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Linear complementarity results for **Z**-matrices on Lorentz cone

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ABSTRACT

Let $\mathcal{K} \subseteq \mathbb{R}^n$ be the n-dimensional Lorentz cone. Given an $n \times n$ matrix M and $q \in \mathbb{R}^n$, the Lorentz-cone linear complementarity problem LCLCP(M, q) is to find an $x \in \mathbb{R}^n$ that satisfies

$$x \in \mathcal{K}, \quad y := Mx + q \in \mathcal{K} \quad \text{and} \quad y^T x = 0.$$

We show that if M is a **Z**-matrix with respect to $\mathcal{K}$, then M is positive stable if and only if LCLCP(M, q) has a non-empty finite solution set for all $q \in \mathbb{R}^n$.

1. Introduction

The *Lorentz*-cone in $\mathbb{R}^n$ is defined by

$$\mathcal{K}^n := \{x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n : x_1 \geq 0,\ x_1^2 \geq x_2^2 + \cdots + x_n^2\}.$$

We will fix $n > 2$. For brevity, we write $\mathcal{K}$ instead of $\mathcal{K}^n$. Further, we will always write if $x \in \mathbb{R}^n$, then $x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}$. Given an $n \times n$ matrix M and a vector $q \in \mathbb{R}^n$, the Lorentz-cone

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linear complementarity problem LCLCP(M, q) is to find an $x \in \mathbb{R}^n$ such that:

$$x \in \mathcal{K}, \quad y := Mx + q \in \mathcal{K} \ \text{ and } \ y^T x = 0.$$

Complementarity problems appear in various areas that include game theory, optimization, and economics. LCLCP is a classical example of a linear complementarity problem defined on a non-polyhedral cone. Complementarity problems are special cases of variational inequality problems that have an extensive literature and wide applications (see, for example, Facchinei and Pang [2]). There is a particular algebra associated with the second order cone. This algebra is well-known and is a special case of a so-called Euclidean Jordan algebra. We refer to Faraut and Korányi [3] for a comprehensive study. The rank of Euclidean Jordan algebra associated with the second order cone is 2 and has extra topological structure which allows us to go beyond the general study of variational inequalities. For further discussion, it is convenient to introduce the following definition.

Definition 1. Let M be an $n \times n$ real matrix. Then

1. M is called a **Z**-matrix on $\mathcal{K}$ if the following condition is satisfied:

$$x \in \mathcal{K}, \quad y \in \mathcal{K} \ \text{ and } \ x^T y = 0 \implies y^T Mx \leq 0.$$

   In this case, we will write $M \in \mathbf{Z}$.
2. M is called a **Q**-matrix if LCLCP(M, q) has a solution for all $q \in \mathbb{R}^n$. In this case, we write $M \in \mathbf{Q}$.
3. $M \in \mathbf{Z} \cap \mathbf{Q}$, if $M \in \mathbf{Z}$ and $M \in \mathbf{Q}$.

The class of **Z**-matrices is very broad. If S is an $n \times n$ matrix such that $S(\mathcal{K}) \subseteq \mathcal{K}$, then S is said to be *positive* on $\mathcal{K}$. Characterization of matrices that are positive on $\mathcal{K}$ appears in Loewy and Schneider [4]. If S is positive on $\mathcal{K}$, then $I - S$ is a **Z**-matrix on $\mathcal{K}$. The most interesting theorem about **Z**-matrices in connection with LCLCP is the following:

Theorem 1. *If $M \in \mathbf{Z}$, then the following are equivalent.*

(A1) $M \in \mathbf{Q}$.
(A2) *Let $e = (1, 0, \cdots, 0)^T$. Then zero is the only solution to* LCLCP$(M, 0)$ *and* LCLCP(M, e).
(A3) *M is positive stable (i.e. every eigenvalue of M has a positive real part).*
(A4) *There exists $v \in \mathrm{int}(\mathcal{K})$ such that $Mv \in \mathrm{int}(\mathcal{K})$.*
(A5) *M^{-1} exists and $M^{-1}(\mathcal{K}) \subseteq \mathcal{K}$.*
(A6) $M^T \in \mathbf{Q}$.

(Here, we have specialized the above result to $\mathcal{K}$. More equivalent conditions in a general setting are found in Theorem 6 of Gowda and Tao [6].) It is natural to ask the following question now: Does $M \in \mathbf{Z} \cap \mathbf{Q}$ imply LCLCP(M, q) has only finitely many solutions for any vector q? Our main result answers the above question precisely. We show that if $M \in \mathbf{Z} \cap \mathbf{Q}$, then LCLCP$(M, q)$ has finitely many solutions for all $q \in \mathbb{R}^n$. For an introduction to $\mathbf{Z}$-matrices with respect to the cone $\mathbb{R}^n_+$ and for an analog of Theorem 1, we refer to Bapat and Raghavan [1].

2. Preliminaries

We use the following notations.

2.1. Notations

(i) To denote the interior and the boundary of $\mathcal{K}$, we use $\text{int}(\mathcal{K})$ and $\partial\mathcal{K}$ respectively.
(ii) Let J denote the $n \times n$ diagonal matrix $\text{diag}(1, -1, -1, \cdots, -1)$.
(iii) Let M be an $n \times n$ matrix.
  (a) Columns of M are denoted by $m^1, m^2, \cdots, m^n$ respectively.
  (b) We write M by $[[m^1, m^2, \cdots, m^n]]$.
  (c) Let $b \in \mathbb{R}^n$. Then, $M^{(i)}(b) := [[m^1, \cdots, m^{i-1}, b, m^{i+1}, \cdots, m^n]]$.
  (d) For each $\tau \in \mathbb{R}$, we define $M_\tau := M - \tau J$.
(iv) I will stand for the identity matrix.
(v) SOL(M, q) will represent the set of all solutions to LCLCP(M, q).
(vi) e will denote the vector $(1, 0, \cdots, 0)^T$ in $\mathbb{R}^n$.
(vii) Let F be a non-empty subset of $\mathbb{R}^n$. Then, $-F := \{-x : x \in F\}$.

2.2. An elementary lemma

In the following elementary lemma, we summarize some basic properties of the second order cone. We refer to Tao [7] for details.

Lemma 1. *The following are true:*

(1) Let $x \in \partial\mathcal{K}$ and $y \in \partial\mathcal{K}$. Then the following are equivalent:
  (a) $x^T y = 0$.
  (b) There exists $\mu \geq 0$ such that $y = \mu J x$.
(2) $x \in \mathcal{K}$ if and only if $x^T y \geq 0$ for all $y \in \mathcal{K}$.
(3) $x \in \text{int}(\mathcal{K})$ if and only if $x^T y > 0$ for all non-zero $y \in \mathcal{K}$.
(4) Let M be an $n \times n$ matrix and $q \in \mathbb{R}^n$. Suppose $x \in \text{SOL}(M, q) \cap \partial\mathcal{K}$. Then there exists a $\mu \geq 0$ such that $Mx + q = \mu J x$. Conversely, if there exist $\mu \geq 0$ and $x \in \partial\mathcal{K}$ such that $Mx + q = \mu J x$, then $x \in \text{SOL}(M, q)$.
(5) If $x \in \text{int}(\mathcal{K})$, $y \in \mathcal{K}$ and $x^T y = 0$, then $y = 0$.

(6) Let $x \in \mathcal{K}$. Then $x \in \text{int}(\mathcal{K})$ if and only if $Jx \in \text{int}(\mathcal{K})$. Similarly $x \in \partial\mathcal{K}$ if and only if $Jx \in \partial\mathcal{K}$.

3. Some preliminary results

The following lemma will be very useful and will be invoked often in the proofs of our main results.

Lemma 2. *Let $x, y \in \partial\mathcal{K}$. If x and y are linearly independent, then $x - y \notin \mathcal{K}$.*

Proof. Let $z = x - y$. Suppose $z \in \mathcal{K}$. If $z \in \text{int}(\mathcal{K})$, then note that $x \in \text{int}(\mathcal{K})$ which is not possible. So, $z \in \partial\mathcal{K}$. Let

$$x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z_0 \\ \bar{z} \end{bmatrix}.$$

Since x, y and z belong to $\partial\mathcal{K}$, we have

$$\|\bar{y} + \bar{z}\| = y_0 + z_0 = \|\bar{y}\| + \|\bar{z}\|.$$

The above equality is true if and only if $\text{span}\{\bar{y}\} = \text{span}\{\bar{z}\}$. Because y and z lie in $\partial\mathcal{K}$, it now follows that y and z are linearly dependent. But $x = y + z$. So, x and y are linearly dependent. This contradicts our assumption. Therefore, $z \notin \mathcal{K}$. □

Lemma 3. *Let M and q satisfy the following conditions:*

1. *$M \in \mathbf{Z} \cap \mathbf{Q}$.*
2. *$0 \neq x \in \text{SOL}(M, q)$ and $0 \neq y \in \text{SOL}(M, q)$.*

If $x \neq y$, then x and y are linearly independent.

Proof. If x and y belong to $\text{int}(\mathcal{K})$, then by Lemma 1(5), it follows that $Mx + q = 0$ and $My + q = 0$. Since M is non-singular, $x = y$. Thus both x and y cannot be in $\text{int}(\mathcal{K})$.

Assume that $x \in \partial\mathcal{K}$. If x and y are linearly dependent, then for some $\alpha > 0$, $y = \alpha x$. Thus, $y \in \partial\mathcal{K}$. By Lemma 1(2), there exist $\mu \geq 0$ and $\mu' \geq 0$ such that

$$Mx + q = \mu Jx, \tag{1}$$
$$M(\alpha x) + q = \mu' J(\alpha x). \tag{2}$$

Multiplying (1) by α and subtracting from (2), we get

$$(\alpha - 1)q = (\mu\alpha - \mu'\alpha)Jx.$$

Since x and y are distinct, $\alpha \neq 1$. Thus, $q = \frac{\alpha(\mu-\mu')}{\alpha-1}Jx$. Hence by (1), there exists $k \in \mathbb{R}$ such that $Mx = kJx$. If $k \geq 0$, then $x \in \text{SOL}(M, 0)$ and by (A2) in Theorem 1, $x = 0$. If

$k < 0$ then, by (A5) in Theorem 1, $x \in -\mathcal{K}$ and therefore $x = 0$. This is a contradiction. Therefore x and y are linearly independent. $\square$

Lemma 4. *Let $M \in \mathbf{Z} \cap \mathbf{Q}$. Then the following are true:*

(i) *If* $\mathrm{LCLCP}(M, q')$ *has a solution in* $\mathrm{int}(\mathcal{K})$*, then* $\mathrm{LCLCP}(M, q')$ *has a unique solution.*
(ii) $\mathrm{LCLCP}(M, q)$ *has a unique solution for all* $q \in -\mathcal{K}$*.*

Proof. We first prove (i). Let $x \in \mathrm{SOL}(M, q') \cap \mathrm{int}(\mathcal{K})$. If $y \in \mathrm{SOL}(M, q') \in \mathrm{int}(\mathcal{K})$, then by Lemma 1(5), we have $Mx + q' = 0$ and $My + q' = 0$. As M is non-singular, $x = y$.

Suppose $y \in \partial\mathcal{K}$. Then by Lemma 1(4), there exists a $\mu \geq 0$ such that $My + q' = \mu Jy$ and thus we have $M(y-x) = \mu Jy$. Now Theorem 1(A5) implies, $y - x \in \mathcal{K}$. As $x \in \mathrm{int}(\mathcal{K})$, we deduce that $y \in \mathrm{int}(\mathcal{K})$ which is a contradiction. This proves (i).

We now prove (ii). Note that if $x = -M^{-1}q$ then $x \in \mathcal{K}$. Let $y \in \mathrm{SOL}(M, q)$. Then, $v := My + q \in \mathcal{K}$ and $y^T v = 0$ implies $(My)^T v = 0$. So,

$$0 \geq q^T v = \|v\|^2 - v^T My \geq 0.$$

Thus, $v = 0$. Hence, $q = -My$ which shows that $y = x$ as M is invertible. This proves (ii). $\square$

4. Main result

We will prove our main result: If $M \in \mathbf{Z} \cap \mathbf{Q}$, then $\mathrm{LCLCP}(M, q)$ has finitely many solutions for all $q \in \mathbb{R}^n$. Recall that $x \in \mathrm{SOL}(M, q) \cap \partial\mathcal{K}$ if and only if there exists a $\mu \geq 0$ such that $(JM - \mu I)x = -Jq$. We will make use of this fact in the sequel. The proof of the main theorem will follow from the next lemma.

Lemma 5. *Let $M \in \mathbf{Z} \cap \mathbf{Q}$ and $0 \neq b \in \mathbb{R}^n$. Then*

$$\Omega := \{\mu \in \mathbb{R} : (JM - \mu I)x = b,\ x \in \partial K\}$$

is a finite set.

Proof. Let $M_\mu := JM - \mu I$. Define

$$U := \{\mu \in \Omega : \det(M_\mu) = 0\} \quad \text{and} \quad V := \{\mu \in \Omega : \det(M_\mu) \neq 0\}.$$

Then $\Omega = U \cup V$ and U is finite. We claim that V is finite. Suppose V contains infinitely many elements. For each $\mu \in \mathbb{R}$, we define

$$f^i(\mu) := \det(M_\mu^{(i)}(b))^2 \quad \text{and} \quad P(\mu) := f^1(\mu) - \sum_2^n f^i(\mu).$$

We now claim that if $\mu \in V$, then $P(\mu) = 0$. Let $\beta \in V$. Since M_β is non-singular, there exists a unique $x \in \partial K$ such that $M_\beta x = b$. By Cramer's rule,

$$x_i = \frac{\det(M_\beta^{(i)}(b))}{\det(M_\beta)}.$$

As $x \in \partial\mathcal{K}$, we have $x_1^2 = x_2^2 + \cdots + x_n^2$ and hence we see that

$$f^1(\beta) = \sum_2^n f^i(\beta).$$

Thus, $P(\beta) = 0$ and this proves our claim.

Since P is a polynomial and V has infinitely many elements, we deduce that $P(\mu) = 0$ for every $\mu \in \mathbb{R}$. We now claim that if $\mu \in V$, M_μ is invertible, and $M_\mu p = b$, then $p \in \partial\mathcal{K} \cup -\partial\mathcal{K}$. Let $\alpha \in V$. Define

$$p_i = \frac{\det(M_\alpha^{(i)}(b))}{\det(M_\alpha)}.$$

By Cramer's rule, $p := (p_1, \cdots, p_n)^T$ is the unique solution for the system $M_\alpha(x) = b$. As $P(\alpha) = 0$, we get

$$p_1^2 = \sum_2^n p_i^2.$$

The above equation implies that $p \in \partial\mathcal{K} \cup -\partial\mathcal{K}$. This proves our claim.

Let $\{\mu_k\}$ be a decreasing sequence of positive numbers converging to 0 such that each M_{μ_k} is non-singular. Let $M_{\mu_k}(x_k) = b$. Then, $x_k \in \partial\mathcal{K} \cup -\partial\mathcal{K}$. We now claim that the sequence $\{x_k\}$ is bounded. Suppose $\{x_k\}$ is unbounded. Because b is non-zero, each x_k is non-zero. We now have

$$M_{\mu_k}(\frac{x_k}{\|x_k\|}) = \frac{b}{\|x_k\|}.$$

Without loss of generality assume that $\frac{x_k}{\|x_k\|} \to y$. Applying limits, we find that $JMy = 0$. This contradicts that JM is non-singular. Thus $\{x_k\}$ is bounded.

Let, without loss of generality, $x_k \to u$. Since $M_{\mu_k}(x_k) = b$, by applying limits we get $JMu = b$. As $b \neq 0$, we see that $u \neq 0$. Thus, $JM(x_k - u) = \mu_k x_k$ for all k and so $x_k - u = \mu_k M^{-1}(Jx_k)$ for all k. Since $x_k \in \partial\mathcal{K} \cup -\partial\mathcal{K}$ for each k, we may assume, without loss of generality, that, (i) $x_k \in \partial\mathcal{K}$ for all k or (ii) $x_k \in -\partial\mathcal{K}$ for all k. In the first case $u \in \partial\mathcal{K}$, $x_k - u \in \mathcal{K}$ and so by Lemma 2, $x_k \in \mathrm{span}\{u\}$ for all k. In the second case, $u \in -\partial\mathcal{K}$ and $-x_k - (-u) = M^{-1}(-Jx_k) \in \mathcal{K}$. By Lemma 2, it now follows that $x_k \in \mathrm{span}\{u\}$.

As x_k is a multiple of u, we see that $x_k - u = \theta_k u$ and so $JM(x_k - u) = \mu_k x_k$. This implies $M(\theta_k u) = \mu_k(1+\theta_k)Ju$. By (A2) and (A5) in Theorem 1, either $\theta_k = 0$ or $u = 0$. But u is non-zero. So, $x_k = u$ for all k. Since $x_k - u = \mu_k M^{-1}(Jx_k)$, we get $M^{-1}(Jx_k) = 0$ and hence $x_k = 0$ which is a contradiction. □

The main result follows now.

Theorem 2. *Let $M \in \mathbf{Z} \cap \mathbf{Q}$. Then,* LCLCP$(M, q)$ *has finitely many solutions for all $q \in \mathbb{R}^n$.*

Proof. Since $M \in \mathbf{Z} \cap \mathbf{Q}$, we see that zero is the only solution to LCLCP$(M, 0)$. Let $q \in \mathbb{R}^n$ be a non-zero vector. In view of Lemma 4, it is enough to show that SOL$(M, q) \cap \partial\mathcal{K}$ is finite. Let $\Omega := \{\mu \geq 0 : M_\mu(x) = -q,\ x \in \partial\mathcal{K}\}$. Then by the previous lemma, Ω is finite. For any $\mu \in \Omega$, let $x \in \partial\mathcal{K}$ such that $M_\mu x = -q$. We claim that x is the only solution to LCLCP(M, q). If $y \in$ SOL$(M, q) \cap$ int$(\mathcal{K})$, then by Lemma 4, $y = x$.

Suppose there exists $y \in \partial\mathcal{K}$ such that $y \neq x$ and $My + q = \mu Jy$. Then, $M(x - y) = \mu J(x - y)$. If $\mu = 0$, then $M(x - y) = 0$. As M is non-singular and $x \neq y$, this is not possible. So, $\mu > 0$. Put $z = x - y$. We claim that $z \in \mathcal{K}$. Suppose $z \notin \mathcal{K} \cup -\mathcal{K}$. Then by Moreau decomposition, there exist $u, v \in \partial\mathcal{K}$ such that $z = u - v$ where u and v are orthogonal. Then $J(u - v) = \alpha v - \beta u$ for some $\alpha > 0$ and $\beta > 0$. Now $(M + \eta I)z = ku$ for some $k \in \mathbb{R}$ and $\eta > 0$. By an easy verification, $M + \eta I \in \mathbf{Z} \cap \mathbf{Q}$. By (A5) in Theorem 1, $\pm z \in \mathcal{K}$. By Lemma 2, x and y are linearly dependent. But this contradicts Lemma 3. This completes the proof. □

5. A note on semidefinite linear complementarity problems

Let $\mathcal{S}^{n\times n}$ be the space of all real symmetric $n \times n$ matrices. Given a linear transformation $L : \mathcal{S}^{n\times n} \to \mathcal{S}^{n\times n}$ and $Q \in \mathcal{S}^{n\times n}$ the semidefinite linear complementarity problem SDLCP(L, Q) is to find an $X \in \mathcal{S}^{n\times n}$ such that

$$X \succeq 0, \quad Y := L(X) + Q \succeq 0 \ \text{ and } \ \langle Y, X\rangle = \text{trace}(XY) = 0.$$

Here $X \succeq 0$ means that $X \in \mathcal{S}^{n\times n}$ and X is positive semidefinite. For motivation and importance of SDLCP, we refer to [5]. The following problem is posed in Gowda and Song [5]. For an $n \times n$ matrix A, the Lyapunov transformation is defined by $L_A(X) := AX + XA^T$. If A is positive stable then show that SDLCP(L_A, Q) has only finitely many solutions for all $Q \in \mathcal{S}^{n\times n}$. Since L_A is a **Z**-transformation with respect to the positive semidefinite cone and $\mathcal{S}^{2\times 2}$ is isomorphic to $\mathcal{K}^3$, we have the following result:

Theorem 3. *Let A be a 2×2 matrix. If A is positive stable then* SDLCP(L_A, Q) *has only finitely many solutions for all $Q \in \mathcal{S}^{2\times 2}$.*

6. An example

It is natural to ask if an $n \times n$ matrix $A \in \mathbf{Z} \cap \mathbf{Q}$, then is it true that LCLCP(A, q) has a unique solution for all $q \in \mathbb{R}^n$. If this is true then by the previous note A is positive stable if and only if SDLCP(L_A, Q) has a unique solution for all $Q \in \mathscr{S}^{2\times 2}$. But this will contradict the following well-known theorem.

Theorem 4. *(See Gowda and Song [5].) Let A be an $n \times n$ matrix. Then the following are equivalent for the Lyapunov transformation $L_A(X) := AX + XA^T$.*

(i) *A is positive stable and positive semidefinite.*
(ii) SDLCP(L_A, Q) *has a unique solution for all $Q \in \mathscr{S}^{n\times n}$.*

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