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Large elaso-plastic deflection of micro-beams using strain gradient plasticity theory

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Abstract

Metallic micro-scale structures are observed to display more stiffness unlike the predictions of the classical plasticity theories, thus showing size-dependency. Hence, for accurate predictions of the behavior of micro-scale structures, non-classical theories are developed with intrinsic material length in their constitutive relation. Out of the many developed models, strain gradient plasticity theory is the one which is extensively followed. Power law based moment curvature relationship obtained from strain gradient plasticity is available in literature which models rigid-plastic behavior of micro-beam bending. In this article, elastic deformation is appended to the power law based moment curvature relationship. The derived law is employed to obtain the governing differential equation of a micro-cantilever under a normal terminal follower load undergoing large deflection. The non-linear differential equation is then solved by employing an efficient semi-incremental approach. In this approach the equation is solved by using Runge-Kutta 4th order method considering load to be at the current value but the material modulus at its previous value. Adoption of such a technique gave considerable computational advantage over the completely incremental method. The elasto-plastic force-displacement response curves of micro-beams showed stiffer results as compared to their classical counterpart. Also the elasto-plastic divergence from their corresponding elastic response curves are observed to happen at the same displacement for different micro-beams with same length to thickness ratio.

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Keywords: Large elasto-plastic deflection; Moment-curvature based constitutive law; Micro-cantilever bending; Strain gradient plasticity theory

1. Introduction

Micro-level mechanical structures are extensively used in applications based on MEMS (Micro-Electro-Mechanical Systems) for instance the micro-actuators, bio-sensors, micro-pumps, etc. Thus, analysing the behavior of these micron level mechanical elements has received importance making it a field of active research. However it has been observed by Fleck et al [1] that the mechanical response of these micro-level structures is considerably different from that of the macro-level structures. An increase in structural stiffness is observed with the decrease in the characteristic

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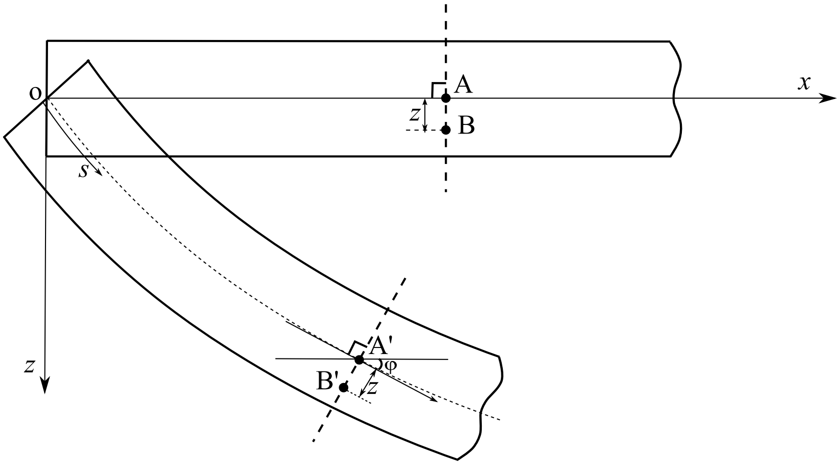

Fig. 1. Thin slender beam undergoing large elasto-plastic deflection

length. Also, the classical theories are incapable of capturing this size-effect which leads to the development of non-conventional size-dependent models. These models consists of material length-scale parameters in the constitutive equations. Fleck and Hutchinson [2], Gao et al [3], Fleck and Hutchinson [4], Hwang et al [5] proposed the strain gradient plasticity theories for predicting the size-dependent behavior. With the knowledge of the developed strain gradient theories, Stolken and Evans [6], Haque and Saif [7] presented micro-beam bending analysis. In addition to this, Wang et al [8], Chen and Wang [9] studied the micro-bend tests under small deformation by considering rigid-plastic material models i.e. the elastic deformations are ignored. Such an analysis is suitable when large plastic strains are involved. However, for structures with small plastic deformations and moderately large deflections, the contribution of elastic strain is considerable and thus its inclusion in the model becomes important.

In this article, a moment-curvature based constitutive model involving linear elastic behavior followed by power law hardening plastic deformation similar to Wang et al [8] is presented. Subsequently the model is employed to solve the large deflection bending of a micro-cantilever under a terminal follower load as described in section 2. The problem is solved by an efficient semi-incremental procedure instead of a full incremental approach as used in a similar problem as in Pandit and Srinivasan [10] and is explained in section 3. It is followed by section 4 where the deformation behavior of the micro-cantilever is presented. The paper is concluded in section 5.

2. Theoretical formulation

A thin slender prismatic beam with length, width and thickness along x, y and z-axis (right handed Cartesian coordinate) respectively, is considered. The beam is assumed to undergo plane-strain bending and large elasto-plastic deflection with small strain in the x-z plane. Euler-Bernoulli beam model is considered. A deformed beam at any pseudo time instant t is as shown in figure 1. The arc-length s is measured along the deformed neutral axis. The tangent at any s makes an angle of ϕ with the horizontal and the scalar curvature by considering sagging to be positive is given by:

$$\kappa(s,t) = -\frac{\partial\phi(s,t)}{\partial s} \tag{1}$$

The beam is assumed to follow the material model developed by Wang et al [8] which is in the form of moment-curvature relationship. However, the rigid-plastic material model is amended to take into account the elastic behavior of the material. Also for the smooth transition from elastic to plastic region, a linear variation for tangent modulus D of the moment-curvature relation is considered for curvature in the range of $\kappa_{y0} \leq \kappa \leq \kappa_s$ where κ_s is assumed to be close to the initial yield curvature κ_{y0}. Thus making the bending moment a quadratic function of curvature in that range. The new developed elasto-plastic version of the moment-curvature relationship is as mentioned below:

$$M = EI_1^c\kappa, \quad \kappa \le \kappa_{y0}$$
$$M = \Big(\frac{EI_1^c - D_s}{\kappa_{y0} - \kappa_s}\Big)\frac{(\kappa - \kappa_{yo})^2}{2} + EI_1^c\kappa, \quad \kappa_{y0} \le \kappa \le \kappa_s$$
$$M = \frac{\sigma_{y0}}{\epsilon_{y0}^{1/n}} I\kappa^{1/n}, \quad \kappa \ge \kappa_s \tag{2}$$

where M is the bending moment (per unit width of the beam), E is the elastic modulus, D_s is the slope of moment-curvature curve when curvature is κ_s, σ_{y0} and ϵ_{y0} are the initial tensile yield stress and strain respectively, n is the plastic work hardening exponent. Also I is the modified moment of inertia as defined below:

$$I = \int_{-\frac{H}{2}}^{\frac{H}{2}} \Big(\frac{4}{3}z^2 + \frac{2}{3}l^2\Big)^{\frac{n+1}{2n}} dz \tag{3}$$

where H is the thickness of the beam and l is the material length scale parameter. The value of $n = 1$ corresponds to an elastic behavior whereas $n = \infty$ gives results for a perfectly plastic behavior. For elastic analysis, the modified moment of inertia I is denoted by $I_1^c = \frac{H^3}{12} + l^2H$. While $I_1 = \frac{H^3}{12}$ is obtained for elastic solid by neglecting the material length scale parameter.

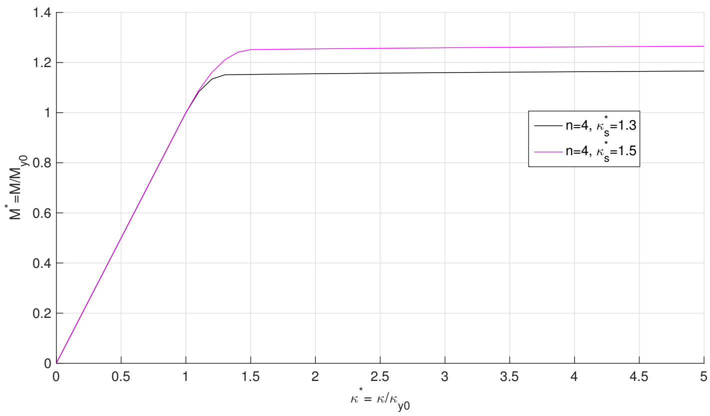

Fig. 2. Normalised constitutive law for a rectangular beam cross-section

The moment-curvature relationship in eq. 2 is non-dimensionalised by using the following normalised quantities: $M^* = \frac{M}{M_{y0}}$, $\kappa^* = \frac{\kappa}{\kappa_{y0}}$ where the initial yield moment is $M_{y0} = EI_1^c\kappa_{y0}$ and the initial yield curvature is $\kappa_{y0} = \frac{2\epsilon_{y0}}{H}$. Thus, the normalised moment-curvature relation is as written below:

$$M^* = \kappa^*, \quad \kappa^* \le 1$$
$$M^* = \Big(\frac{1 - D_s^*}{1 - \kappa_s^*}\Big)\frac{(\kappa^* - 1)^2}{2} + \kappa^*, \quad 1 \le \kappa^* \le \kappa_s^*$$
$$M^* = C\kappa^{*\,1/n}, \quad \kappa^* \ge \kappa_s^* \tag{4}$$

where $C = \frac{\sigma_{y0}}{E\epsilon_{y0}}\big(\frac{H}{2}\big)^{\frac{n-1}{n}}\frac{I}{I_1^c}$. The variation of normalised moment versus curvature is shown in figure 2 where its response for two different κ_s^* with the plastic work hardening exponent n=4 is displayed. It portrays the sensitivity of moment-curvature response on the choice of κ_s^*.

The above mentioned moment-curvature law is now used to solve a structural problem involving a slender, end loaded micro-cantilever of length L, width B and thickness H. It is considered to be undergoing large elasto-plastic deflection caused by a follower load P (per unit width of the beam) acting normal to it as shown in the figure 3. At the free-end i.e. $s = L$, the angle ϕ is denoted by ψ. A quasi-static analysis of the micro-cantilever is carried out. The micro-cantilever deformations are considered to be within the small strain framework with the presence of moderately large curvature. The bending moment $M(s,t)$ (per unit width of the beam) developed at any section s on the beam can

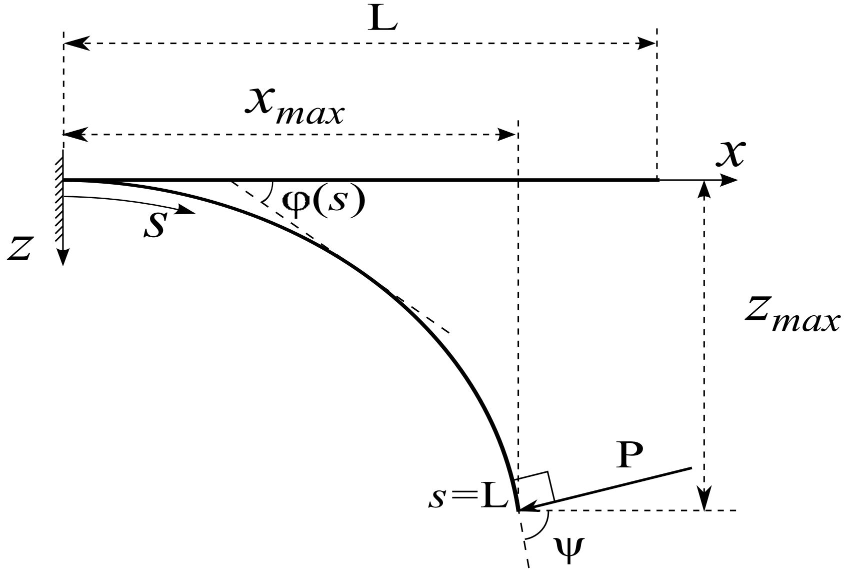

Fig. 3. Micro-cantilever subjected to a normal follower load acting at the free-end

be written in terms of the external force P with the aid of the moment equilibrium equation as follows:

$$M(s,t) = P sin\psi(z_{max} - z(s,t)) + P cos\psi(x_{max} - x(s,t)) \quad (5)$$

where x_{max} and z_{max} are the horizontal and vertical co-ordinates of the free-end of the beam respectively.

A general rate independent constitutive law which governs the moment-curvature relation is given by:

$$D = \frac{dM}{d\kappa} \quad (6)$$

where D is the flexural rigidity of the beam. From eqs. 2 and 6, for the linear elastic case i.e. when the curvature $\kappa \leq \kappa_{y0}$, it can be obtained that D is a constant and is equal to EI_1^c. Whereas when curvature is in the range $\kappa_{y0} < \kappa \leq \kappa_s$, D is assumed to linearly vary with curvature. Also for the part of the moment-curvature curve beyond κ_s, the variation of D is obtained from eqs. 2 and 6. Now combining eqs. 1, 5 and 6 and then by comparing the co-efficients of ds, the non-linear differential equation governing the slope of the deformed beam is obtained as mentioned below similar to Pandit and Srinivasan [10]:

$$D\frac{\partial^2 \phi}{\partial s^2} + P cos(\psi - \phi) = 0, \quad 0 \leq s \leq L \quad (7)$$

with the following boundary conditions:

$$\phi|_{s=0} = 0, \quad \frac{\partial \phi}{\partial s}|_{s=L} = 0 \quad (8)$$

The differential eq. 7 is normalised by using the following quantities: $\bar{D} = \frac{D}{EI_1}$, $\bar{s} = \frac{s}{L}$ and and $\bar{P} = \frac{PL^2}{EI_1}$ and is as written below:

$$\bar{D}\frac{\partial^2 \phi}{\partial \bar{s}^2} + \bar{P} cos(\psi - \phi) = 0 \quad (9)$$

subject to the following normalised boundary conditions:

$$\phi|_{\bar{s}=0} = 0, \quad \frac{\partial \phi}{\partial \bar{s}}|_{\bar{s}_{max}} = 0 \tag{10}$$

The deformed profile of the beam i.e. x and z co-ordinates are obtained after we numerically solve the differential eq. 9 for the angle ϕ and by using the following relations :

$$\bar{x} = \frac{x}{L} = \int_0^{\bar{s}} cos\phi \, d\bar{s}, \quad \bar{z} = \frac{z}{L} = \int_0^{\bar{s}} sin\phi \, d\bar{s} \tag{11}$$

3. Solution Methodology

The deflection governing eq. 9 is a second order differential equation involving material and geometric non-linearity. It is a boundary value problem in its present form. However, using the following variable transformation relation:

$$\alpha = \psi - \phi \tag{12}$$

it is transformed to an initial value problem as given below:

$$\bar{D}\frac{\partial^2 \alpha}{\partial \bar{s}^2} - \bar{P}cos\alpha = 0 \tag{13}$$

with the following initial conditions:

$$\alpha|_{\bar{s}_{max}} = 0, \quad \frac{\partial \alpha}{\partial \bar{s}}|_{\bar{s}_{max}} = 0 \tag{14}$$

The load is applied to the micro-cantilever in steps to reach its final value i.e. a load controlled problem is considered. Also, the length of the beam is equally discretised into $n - 1$ elements of size h, where n is the total number of nodes. The differential eq. 13 is then numerically solved for α's at all nodes and for all the load steps using the Runge-Kutta 4^{th} (RK4) order method till the final load is achieved. A semi-incremental procedure is adopted wherein the differential eq. 13 is solved at the current load step using the variation of D from the previous load step. Ideally D should be also corresponding to the current load step. However that would require finding the rate of change of D with respect to the load. This would increase the computational burden considerably. Also it is noted from figure 2 that D is a decreasing function of curvature. This character of D is utilized in ignoring the rate of change of D term in the governing equation at the current load step.

After obtaining the α's at all the nodes, the corresponding ϕ's are determined using the variable transformation eq. 12 which is followed by determining the curvature κ for the current load step. The normalised quantities obtained by solving the structural differential equation are related to the materially normalised quantities in constitutive law as follows: $\zeta = \kappa_{y0}L$, $\bar{\kappa} = \kappa^*\zeta$, $\bar{D} = (I_1^C/I_1)D^*$, $\bar{M} = M^*(I_1^C/I_1)\zeta$. Subsequently, the flexural rigidity D is updated at the current load step as described in the algorithm 1 by using the eq. 4.

procedure (Quantities κ, D, *and* M implies "*")
    Index 'i' describes the node number from the free end
    Input: κ_i (at current load step)
    Output: M_i (at current load step), D_i (at current load step)
    For i=1 to n
    If $\kappa_i \leq 1$
    $M_i = \kappa_i, D_i = 1$
    elseif $\kappa_i > 1$ *and* $\kappa_i \leq \kappa_s$
    $M_i = \frac{1-D_s}{1-\kappa_s}(\frac{(\kappa_i-1)^2}{2}) + \kappa_i, \; D_i = \frac{1-D_s}{1-\kappa_s}(\kappa_i - 1) + 1$
    elseif $\kappa_i > \kappa_s$
    $M_i = C(\kappa_i^{1/n} - \kappa_s^{1/n}) + M_{\kappa_s}, \; D_i = \frac{C}{n}\kappa_i^{(1-n)/n}$
    End if
    End For
end procedure

Algorithm 1: Evolution of D and M using the constitutive law

M_{κ_s} is the moment when curvature is κ_s and can be obtained from eq. 4. Eventually after obtaining the $\phi(\bar{s})$ for the final load step, the normalised $\bar{x}$ and $\bar{z}$ co-ordinates of any point along the deformed neutral axis of the beam are evaluated from eq. 11 using the Simpson's $1/3^{rd}$ method.

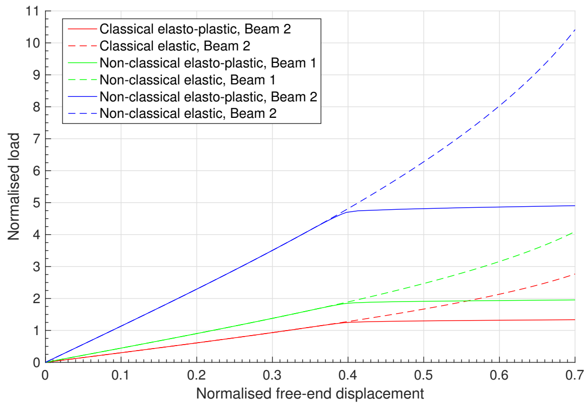

Fig. 4. Normalised load versus free-end displacement for different beams obtained for different theories

4. Results and Discussions

The prediction of the developed model for the deformation behavior of a micro-cantilever is presented in this section. The force-deflection response of two cantilevers with different thicknesses but a constant length to thickness ratio (=240) is studied. The elastic modulus E of the material is 207GPa, uniaxial stress-strain relation is $\sigma = 610\epsilon^{0.56}$MPa and the material length scale parameter is l=6μm as in Wang et al [8]. Also, κ_s is considered to be 1.3.

Two micro-cantilevers beam 1 and beam 2 of thickness 30μm and 12.5μm respectively are considered for analysis. The normalised load versus free-end vertical displacement plots obtained from the developed model are presented in figure 4. Its variation considering classical (l=0) and non-classical, elastic (n=1) and elasto-plastic (n=4) models are plotted. It is observed that classical results display less stiffness as compared to the non-classical results. Also the non-classical results obtained for two different beams show that beam 2 (H=12.5μm) is more stiff in comparison to beam 1 (H=30μm). Also considering the behavior for a particular theory and a beam, occurrence of plastic deformation is seen to make the beam more compliant. The elasto-plastic response curves are seen to diverge out from that of its elastic counterpart. It is of interest to note that this diverging phenomenon occurs at the same normalised displacement for micro-beams with different thicknesses.

5. Concluding Remarks

In this paper a moment-curvature based model for elasto-plastic micro-beam bending is proposed. This bending constitutive law is employed to solve a micro-cantilever subjected to a normal follower load undergoing large deflection. The obtained governing differential equation is in terms of the slope of the deformed beam. The coefficients of the equation are the material tangent modulus and the load. The equation is solved by considering the load to be at its current value but the tangent modulus to be at its previous value. This approach reduced the computational cost considerably while marginally decreasing the accuracy.

The force displacement response curves showed stiffness increase for non-classical theory as compared to its classical counterpart. Also the elasto-plastic response curves diverge from their corresponding elastic curves at the the same displacement for different micro-beams with same length to thickness ratio.

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References

[1] N. Fleck, G. Muller, M. Ashby, J. Hutchinson, Strain gradient plasticity: theory and experiment, Acta Metallurgica et Materialia 42 (1994) 475–487.
[2] N. Fleck, J. Hutchinson, A phenomenological theory for strain gradient effects in plasticity, Journal of the Mechanics and Physics of Solids 41 (1993) 1825–1857.
[3] H. Gao, Y. Huang, W. Nix, J. Hutchinson, Mechanism-based strain gradient plasticityi. theory, Journal of the Mechanics and Physics of Solids 47 (1999) 1239–1263.
[4] N. Fleck, J. Hutchinson, Strain gradient plasticity, Advances in applied mechanics 33 (1997) 296–361.
[5] K. Hwang, H. Jiang, Y. Huang, H. Gao, N. Hu, A finite deformation theory of strain gradient plasticity, Journal of the Mechanics and Physics of Solids 50 (2002) 81–99.
[6] J. Stölken, A. Evans, A microbend test method for measuring the plasticity length scale, Acta Materialia 46 (1998) 5109–5115.
[7] M. Haque, M. Saif, Strain gradient effect in nanoscale thin films, Acta Materialia 51 (2003) 3053–3061.
[8] W. Wang, Y. Huang, K. Hsia, K. Hu, A. Chandra, A study of microbend test by strain gradient plasticity, International Journal of Plasticity 19 (2003) 365–382.
[9] S. Chen, T. Wang, A new hardening law for strain gradient plasticity, Acta Materialia 48 (2000) 3997–4005.
[10] D. Pandit, S. M. Srinivasan, An incremental approach for springback analysis of elasto-plastic beam undergoing contact driven large deflection, International Journal of Mechanical Sciences (2016).