0}\right]u_1 + \frac{\Gamma_0(y)}{a_0^*}u_1^2 = 0. \tag{24}$$

This equation can be reduced to a linear equation as

$$\frac{d}{dy}\left(\frac{1}{u_1}\right) - \left[\frac{3}{2}\frac{a_0^{*\prime}}{a_0^*} + \frac{1}{2}\frac{\rho_0'}{\rho_0} + \frac{1}{2}\frac{A_0'}{A_0}\right]\frac{1}{u_1} = \frac{\Gamma_0(y)}{a_0^*}. \tag{25}$$

The solution to this equation with an initial slope $u_1(0)$ at the leading edge can be written as

$$\frac{1}{u_1(y)} = \frac{IF(0)}{IF(y)u_1(0)} + \frac{1}{IF(y)}\int_0^y \frac{IF(\hat{y})\Gamma_0(\hat{y})}{a_0^*(\hat{y})}d\hat{y}, \tag{26}$$

where, $IF(y)=a_0^*(y)^{-3/2}\rho_0(y)^{-1/2}A_0(y)^{-1/2}$. Equation (26) describes the nonlinear evolution of the leading edge of a wave front of a fast magnetosonic wave traveling into a stationary MHD fluid governed by a general equation of state $a^*=a^*(p^*,\rho)$, in the presence of density gradients. A shock forms when $|u_1(y)| \to \infty$. If y_s indicates the shock formation distance, then the shock formation time can be obtained from

$$t_s = \int_0^{y_s} \frac{dy}{a_0^*(y)}. \tag{27}$$

Equation (26) can be used to track the evolution of the slope of a wave front, and shock formation, in variable area ducts and in the presence of density gradients. This will be illustrated in various situations in the following sections.

For a compression wave front, the initial slope is negative; i.e., $u_1(0) < 0$, and can be for convenience written as $u_1(0) = -|u_1(0)|$. Then Eq. (26) becomes

$$\frac{1}{u_1(y)} = \frac{IF(0)}{IF(y)}\left[\frac{1}{|u_1(0)|} - \frac{1}{IF(0)}\int_0^y \frac{IF(\hat{y})\Gamma_0(\hat{y})}{a_0^*(\hat{y})}d\hat{y}\right]. \tag{28}$$

Equation (28) gives the slope of the wave front at a position y. As Γ_0 is positive, there is a possibility that the right-hand side of Eq. (28) could become zero at some finite value of $y=y_s$. This will occur when

$$|u_1(0)| = \frac{1}{\dfrac{1}{IF(0)}\displaystyle\int_0^{y_s} \frac{IF(\hat{y})\Gamma_0(\hat{y})}{a_0^*(\hat{y})} d\hat{y}}. \tag{29}$$

At this stage, the first derivative of the wave front becomes infinite. This phenomenon is referred to as a shock. It can be seen from Eq. (29) that the steepening of a compression wave front into shock is greatly influenced by variations in density and the cross-section area of the duct. Furthermore, it can be seen that only sufficiently steep compression wave fronts steepen into a shock; they must have an initial slope

$$|u_1(0)| > \frac{1}{\dfrac{1}{IF(0)}\displaystyle\int_0^{\infty} \frac{IF(\hat{y})\Gamma_0(\hat{y})}{a_0^*(\hat{y})} d\hat{y}}. \tag{30}$$

It is clear from Eq. (30) that if the integral $\int_0^\infty [IF(\hat{y})\Gamma_0(\hat{y})/a_0^*(\hat{y})]d\hat{y}$ does not converge, all compression wave fronts will steepen into shocks at some finite distance y_s given by Eq. (29). If it converges, only those compression wave fronts having initial slope satisfying Eq. (30) will steepen into shocks.

III. EXAMPLES

All the examples presented in this paper are for an ideal gas where the equation of state $a^*=a^*(p^*,\rho)$ can be written as

$$a^{*2} = \gamma\frac{p^*}{\rho} + \frac{1}{\mu}\left[\frac{B}{\rho}\right]^2\left(1-\frac{\gamma}{2}\right)\rho.$$

For such a gas, Γ_0 can be evaluated as

$$\frac{\gamma+1}{2} + \frac{1}{\mu}\left[\frac{B}{\rho}\right]^2\left(1-\frac{\gamma}{2}\right)\frac{\rho_0}{a_0^{*2}}.$$

However, the analysis in Sec. II is quite general and can be applied to study shock formation in any fluid.

A. Homentropic environment

A homentropic medium is considered first to illustrate the effect of area variation on shock formation.

1. Steepening of a plane wave in a constant area duct

For the sake of comparison, results are presented for a plane wave in an infinite homentropic medium ($\Gamma=\Gamma_0$ =const). This also helps in illustrating the effect of Γ on the distortion of the wave. For a uniform medium, Eq. (28) reduces to

$$u_1(t) = \frac{u_1(0)}{1+\Gamma_0 u_1(0)t}. \tag{31}$$

It can easily be seen from the above expression that only compression waves can steepen to form shocks. Further, the rate of steepening of the leading edge is obtained by differentiating Eq. (31) as

$$\frac{du_1}{dt} = -\Gamma_0 u_1^2(t). \tag{32}$$

It can easily be seen from the above expression that the value of Γ_0 decides the extent of nonlinear distortion. The presence of a magnetic field increases the value of Γ_0, thereby increasing the rate of wave steepening and causing early shock formation. If a shock forms, the time of shock formation (t_s) and the distance traveled by the wave before it turns into a shock (α) are given by

$$t_s = -\frac{1}{u_1(0)\Gamma_0}, \quad \alpha = -\frac{a_0^*}{u_1(0)\Gamma_0}. \tag{33}$$

2. Shock formation in a variable area duct

In this section we investigate shock formation in variable area ducts, in a homentropic environment. The wave distortion and steepening of a wave front into a shock are affected very much by the changes in the cross-sectional area of the duct and changes in density in the duct. In this section, the effect of variation of area of cross section alone on the wave form distortion will be discussed. Although Eq. (28) is applicable for any smooth duct, in this analysis only ducts with monotonically increasing (diverging) or decreasing (converging) cross section will be considered. In fact, any smooth duct consists of many converging and diverging ducts, each of which can be analyzed separately. In this analysis, a right running wave is considered. A left running wave can be analyzed in a similar fashion.

The evolution equation for the slope of a compression wave front can be written as

$$\frac{1}{u_1} = -\sqrt{\frac{A_0(y)}{A_0(0)}}\left[\frac{1}{|u_1(0)|} - \frac{\Gamma_0}{a_0^*}\int_0^y \sqrt{\frac{A_0(0)}{A_0(\hat{y})}}d\hat{y}\right]. \tag{34}$$

The expression for shock formation distance can be written as

$$\int_0^{y_s}\sqrt{\frac{A_0(0)}{A_0(\hat{y})}}d\hat{y} = \frac{a_0^*}{|u_1(0)|\Gamma_0} = \lambda. \tag{35}$$

Here, λ is the shock formation distance for a plane wave front in a constant-area duct in homentropic conditions.

It can easily be seen from Eq. (35) that for a divergent duct, the shock formation is delayed when compared to a duct with constant cross-section area. In the case of a convergent duct, the shock formation distance will be less than λ. For a homentropic environment, the shock condition reduces to

$$|u_1(0)| > \frac{1}{\dfrac{\Gamma_0}{2}\displaystyle\int_0^\infty \sqrt{\frac{A_0(0)}{A_0(\hat{y})}}d\hat{y}}. \tag{36}$$

It can easily be shown using the properties of improper integrals that for a converging duct, the integral $\int_0^\infty \sqrt{A(0)/A(\hat{y})}d\hat{y}$ is divergent. Consequently, any compression wave front traveling in a convergent passage will eventually steepen into a shock. If the area of cross section of a convergent duct becomes zero at some point y^*, the integral

$\int_0^\infty \sqrt{A(0)/A(\hat{y})}d\hat{y}$ should be replaced with $\int_0^{y^*} \sqrt{A(0)/A(\hat{y})}d\hat{y}$. In this case, every compression wave front will steepen into a shock before y^*.

In the case of divergent ducts, there is a possibility that $\int_0^\infty \sqrt{A(0)/A(\hat{y})}d\hat{y}$ may converge. This usually occurs when a duct is diverging rapidly (e.g., an exponential horn). For a compression wave front to steepen into a shock in such a duct, the magnitude of the initial slope must exceed a minimum value. However for a duct that is diverging less rapidly, such as a conical horn, any compression wave front can steepen into a shock, given sufficient length.

These results are identical to the results derived by Tyagi and Sujith[10] in the absence of MHD effects, except that a is replaced here by a^* and $(\gamma+1)/2$ by Γ_0. The analysis for various types of ducts given by Tygai and Sujith[10] can easily be applied for the evolution of fast magnetosonic waves.

B. Effect of stratified medium

In this section, the effect of a change in entropy of the environment on wave form distortion will be investigated. In the present analysis, the entropy is altered by varying the density, which is in turn related to the speed of wave propagation. In order to elucidate the effect of an entropy gradient on the wave propagation, a duct with a constant cross section is considered. In this case, the condition for shock formation for a compression wave becomes

$$|u_1(0)| = \frac{1}{\frac{1}{a_0^*(0)^3}\int_0^{y_s}\left[\frac{a_0^*(0)}{a_0^*(\hat{y})}\right]^{9/2}\left[\frac{\rho_0(0)}{\rho_0(\hat{y})}\right]^{1/2}\left[\frac{\gamma+1}{2}\frac{\gamma p_0^*}{\rho_0}+\frac{1}{\mu}\left[\frac{B}{\rho_0}\right]^2\left(1-\frac{\gamma}{2}\right)\left(\frac{\gamma+3}{2}\right)\rho_0\right]d\hat{y}}. \quad (37)$$

The integral here is of the form

$$I = \int_0^\infty \left[\frac{g}{\rho_0}+h\rho_0\right]^{-9/4}\frac{1}{\rho_0^{1/2}}\left[\frac{e}{\rho_0}+f\rho_0\right]d\hat{y}$$
$$> \min\left[\frac{\rho_0^{3/4}(e+f\rho_0^2)}{(g+h\rho_0^2)^{9/4}}\right]\int_0^\infty d\hat{y}. \quad (38)$$

where $e=\gamma(\gamma+1)/2p_0^*$, $f=1/\mu[B/\rho]^2(1-\gamma/2)(\gamma+3)/2$, $g=\gamma p_0^*$, and $h=1/\mu[B/\rho]^2$. Hence, if $Z=\min[\rho_0^{3/4}(e+f\rho_0^2)/(g+h\rho_0^2)^{9/4}]\neq 0$, the integral will diverge.

As long as the density has a nonzero finite value at infinity, the integrand of I is nonzero, and the integral will diverge, and every compression wave, irrespective of the initial slope, will steepen to form a shock. However, when the density at infinity becomes zero or infinite, the minimum value of above expression is zero. Physically, this means that a^* is infinite if density is zero or infinite. Hence the flow in this case behaves like an incompressible fluid, where the wave form cannot be distorted. However, this does not imply that the shock does not form, since the wave could have steepened before reaching infinity, before the fluid starts behaving like an incompressible fluid (in this case the integral diverges even when the minimum value Z is zero). However, if the wave does not steepen even at large distances (before the fluid behaves like an incompressible fluid), then it will not steepen afterwards due to the above-mentioned reason (in this case the integral converges when Z is zero). It must be noted however that these effects are for waves with small initial slope; waves with large initial slopes steepen since they do not get affected much by these gradients.

First we consider the limiting case of magnetic pressure being zero. Equation (37) then reduces to

$$|u_1(0)| = \frac{1}{\frac{\gamma+1}{2a_0(0)}\int_0^{y_s}\left[\frac{a_0(0)}{a_0(\hat{y})}\right]^{3/2}d\hat{y}}. \quad (39)$$

This is the result given by Tyagi and Sujith[10] for shock formation in the absence of MHD effects. Next, considering the limiting case of compressional Alfvén wave, where the magnetic pressure being much higher than the hydrodynamic pressure ($\beta_e=p_e/p_B\ll 1$), it can be shown that

$$a^{*2} = \frac{1}{\mu}\left[\frac{B}{\rho}\right]^2\rho, \quad \Gamma = \frac{3}{2}. \quad (40)$$

It is interesting to note that the fluid behaves as if it has a γ value of 2 as the magnetic pressure $p_{\mathrm{mag}}\propto\rho^2$. This results in the value of nonlinearity parameter Γ_0 being $(\gamma+1)/2=3/2$.

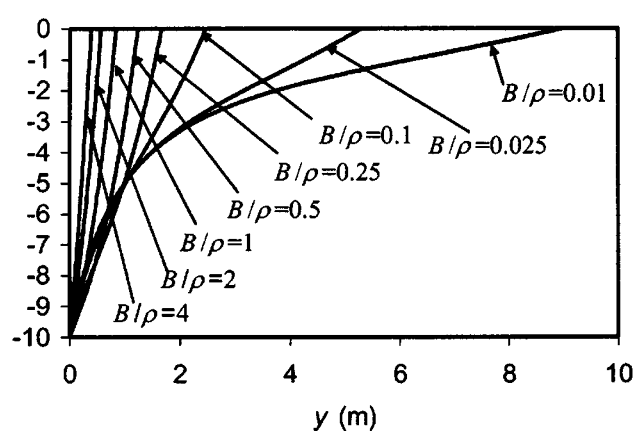

FIG. 1. Evolution of a compression wave front for various values of B/ρ for a linear density profile ($\rho=1+y$); $|u_1(0)|=0.1$.

$$\frac{1}{|u_1(0)|}=\frac{1}{\frac{3}{2}\frac{1}{a_0^*(0)}\int_0^{y_s}\left[\frac{a_0^*(0)}{a_0^*(\hat{y})}\right]^{7/2}d\hat{y}}. \tag{41}$$

When a compression wave moves in an environment where the density is decreasing, it can be shown that $\int_0^\infty[a_0^*(0)/a_0^*(\hat{y})]^{7/2}d\hat{y}$ diverges. Consequently, every compression wave traveling in such an environment will blow up at some finite distance. On the other hand, in the presence of a positive density gradient, the integral $\int_0^\infty[a_0^*(0)/a_0^*(\hat{y})]^{7/2}d\hat{y}$ may diverge or converge depending on the nature of the density variation. If the integral diverges, every compression wave front will steepen into a shock at some finite distance. However, if the integral converges, the initial slope of the wave front must exceed a minimum value in order to steepen into a shock.

For an arbitrary value of the ratio between the magnetic pressure and the hydrodynamic pressure, Eq. (37) has to be evaluated for a given density profile. For a linear density profile $\rho=Ky+K_1$, Eq. (37) reduces to

$$|u_1(0)|=\frac{1}{a^*(0)^{3/2}\rho(0)^{1/2}[\theta(\rho)-\theta(\rho_1)]}, \tag{42}$$

where

$$\theta(\rho)=\rho_0^{7/4}\left[\frac{7eh(5g+3h\rho_0^2)+7fg(5g+7h\rho_0^2)-5(7fg+3eh)(g+h\rho_0^2)\left(1+\frac{h\rho_0^2}{g}\right)_2F_1\left[\frac{7}{8},\frac{1}{4},\frac{15}{8},-\frac{h\rho_0^2}{g}\right]}{K35g^2h(g+h\rho_0^2)^{5/4}}\right]. \tag{43}$$

Figure 1 illustrates the evolution of a wave with a given value of initial slope through a given linear density field for various values of B/ρ. It can be seen that the effect of increasing B/ρ values on the compression waves is to increase the rate of wave steepening, causing early shock formation.

Figure 2 shows the variation of shock formation distance with changes in the value of the initial slope of the disturbance $[u_1(0)]$ for various values of B/ρ for a given linear density profile. Higher values of the initial slope lead to shorter shock formation distance. It can clearly be seen that higher the value of B/ρ, shorter the shock formation distance.

Figure 3 depicts the evolution of a wave with a given initial slope and strength of the magnetic field for various values of density gradient, keeping B/ρ constant. It is seen that as the density gradient increases, the rate of wave steepening decreases, and shock formation is delayed.

Figure 4 shows the dependence of shock formation distance on the value of the initial slope of the disturbance $[u_1(0)]$ for different values of density gradient for a given linear density profile, keeping B/ρ constant. It is seen that the density gradient considerably affects the value of shock formation distance.

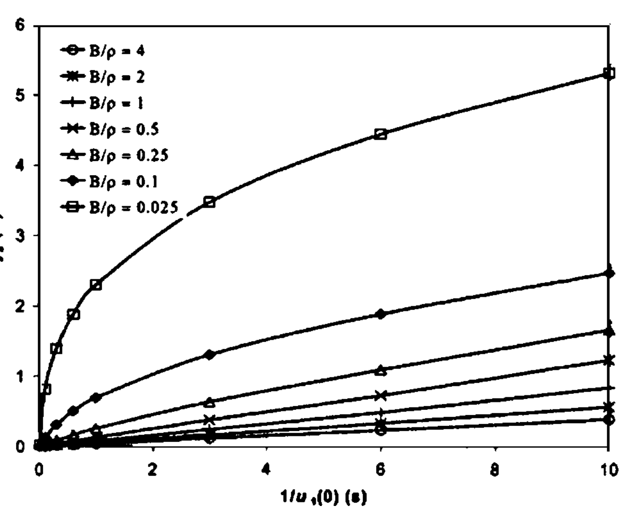

FIG. 2. The dependence of shock formation distance on the initial slope for various values of B/ρ for a linear density profile ($\rho=1+y$).

IV. SHOCK FORMATION IN A WEAKLY CONDUCTING GAS

The basic equations governing the unsteady motion of a weakly conducting, inviscid, non-heat-conducting gas are given by the following:[13]

Continuity,

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}+\frac{\rho u}{A}\frac{dA}{dx}=0. \tag{44}$$

Momentum,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x}=-\sigma B_0^2u. \tag{45}$$

Energy,

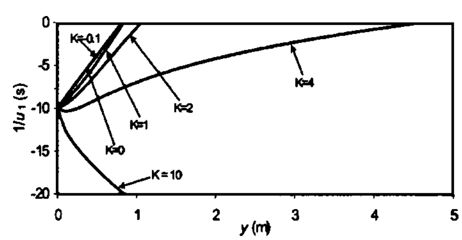

FIG. 3. Evolution of a compression wave front for various values of density gradient (K) for a linear density profile ($\rho=1+Ky$). The value of B/ρ is kept constant as 1; $|u_1(0)|=0.1$.

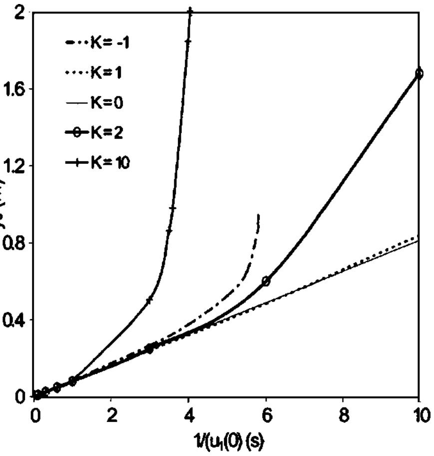

FIG. 4. The dependence of shock formation distance on the initial slope for various values of density gradient (K) for a linear density profile ($\rho=1+Ky$). The value of B/ρ is kept constant as 1. Note that for the case of $K=-1$, shock forms only for high values of $|u_1(0)|$.

$$\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)-a^2\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)=(\gamma-1)\sigma B_0^2u^2. \tag{46}$$

These equations form a hyperbolic system, and along with the equation of state, they completely describe the flow. The undisturbed medium is assumed to be at rest. To study the propagation and distortion of a wave in the medium, these equations are manipulated and written along their characteristics in the (t,x) plane

$$\frac{d^+u}{dt}+\frac{1}{\rho a}\frac{d^+p}{dt}+\frac{ua}{A}\frac{dA}{dx} =\left[\frac{(\gamma-1)u-a}{\rho a}\right]\sigma B_0^2u \quad \text{on } C_+:\frac{d^+x}{dt}=u+a, \tag{47}$$

$$\frac{d^-u}{dt}-\frac{1}{\rho a}\frac{d^-p}{dt}-\frac{ua}{A}\frac{dA}{dx} =-\left[\frac{(\gamma-1)u+a}{\rho a}\right]\sigma B_0^2u \quad \text{on } C_-:\frac{d^-x}{dt}=u-a. \tag{48}$$

Using the wave front expansion technique described in Sec. II B, and comparing the coefficients of ξ^0 and ξ^1, one finds

ξ^0 terms,

$$a_0(\rho_0'-\rho_1)+\rho_0u_1=0, \tag{49}$$

$$p_1=\rho_0u_1a_0, \tag{50}$$

$$(p_0'-p_1)-a_0^2(\rho_0'-\rho_1)=0; \tag{51}$$

ξ^1 terms,

$$(\rho_0'-\rho_1)a_0A_1+A_0(\dot\rho_1-\rho_2a_0)+\rho_0A_0u_2+2\rho_0A_1u_1 +2\rho_1u_1A_0=0, \tag{52}$$

$$-\rho_1u_1a_0+\rho_0(\dot u_1-u_2a_0)+\rho_0u_1^2+p_2=-\sigma B_0^2u_1, \tag{53}$$

$$\dot p_1-a_0p_2+p_1u_1=2a_0a_1(\rho_0'-\rho_1)a_0+a_0^2(\dot\rho_1-\rho_2a_0 +\rho_1u_1). \tag{54}$$

The matrix formed by the coefficients of first derivatives in Eqs. (49)–(51) is singular. Hence, the first set of equations reduce to $p_0'=0$, which is as expected in a quiescent medium.

Similarly, coefficients of the second derivatives in Eqs. (52)–(54) form a singular matrix. Hence on elimination of second derivatives, Eqs. (52)–(54) yield, after some manipulation:

$$\frac{du_1}{dy}+\left[\frac{1}{2}\frac{a_0'}{a_0}+\frac{1}{2}\frac{A_0'}{A_0}+\frac{1}{2}\frac{\sigma B_0^2}{\rho_0a_0}\right]u_1+\frac{\gamma+1}{2a_0}u_1^2=0. \tag{55}$$

This equation can be rewritten in linear form as

$$\frac{d}{dy}\left(\frac{1}{u_1}\right)-\left[\frac{1}{2}\frac{a_0'}{a_0}+\frac{1}{2}\frac{A_0'}{A_0}+\frac{\sigma B_0^2}{2\rho_0a_0}\right]\frac{1}{u_1}=\frac{\gamma+1}{2a_0}. \tag{56}$$

The solution of Eq. (53) is given by

$$\frac{1}{u_1(y)}=\frac{1}{u_1(0)}\frac{IF(0)}{IF(y)}+\frac{\gamma+1}{2a_0}\frac{1}{IF(y)}\int_0^y\frac{IF(\hat y)}{a_0}d\hat y, \tag{57}$$

where

$$IF=\frac{1}{\sqrt{a_0A_0}}\exp\left[-\frac{\sigma B_0^2}{2}\int_0^y\frac{d\hat y}{\rho_0a_0}\right]. \tag{58}$$

Shock forms when $|u_1(0)|\to\infty$. Therefore the location for shock formation can be obtained from the following expression:

$$\frac{1}{|u_1(0)|}=\frac{\gamma+1}{2a_0(0)}\int_0^{y_s}\left[\frac{a_0(0)}{a_0(y)}\right]^{3/2}\left[\frac{A_0(0)}{A_0(y)}\right]^{1/2} \times\exp\left(-\frac{\sigma B_0^2}{2}\int_0^y\frac{d\hat y}{\rho_0(\hat y)a_0(\hat y)}\right)dy. \tag{59}$$

The exponential term in Eq. (59) can be evaluated for specific density profiles. For example, if $\rho_0(y)=(my+c)^2$, Eq. (59) reduces to

$$\frac{1}{|u_1(0)|}=\frac{\gamma+1}{2a_0}\rho_0(0)^\delta\int_0^y\left[\frac{\rho_0(y)}{\rho_0(0)}\right]^\varepsilon\left[\frac{A_0(0)}{A_0(y)}\right]^{1/2}dy, \tag{60}$$

where $\delta=-\sigma B_0^2/4m\sqrt{\gamma P_0}$ and $\varepsilon=3/4+\delta$.

First, the case of a constant-area duct is considered. If ε is greater than zero, when the density is increasing, the integral diverges. Consequently, every compression wave traveling in such an environment will steepen after some finite distance. This also leads to a decrease in shock formation distance as compared to the case with constant density. On the contrary, the shock formation distance increases as compared to the constant density case, when ε is negative. In the presence of a negative density gradient, ε can only be positive. In this situation, the integral will converge, and therefore, the initial slope of the wave front must exceed a minimum value in order to steepen into a shock.

For the case of a wave moving into a gas with uniform density, it can easily be seen that shock formation distance in a convergent duct is less than shock formation in a constant area duct. On the other hand, in the case of divergent ducts, compression waves may or may not become shocks depending on the nature of area variation.

V. SUMMARY

The steepening of fast magnetosonic waves into shocks is studied using the wave front expansion technique. A closed form solution for the steepening of the leading edge of wave front is obtained for a perfectly conducting fluid. Also, expressions for time and location of shock formation are obtained.

It is shown that the shock formation distance in a convergent duct is always less than the shock formation distance in a constant-area duct with uniform density. However, for divergent ducts, compression waves may or may not become shocks depending on the area variation of the duct. For ducts diverging very rapidly, a compression wave front will steepen to form a shock if the magnitude of the initial slope exceeds a certain value. However for a duct that is diverging less rapidly (such as a conical horn), any wave front can steepen into a shock given sufficient length.

For a stratified medium, it is shown that as long as the density has a nonzero, finite value at infinity, any disturbance will steepen to form a shock. It is shown that the effect of magnetic field is to cause early shock formation. It is also shown that as the density gradient increases, shock formation is delayed. In the limit of magnetic pressure being much larger than the hydrodynamic pressure, the nonlinearity parameter reduces to $3/2$, as the fluid behaves as if it has a ratio of specific heats of 2. Finally an expression for the evolution of the leading edge of wave front traveling in a weakly conducting gas is obtained.

The results obtained from this analysis can be used as bench marks for checking the accuracy of numerical codes for MHD flows.

ACKNOWLEDGMENTS

This work was supported by the Alexander Von Humboldt Foundation through a Humboldt Fellowship to the author. The author wishes to thank Professor Wolfgang Polifke of Technische Universität München for hosting him during this stay. The author also wishes to thank Koushik Balasubramanian, Manav Tyagi, and Srevatsan Muralidharan of IIT Madras, with whom he had many interesting discussions.

[1] B. Vrsnak and S. Lulic, Sol. Phys. **196**, 157 (2000).
[2] B. Vrsnak and S. Lulic, Sol. Phys. **196**, 181 (2000).
[3] J. R. Spreiter and S. S. Stahara, Adv. Space Res. **15**, 443 (1995).
[4] L. Stenflo, A. B. Shvartsburg, and J. Weiland, Phys. Lett. A **225**, 113 (1997).
[5] A. Ödblom, Phys. Lett. A **249**, 93 (1998).
[6] D. Bharadwaj, Int. J. Eng. Sci. **38**, 1197 (2000).
[7] J. A. Orta, M. A. Huerta, and G. C. Boynton, Astrophys. J. **596**, 646 (2003).
[8] C. Appert, C. Tennaud, X. Chavanne, S. Balibar, F. Caupin, and D. d'Humieres, Eur. Phys. J. B **35**, 531 (2003).
[9] H. Lin and A. J. Szeri, J. Fluid Mech. **431**, 161 (2001).
[10] M. Tyagi and R. I. Sujith, J. Fluid Mech. **492**, 1 (2003).
[11] G. B. Whitham, Commun. Pure Appl. Math. **12**, 113 (1959).
[12] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[13] J. Tyl, J. Tech. Phys. **33**, 205 (1992).
[14] S. Muralidharan and R. I. Sujith, Phys. Fluids **16**, 4121 (2004).
[15] W. D. Hayes, *Gasdynamic Discontinuities*, Princeton series on high speed aerodynamics and jet propulsion (Princeton University Press, Princeton, NJ, 1960).