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Instabilities in superfluid plane Poiseuille flow

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Abstract. Modal and non-modal analysis of superfluid He-II is carried out in this paper. Landau's two fluid model for superfluid flow is assumed and stability of both normal fluid and superfluid are considered separately by taking the effect of the other through mutual friction. Stability of normal fluid, with a numerically calculated base flow, shows Tollmien-Schlichting mode of instability similar to classical fluids. Non-modal analysis shows the transient growth of perturbations at sub-critical Reynolds numbers. Mutual friction is found to have a stabilizing effect on normal fluid. Superfluid, with an inviscid uniform profile is found to be linearly stable for all wave numbers. Non-modal analysis rules out the possibility of transient growth of perturbations and sub-critical transition. The presence of turbulence in superfluid brings in a possibility that an initial uniform base flow is getting altered due to the action of mutual friction and this altered profile is likely to become unstable.

1. Introduction

The hydrodynamical properties exhibited by superfluid Helium-II is phenomenologically explained using Landau's two-fluid model. According to this model, He-II is an intimate mixture of two fluid components, a viscous normal fluid and an inviscid superfluid, with different velocity fields ($\mathbf{V_n}$ & $\mathbf{V_s}$) and densities (ρ_n & ρ_s). System is governed by two separate equations for momentum conservation of the two fluids, which are coupled through a mutual friction force (Gorter & Mellink, 1949). Vorticity in superfluid is quantized and turbulence in it is considered to be a tangle of quantized vortex lines (Feynman, 1955). Because of the existence of two velocities, turbulence in He-II can be coflow turbulence or counterflow turbulence, based on whether the two components are moving in same direction or in opposite directions. Coflow turbulence is analogous to classical turbulence, where as counterflow turbulence is excited by thermal counterflow (Vinen, 1957). Tough (1982), argued that there exists two different turbulent states T1 and T2 , marked by sudden jumps in superfluid vortex line density. T1 state is triggered when the relative velocity of normal fluid and superfluid, V_{ns}, exceeds a critical value V_{c1}, and T2 state is triggered at a higher critical value V_{c2}. Melotte & Barenghi (1998*a*,*b*) numerically studied the effect of superfluid vortex tangle on the stability of normal fluid in thermal counterflow. They considered a circular pipe closed at one end and open to helium bath at other end. It was observed that the normal fluid becomes unstable by the action of superfluid vortices by linear mechanism. Superfluid vortex line density for unstable regimes were calculated and compared with experimental results, and concluded that "transition from T1 to T2 state is an instability of normal fluid". Godfrey *et al.* (2001) carried out a linear stability analysis numerically, in plane Poiseuille flow of He-II, where the two fluids were flowing in the same direction. Normal fluid and superfluid components were considered separately by taking the effect of the other through a mutual friction term. They found that normal fluid is linearly unstable similar to classical fluids, and mutual friction has a destabilizing effect. They also found a new regime of stability at lower streamwise wave number which they termed as

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lower unstable branch. Interestingly, superfluid part was observed to be linearly stable for all Reynolds numbers. The classical Tollmien-Schlichting waves (linear mechanism) are shown not to be the cause of instability in a superfluid flow. However, superfluid is known to become turbulent and here we attempt to see algebraic transient growth mechanism as a possible route to instability.

2. Formulation

This paper computes instabilities in a plane channel He-II superfluid flow. $\mathbf{V_n}$ & $\mathbf{V_s}$ are the velocities and ρ_n & ρ_s are the densities of normal and superfluid respectively. The total density of He-II is,

$$\rho = \rho_n + \rho_s \tag{1}$$

The two fluid model requires two sets of equations, one for normal fluid and one for superfluid. Considering isothermal flow, and using the assumption of incompressibility, we can write the mass conservation equations as,

$$\nabla \cdot \mathbf{V_n} = \nabla \cdot \mathbf{V_s} = 0 \tag{2}$$

The intimate mixture of two fluids interact with each other through mutual friction, which occurs due to the scattering of the normal fluid by superfluid vortices, Hall & Vinen (1956*a*,*b*). Hence the momentum conservation equations for the two fluids will have a coupling term to account for this mutual friction between the fluids. Assuming that there is no temperature gradient, the momentum conservation equations for the two fluids in non-dimensional form are (Godfrey *et al.*, 2001; Chandrasekhar & Donnelly, 1957; Chandrasekhar, 1957),

$$\frac{\partial}{\partial t}\mathbf{V_n} + (\mathbf{V_n} \cdot \nabla)\mathbf{V_n} = -\nabla P + \frac{1}{Re}\nabla^2\mathbf{V_n} - \mathbf{F_{mf}} \tag{3}$$

$$\frac{\partial}{\partial t}\mathbf{V_s} + (\mathbf{V_s} \cdot \nabla)\mathbf{V_s} = -\nabla P + \frac{\rho_n}{\rho_s}\mathbf{F_{mf}} \tag{4}$$

Here Re is the normal fluid Reynolds number defined as $Re = \dfrac{LV_0}{\eta_n}$. L is the channel half width and V_0 is the centre line velocity of the normal fluid, and are used as the length and velocity scales for non-dimensionalizing. η_n is the kinematic viscosity of the normal fluid defined using ρ_n. Pressure is non-dimensionalized using $\rho {V_0}^2$. $\mathbf{F_{mf}}$ is the is the non-dimensionalized mutual friction force per unit mass of the normal fluid given by (Hall & Vinen, 1956*a*,*b*; Godfrey *et al.*, 2001),

$$\mathbf{F_{mf}} = \left(\frac{B\rho_s L}{\rho V_0}\right)\hat{\mathbf{\Omega}}_\mathbf{s} \times [\mathbf{\Omega_s} \times (\mathbf{V_n} - \mathbf{V_s})] + \left(\frac{B'\rho_s L}{\rho V_0}\right)\mathbf{\Omega_s} \times (\mathbf{V_n} - \mathbf{V_s}) \tag{5}$$

where B and B' are the mutual friction coefficients and $\mathbf{\Omega_s}$ is the vorticity vector of quantized superfluid vortex lines and $\hat{\mathbf{\Omega}}_\mathbf{s}$ is the unit vector in direction of superfluid vorticity. The first term gives the mutual friction force in streamwise direction and the second term gives that of transverse direction. The complex mutual friction term given above is approximated as given by Godfrey *et al.* (2001) as,

$$\mathbf{F_{mf}} = f(z)[\mathbf{V_n} - \mathbf{V_s}] \tag{6}$$

where $f(z)$ is a coefficient to account for the spatial dependence for mutual friction, which can be regarded as the distribution of superfluid vortex lines along the wall-normal or transverse direction, **z**, and is given by,

$$f(z) = f_{max}\{exp[-(z - z_-)^2/2\sigma^2] + exp[-(z - z_+)^2/2\sigma^2]\} \tag{7}$$

which gives a Gaussian distribution centered at points where superfluid and normal fluid velocities are equal, where f_{max} is the temperature dependent coefficient, $z_\pm = \pm 1/\sqrt{3}$ and σ is the standard deviation of the distribution. The velocity profile for normal fluid is calculated numerically from the modified Navier-Stokes equation (Equation 3), assuming steady flow and $\mathbf{V_n}$ has only transverse dependence ($\mathbf{V_n} = U_n(z)\hat{\mathbf{x}}$) with no slip boundary condition at the walls, Figure 1. A uniform velocity profile with same volume flow rate as normal fluid is assumed for superfluid component.

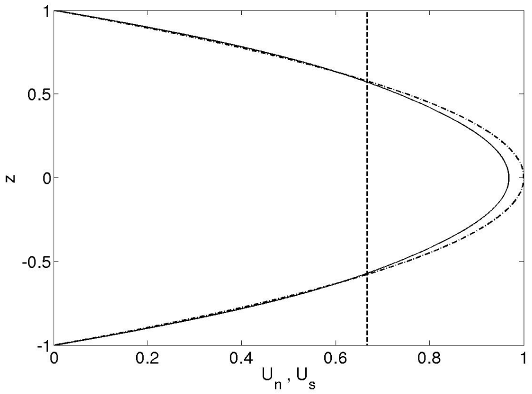

Figure 1. Sample velocity profiles of normal and superfluid. —— velocity profile for normal fluid with $f_{max} = 0.03$ and $\sigma = 0.15$, -·-·- velocity profile for classical fluids. ----- uniform profile for superfluid.

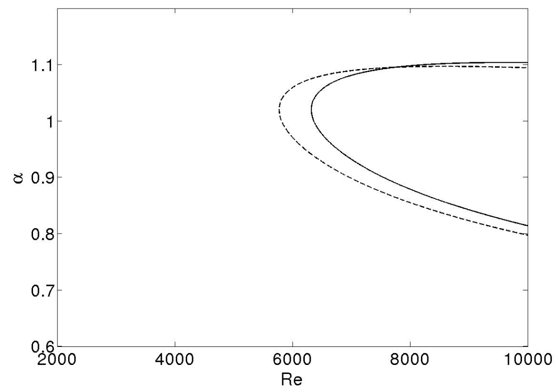

Figure 2. Neutral stability curve for normal fluid with $f_{max} = 0.005$, $\sigma = 0.15$. ----- line represent the neutral stability for classical fluids.

3. Results and Discussion

3.1. Stability of Normal Fluid

The stability of normal fluid is studied here by adding perturbations, keeping superfluid undisturbed, and observing the subsequent evolution of those perturbations. Following the standard procedure for linear stability analysis as in Schmid & Henningson (2001), modified Orr-Sommerfeld and Squire's equations can be written in matrix form as,

$$-i\omega\begin{pmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} + \begin{pmatrix} L_{OS} & 0 \\ i\beta U_n' & L_{SQ} \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0 \tag{8}$$

where

$$L_{OS} = i\alpha U_n(k^2 - D^2) + i\alpha U_n'' + \frac{1}{Re}(k^2 - D^2)^2 - [f'(z)D + f(z)D^2 - k^2 f(z)] \tag{9}$$

$$L_{SQ} = i\alpha U_n + \frac{1}{Re}(k^2 - D^2) + f(z) \tag{10}$$

where $v' = \tilde{v}(z)e^{i(\alpha x + \beta y - \omega t)}$ and $\eta' = \tilde{\eta}(z)e^{i(\alpha x + \beta y - \omega t)}$ are the normal perturbation velocity and vorticity respectively, α & β are the wave numbers in x and y directions, $k^2 = \alpha^2 + \beta^2$, $\omega = c\alpha$ where c is the complex wave speed whose real part c_r represents the phase speed of perturbation and imaginary part multiplied by streamwise wave number, αc_i represents the growth or decay of perturbations and D stands for differentiation in z direction $\left(\frac{\partial}{\partial z}\right)$. Here x, y, z axes represents streamwise, spanwise and wall normal directions respectively. Equation (8), an eigenvalue problem, is solved with the boundary conditions $\tilde{v} = D\tilde{v} = \tilde{\eta} = 0$, using Chebyshev spectral collocation method. The above linear stability results gives neutral stability curve similar to those of classical fluids, but the critical Reynolds number being shifted to higher values, showing that the mutual friction is stabilizing the normal fluid flow, Figure 2. A Tollmien-Schlichting(TS) mode of instability is present but it requires higher Reynolds number to trigger the first TS mode. Our stability results are different from that of Godfrey *et al.* (2001), in which they had two unstable regions, a top neutral curve similar to classical fluids and a bottom curve showing low wave number instability. Mutual friction was reported to destabilize the normal fluid flow. In their formulation they have added the mutual friction term $\mathbf{F_{mf}}$ to the R.H.S. of momentum equation for normal fluid, while we have subtracted $\mathbf{F_{mf}}$ from the R.H.S. of normal fluid equation. We have followed formulations in Tilley & Tilley (2005), and this seems to represent the physics better because whenever normal fluid velocity is greater than superfluid velocity, mutual friction force will have a decelerating effect on normal fluid, Hall & Vinen (1956*a*,*b*).

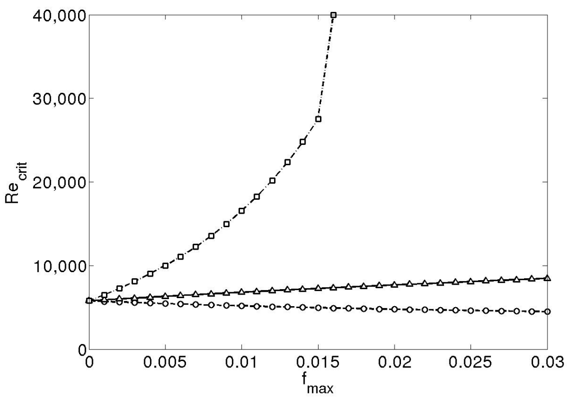

Figure 3. Variation of Critical Reynolds number with f_{max} for different values of σ. ○ for σ = 0.1, △ for σ = 0.15, □ for σ = 0.25.

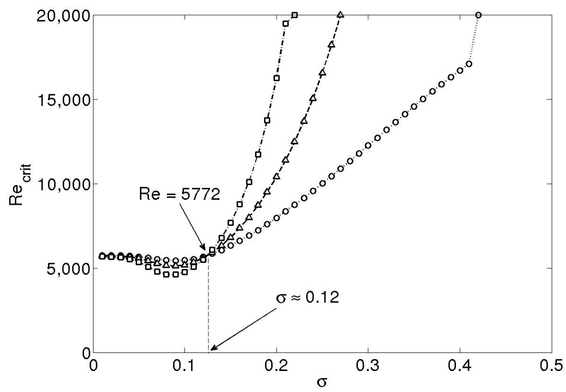

Figure 4. Variation of Re_{crit} with standard deviation of superfluid vortex line distribution, σ for different f_{max}. ○ for f_{max} = 0.005, △ for f_{max} = 0.1, □ for f_{max} = 0.2.

The variation of critical Reynolds number with f_{max} for different σ is plotted in Figure 3. It shows that increasing f_{max} has a stabilizing effect for σ values 0.12 and higher, but has a destabilizing effect for $\sigma = 0.1$. It may be infered that the mutual friction have a destabilizing effect on the normal fluid only for a very localized superfluid vortex line distribution. Note that $Re_{crit} \approx 5772$ at $f_{max} = 0$, which gives a validation of the code that we have used. The variation of Re_{crit} with σ is plotted in Figure 4. The plot shows that at $\sigma \approx 0.12$ the mutual friction has no effect on the stability of normal fluid, for σ>1.2 mutual friction has stabilizing effect and for σ<0.12 mutual friction destabilizes the flow.

Now, as in the case of classical fluids we can expect modified stability operator to be non-normal, due to which perturbations can have amplification of the order of 10^5 (for Re just below Re_{crit}) by a linear mechanism even though all the eigenvalues decay asymptotically and this intermediate growth can cause a sub-critical transition to turbulence (Schmid & Henningson, 2001). We have conducted a non-modal analysis on normal fluid equations to check for a possible transient growth and sub-critical transition to turbulence.

The above eigenvalue problem (8) can be written using vector notation as,

$$\mathbf{L}\tilde{\mathbf{q}} = i\omega\tilde{\mathbf{q}} \tag{12}$$

where

$$\tilde{\mathbf{q}} = \begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \times \begin{pmatrix} L_{OS} & 0 \\ i\beta U' & L_{SQ} \end{pmatrix} \tag{13}$$

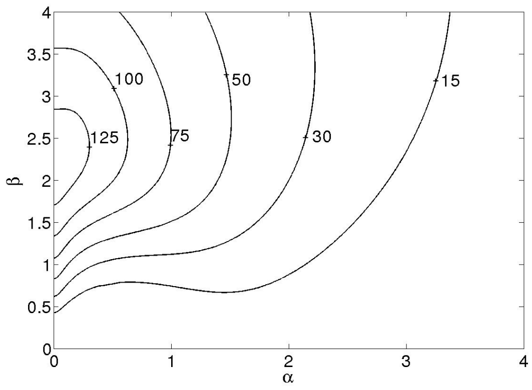

Figure 5. Contours of G_{max} with $Re = 1000$, $f_{max} = 0.005$ and standard deviation of superfluid vortex line distribution, $\sigma = 0.15$.

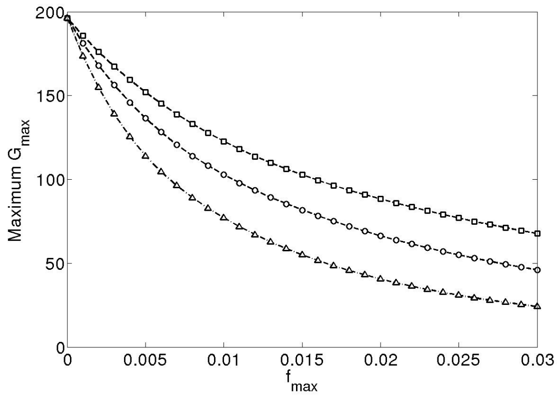

Figure 6. Maximum G_{max} plotted against f_{max}, for $Re = 1000$ and different σ. □ for $\sigma = 0.1$, ○ for $\sigma = 0.15$, △ for $\sigma = 0.25$.

Since **L** is a linear operator Equation (12) is valid for any disturbance **q**, which is a linear combination of eigen vectors $\tilde{\mathbf{q}}$. The eigenvalue problem is actually a simplification of the initial value problem

$$\frac{\partial \mathbf{q}}{\partial t} = \mathbf{L}\mathbf{q} \tag{14}$$

with solutions of the form $\mathbf{q} = \tilde{\mathbf{q}} \times e^{-i\omega t}$. We define the maximum possible amplification, $G(t)$, of an initial disturbance $\mathbf{q_0}$, which is the optimized one over all possible initial conditions, as

$$G(t) = \max_{\mathbf{q_0} \neq 0} \frac{\mathbf{q}(t)^2}{\mathbf{q_0}^2} \tag{15}$$

The maximum growth rate function, G_{max}, is defined as largest possible energy amplification for all times. The contours of maximum growth rate, G_{max}, in α-β plane is shown in Figure 5. The figure shows that normal fluid has a transient growth of perturbations at sub-critical Reynolds numbers, but the growth rate is less compared to classical fluids due to mutual friction. As in the case of classical fluids maximum value of G_{max} occurs at $\alpha = 0$ and $\beta \approx 2$, showing that stream wise vortices have higher transient growth and the possible cause for sub-critical turbulence. The effect of f_{max} and σ on maximum G_{max} are shown in Figures 6 and 7. The above results shows a stabilizing effect for mutual friction in the case of normal fluid. In normal mode analysis f_{max} and σ were found to have a stabilizing effect. The values of σ and f_{max} have significant effects on the stability results.

3.2. Stability of Superfluid

This section deals with the instability of superfluid, by adding perturbations to superfluid keeping normal fluid undisturbed to see if superfluid can trigger transition. Modified Rayleigh and Squire's equations for superfluid can be written in matrix form as,

$$-i\omega \begin{pmatrix} k^2 - D^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{v}_s \\ \tilde{\eta}_s \end{pmatrix} + \begin{pmatrix} L_R & 0 \\ i\beta U' & L_{SQ} \end{pmatrix} \begin{pmatrix} \tilde{v}_s \\ \tilde{\eta}_s \end{pmatrix} = 0 \tag{16}$$

where

$$L_R = i\alpha U_s(k^2 - D^2) + i\alpha U_s'' - [f'(z)D + f(z)D^2 - k^2 f(z)] \tag{17}$$

$$L_{SQ} = i\alpha U_s + f(z) \tag{18}$$

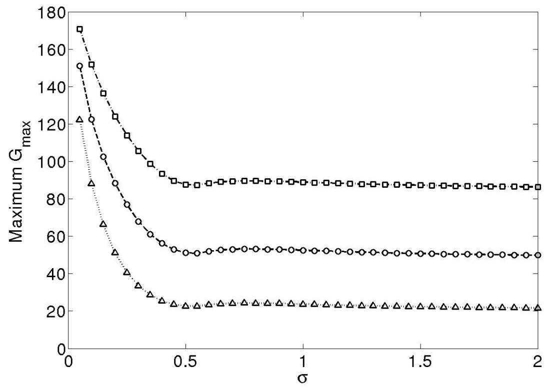

Figure 7. Variation of maximum G_{max} with σ for Re = 1000 and different f_{max}. □ for f_{max} = 0.005, ○ for f_{max} = 0.1, △ for f_{max} = 0.2.

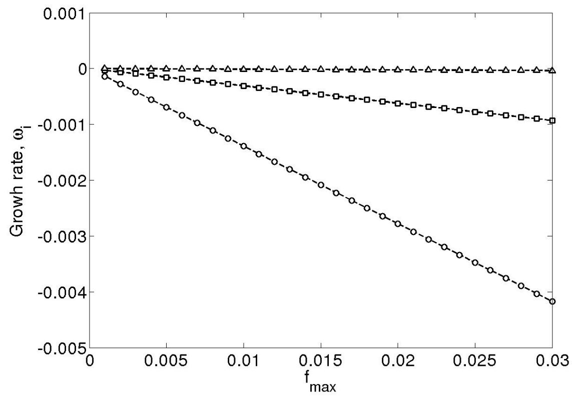

Figure 8. Variation of maximum growth rate, ω_i with f_{max} for different σ. ○ for σ = 0.25, □ for σ = 0.2, △ for σ = 0.15.

where we have assumed $\frac{\rho_n}{\rho_s} = 1$. Linear modal and non-modal analysis is carried out as in section 3.1 with vanishing perturbations as boundary conditions. Here the base flow velocity of superfluid is assumed to be uniform. A uniform profile for inviscid channel flow is neutrally stable for all wave numbers. As we add mutual friction, perturbations start decaying which indicates mutual friction has a stabilizing effect on superfluid. Owing to the possibility of transient growth due to non-normality of eigenfunctions of stability operator as in classical fluids, a non-modal analysis for superfluid is performed. The results do not show any non-normal behaviour of eigenfunctions and the perturbations are found not to have any transient growth. Thus linear modal and non-modal analysis predicts the stability of superfluid with a uniform base flow profile for all wave numbers of perturbation. Experimental results in Baehr *et al.* (1983); Baehr & Tough (1984) shows that dissipationless flow of superfluid breaks down at some critical velocity and this has been attributed to the onset of turbulence in superfluid. Therefore, it is plausible that a linear instability mechanism is absent in superfluid and transition to turbulence in superfluid is different from classical fluids. However, the assumption of uniform base flow profile may not be valid in the case of superfluid and has to be reviewed. A more convincing argument may be to compute the base flow profile numerically from superfluid equation (Equation 4), similar to the case of normal fluid (Equation 3).

4. Conclusion

The effect mutual friction is shown to have a stabilizing effect on both normal and superfluid components. Tough (1982), showed that two different regimes of turbulence, T1 and T2 states, exists in flow of He-II in circular pipes and this has been attributed to the onset of turbulence in superfluid and normal fluid respectively, at different critical velocities. Our study on normal fluid reveals that not only a TS mode of instability but also a sub-critical transition to turbulence due to the non-normal behaviour of eigen vectors is possible. It is also showed that a low wave number instability as proposed by Godfrey *et al.* (2001) is not found in the present computation. The superfluid which is known to become turbulent, is stable for all wave numbers in our analysis, showing the absence of modal and non modal mechanisms of instability in superfluid. This stable behaviour of superfluid equations is mainly because of the uniform inviscid base flow profile which eliminates the terms containing first and second order derivatives of base flow in the stability equations. The presence of turbulence in superfluid brings in a possibility that uniform base flow is getting altered due to the action of mutual friction and this altered profile becomes unstable. The altered profile has to be numerically calculated from the equation for superfluid with steady and parallel flow assumptions. However, this will end up contradicting the earlier assumption of both fluid with same volume flow rate. Consequently, the base flow equations need to be modified suitably to apply it for plane Poiseuille coflow case, where mass and entropy flow are in the same direction.

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