International Journal of Solids and Structures 63 (2015) 289–297

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Identifying hyperelastic and isotropic materials by examining the variation of principal direction of left Cauchy–Green deformation tensor in uniaxial loading ☆

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ARTICLE INFO

Article history:
Received 30 December 2013
Received in revised form 3 March 2015
Available online 24 March 2015

Keywords:
Inhomogeneous body
Isotropy
Transverse isotropy
Elasticity

ABSTRACT

Many bodies of biological and engineering interest have a fibrous and layered structure. Hence, it is believed that these bodies are inhomogeneous and made up of anisotropic material. Classical mechanical experiments used to find the required material symmetry in the constitutive relation cannot distinguish inhomogeneous bodies made of isotropic material and homogeneous bodies made of anisotropic material. Therefore, it is of interest to find an alternative hypothesis so that inhomogeneity and anisotropy can be determined independent of the other. This study finds that the principal (or eigen) direction of the left Cauchy–Green deformation tensor, **B** does not vary with the magnitude of the applied uniaxial load at a given location whenever the body – homogeneous or inhomogeneous – is made of isotropic and hyperelastic material and the deformations are measured from a stress free reference configuration. In general, the principal direction of the left Cauchy–Green deformation tensor varies with the magnitude of the uniaxial load when the body is made up of anisotropic material. Thus, it is concluded that if the variation in the principal direction of **B** with the magnitude of the applied uniaxial load is experimentally investigated then one could ascertain whether the body is made up of isotropic or anisotropic material.

1. Introduction

Many engineering materials have a fibrous microstructure. It is also known that (Fung, 1990, 1993; Holzapfel, 2000) many biological tissues have a layered and a fibrous structure. Hence, it is believed that these materials have to be modeled as inhomogeneous bodies made of anisotropic material (Holzapfel, 2000). However, a systematic methodology using experimental observations from mechanical experiments to decide whether these bodies need to be approximated as homogeneous bodies made of anisotropic material or inhomogeneous bodies made of isotropic material[1] or inhomogeneous bodies made of anisotropic material is absent in the literature.

Traditionally to identify anisotropy from mechanical experiments one examines whether the response of the body under investigation changes with the direction of the applied uniaxial load or by how much the ratio of the applied normal stress in the (say) x and y directions in the equal biaxial experiment differ from 1 (Strumpf et al., 1993). Following Saravanan and Rajagopal (2005), it is known that the change in the response with the direction of the applied load would happen even in case of inhomogeneous bodies made of isotropic material. Also, we show that in case of inhomogeneous bodies made of isotropic material subjected to equal biaxial experiment, the ratio of the nominal stress in the x and y direction would differ from 1, provided the material parameters vary along two directions. Only in case of homogeneous bodies whose deformations are measured from a stress free reference configuration would a change in response with the direction of the applied uniaxial load imply that the body is made of anisotropic material. Thus, the classical mechanical experiment to identify material symmetry assumes that the tested body is homogeneous. While it may be possible to make homogeneous bodies out of man made materials and test them for material symmetry, in case of naturally occurring bodies one can test only what is available. Therefore, it would be of value if experimental

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[1] To be clearer, when we say inhomogeneous body is made of isotropic material, we mean that the inhomogeneous body is made of different constituents which are isotropic.

http://dx.doi.org/10.1016/j.ijsolstr.2015.03.007
0020-7683/

observations can reveal the material symmetry during mechanical testing independent of whether the body being tested is homogeneous or inhomogeneous.

It is known (Truesdell and Noll, 1965) that the symmetry group of the material that a homogeneous body is made up of depends on the configuration in which the body is in. For simple materials Noll's rule (Noll, 1958) tells how the symmetry group changes with the configuration of the homogeneous body. Consistent with the prevalent usage, a material is said to be isotropic if its symmetry group in a configuration of the homogeneous body, probably the undistorted (unstressed) state, coincides with the proper orthogonal group. Similarly, it also known that (Truesdell and Noll, 1965) if there exist a configuration for the entire body such that the mechanical response of any arbitrary subpart from this configuration is identical, then the body is said to be homogeneous. A body that is not homogeneous is said to be inhomogeneous. This definition for homogeneous body allows two distinct classes of inhomogeneous body. Different subparts of a body having different chemical composition and hence mechanical response is one class of inhomogeneous bodies. In another class, all subparts of the body have the same chemical composition but there exist no configuration in which the state of stress in the body would be uniform; in other words, in this case the internal structure variation causes a change in the mechanical response. Example of this class of inhomogeneous bodies where the state of stress is non-uniform in any configuration is residually stressed bodies; bodies which have stresses in the interior even though there is no boundary traction. Of course, a body could be inhomogeneous for both the above reasons, variation in the chemical composition and internal structure.

As discussed above, even though in mechanics the material symmetry and inhomogeneity have a precise meaning, there is a lot of ambiguity in application of these definitions to obtain constitutive relations for engineered and naturally occurring bodies. While some issues related to identifying whether a given body is homogeneous or inhomogeneous is discussed in Saravanan (2014), this article addresses the problems in finding material symmetry.

When one talks about material symmetry in continuum mechanics, one refers to the symmetry that the constitutive relation should have. It is customary to require that the set of rotations of the reference configuration that leaves the mechanical response unchanged should not alter the functional form of the constitutive relation also. There is a conundrum here. While the constitutive relation is for a material point, symmetry restriction on this constitutive relation at a point depends on the unidentifiable rotations of a set of points, the configuration. It is tacitly assumed that the set of points under consideration is materially uniform. Otherwise, a set of rotations would become identifiable due to just the arrangement of the set of points under consideration. Because material symmetry, in continuum mechanics, is the inherent material property of the point under consideration and not the arrangement or the nature of the neighboring points, the requirement of the material uniformity arises. Further, the framework of continuum mechanics allows one to have homogeneous body made of anisotropic material or inhomogeneous body made of isotropic material.

Alternatively for some, material symmetry is related to the materials internal structure. Material symmetry, in this point of view, is the set of rotations of the body that leave its internal structure unaltered. In this definition of material symmetry based on its internal structure, homogeneous body made of anisotropic material or inhomogeneous body made of isotropic material is not possible. Also, in this case, the material symmetry is not a concept associated with a material point in the body.

As articulated by Lekhnitskii (1981), it is necessary to distinguish the symmetry in the constitutive relation versus the symmetry of the material based on its internal structure. Lekhnitskii (1981) sights Neumann (1885) for the assumption that the symmetry in the constitutive relation to be not inferior to that of the symmetry in the crystallographic structure. That is the set of rotations that form the symmetry group for a given crystallographic structure should always be contained in the set of rotations for which the constitutive relation remains invariant. Lekhnitskii (1981) then states that this assumption is extended to bodies that are not made of crystals but still have an internal structure like wood, glass fiber reinforced plastics. However, to the knowledge of the authors there has been no experimental validation of this assumption of the symmetry in the constitutive relation be not inferior to the symmetry in the internal structure. Hence, here an hypothesis using which the symmetry requirement of the constitutive relation could be established is sought.

It is found that when a body is made of isotropic material and is deforming from its undistorted state in a non-dissipative manner, then the principal direction of the left Cauchy–Green deformation tensor does not change with the magnitude of the applied uniaxial load. Further, it is also inferred that if any of the principal directions of the stress tensor does not coincide with the fiber direction then the principal direction of the left Cauchy–Green deformation tensor changes with the magnitude of the applied uniaxial stress. Therefore, if the principal direction of the left Cauchy–Green deformation tensor does not change with the magnitude of the applied uniaxial load while the body is being subjected to a non-dissipative process, for two different directions of the applied uniaxial load which are not orthogonal to each other, then the tested material could be inferred as being isotropic.

Homogenization procedures used to generate homogeneous constitutive relations for inhomogeneous bodies made of isotropic materials result in anisotropic constitutive relations (Nemat-Nasser and Hori, 1993). Therefore, it would be of interest to examine the quality of approximating inhomogeneous bodies made of isotropic materials with homogeneous but anisotropic constitutive relations. This study shows that the way in which the principal direction of the left Cauchy–Green deformation tensor vary with the magnitude of the applied uniaxial load depends on the material symmetry of the constitutive relation. Thus, if the direction in which the maximum stress and/or change in length occurs could not be predicted by these homogeneous anisotropic models for inhomogeneous bodies made of isotropic material, the engineering usefulness of these homogenization procedures is limited.

This article is organized in four sections including this introduction. In Section 2 the notations and well established relationships are documented for further reference. Then, in Section 3, it is established that the principal direction of left Cauchy–Green deformation tensor measured in an inhomogeneous body made of isotropic material would not change with the magnitude of the applied uniaxial load as long as it deforms in a non-dissipative manner and the deformation is measured from a stress free reference configuration. On the other hand, a body made of transversely anisotropic material the principal direction of left Cauchy–Green deformation tensor does change with the magnitude of uniaxial stress, provided that any of the principal directions of the stress does not coincide with the fiber direction. Finally, issues in testing this hypothesis experimentally is discussed.

2. Preliminaries

Let $\mathbf{X}$ denote the position vector of a typical particle belonging to the reference configuration of the body. Similarly, let $\mathbf{x}$ denote the position vector of the same particle in the current configuration. The deformation field of the body is defined through a one to one mapping $\boldsymbol{\chi}$ that tells the current position of the particle that occupied the point whose position vector is $\mathbf{X}$ in the reference configuration,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}). \tag{1}$$

Then, the deformation gradient, $\mathbf{F}$ is defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}. \tag{2}$$

The left and right Cauchy–Green deformation tensor, $\mathbf{B}$ and $\mathbf{C}$ respectively, is defined as:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^t, \quad \mathbf{C} = \mathbf{F}^t\mathbf{F}, \tag{3}$$

where the superscript t denotes the transpose operator. $\mathbf{C}$ being positive definite, we find that using the following set of invariants

$$J_1 = tr(\mathbf{C}), \quad J_2 = tr(\mathbf{C}^{-1}), \quad J_3 = \sqrt{det(\mathbf{C})} = det(\mathbf{F}), \tag{4}$$

minimizes the complexity of the ensuing analysis.

Here we study the case wherein the constitutive relation for Cauchy stress, $\boldsymbol{\sigma}$ in an isotropic material deforming from a stress free reference configuration is given by (see for example Truesdell and Noll, 1965):

$$\boldsymbol{\sigma} = \begin{cases} \alpha_0\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}, & \text{for compressible materials} \\ -p\mathbf{1} + \beta_1\mathbf{B} + \beta_2\mathbf{B}^{-1}, & \text{for incompressible materials} \end{cases}, \tag{5}$$

where $\alpha_i = \hat{\alpha}_i(J_1, J_2, J_3)$ and $\beta_i = \hat{\beta}_i(J_1, J_2)$ are material response functions and p is the Lagrange multiplier used to enforce the incompressibility condition that needs to be found such that the equilibrium equations and boundary conditions are satisfied.

In case of transversely isotropic materials with fibers running along the direction $\mathbf{M}$ in the stress free reference configuration, the constitutive relation for Cauchy stress takes a form (see for example Holzapfel, 2000):

$$\boldsymbol{\sigma} = \alpha_0^a\mathbf{1} + \alpha_1^a\mathbf{B} + \alpha_2^a\mathbf{B}^{-1} + \alpha_3^a\mathbf{F}\mathbf{M}\otimes\mathbf{M}\mathbf{F}^t + \alpha_4^a\mathbf{F}[\mathbf{C}\mathbf{M}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{C}\mathbf{M}]\mathbf{F}^t, \tag{6}$$

for compressible materials and

$$\boldsymbol{\sigma} = -p\mathbf{1} + \beta_1^a\mathbf{B} + \beta_2^a\mathbf{B}^{-1} + \beta_3^a\mathbf{F}\mathbf{M}\otimes\mathbf{M}\mathbf{F}^t + \beta_4^a\mathbf{F}[\mathbf{C}\mathbf{M}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{C}\mathbf{M}]\mathbf{F}^t, \tag{7}$$

for incompressible materials, where $\alpha_i^a = \hat{\alpha}_i^a(J_1, J_2, J_3, J_4, J_5)$ and $\beta_i^a = \hat{\beta}_i^a(J_1, J_2, J_4, J_5)$ are material response functions,

$$J_4 = \mathbf{M}\cdot\mathbf{C}\mathbf{M}, \quad J_5 = \mathbf{M}\cdot\mathbf{C}^2\mathbf{M}, \tag{8}$$

and p as before is the Lagrange multiplier. Note that, the constitutive relations (6) and (7) assumes that the deformation gradient is computed from a stress free reference configuration.

If the constitutive relation for Cauchy stress in an isotropic material deforming from a stress free reference configuration is given by (5), then the constitutive relation for the same material deforming from a stressed reference configuration can be written as (see Saravanan, 2011 for details):

$$\boldsymbol{\sigma} = \begin{cases} \alpha_0^r\mathbf{1} + \alpha_1^r\mathbf{F}\mathbf{B}_o\mathbf{F}^t + \alpha_2^r\mathbf{F}^{-t}\mathbf{B}_o^{-1}\mathbf{F}^{-1}, & \text{for compressible materials} \\ -p\mathbf{1} + \beta_1^r\mathbf{F}\bar{\mathbf{B}}_o\mathbf{F}^t + \beta_2^r\mathbf{F}^{-t}\bar{\mathbf{B}}_o^{-1}\mathbf{F}^{-1}, & \text{for incompressible materials} \end{cases}, \tag{9}$$

where as before p is the Lagrange multiplier used to enforce the incompressibility condition, $\alpha_i^r = \hat{\alpha}_i^r(J_{m1}, J_{m2}, J_{m3})$ and $\beta_i^r = \hat{\beta}_i^r(\bar{J}_{m1}, \bar{J}_{m2})$,

$$\mathbf{B}_o = \delta_0\mathbf{1} + \delta_1\boldsymbol{\sigma}^o + \delta_2(\boldsymbol{\sigma}^o)^2, \quad \mathbf{B}_o^{-1} = \kappa_0\mathbf{1} + \kappa_1\boldsymbol{\sigma}^o + \kappa_2(\boldsymbol{\sigma}^o)^2, \tag{10}$$

$$\bar{\mathbf{B}}_o = \bar{\delta}_0\mathbf{1} + \bar{\delta}_1\boldsymbol{\sigma}^o + \bar{\delta}_2(\boldsymbol{\sigma}^o)^2, \quad \bar{\mathbf{B}}_o^{-1} = \bar{\kappa}_0\mathbf{1} + \bar{\kappa}_1\boldsymbol{\sigma}^o + \bar{\kappa}_2(\boldsymbol{\sigma}^o)^2, \tag{11}$$

$\boldsymbol{\sigma}^o$ is the Cauchy stress in the reference configuration,

$$J_{m1} = \delta_0 J_1 + \delta_1\mathbf{C}\cdot\boldsymbol{\sigma}^o + \delta_2\mathbf{C}\cdot(\boldsymbol{\sigma}^o)^2, \tag{12}$$

$$J_{m2} = \kappa_0 J_2 + \kappa_1\mathbf{C}^{-1}\cdot\boldsymbol{\sigma}^o + \kappa_2\mathbf{C}^{-1}\cdot(\boldsymbol{\sigma}^o)^2, \tag{13}$$

$$J_{m3} = J_3^r J_3, \tag{14}$$

$$\bar{J}_{m1} = \bar{\delta}_0 J_1 + \bar{\delta}_1\mathbf{C}\cdot\boldsymbol{\sigma}^o + \bar{\delta}_2\mathbf{C}\cdot(\boldsymbol{\sigma}^o)^2, \tag{15}$$

$$\bar{J}_{m2} = \bar{\kappa}_0 J_2 + \bar{\kappa}_1\mathbf{C}^{-1}\cdot\boldsymbol{\sigma}^o + \bar{\kappa}_2\mathbf{C}^{-1}\cdot(\boldsymbol{\sigma}^o)^2, \tag{16}$$

and $J_3^r, \delta_i, \bar{\delta}_i, \kappa_i$ and $\bar{\kappa}_i$ are functions of the principal invariants of $\boldsymbol{\sigma}^o$ obtained from the requirements that $\mathbf{B}_o\mathbf{B}_o^{-1} = \mathbf{1}$ and that $\boldsymbol{\sigma}(\mathbf{1}, \boldsymbol{\sigma}^o) = \boldsymbol{\sigma}_o$ refer to Saravanan (2011) for details.

3. Theoretical observations

Theoretically the following could be observed when a body is tested in static conditions in the absence of body forces:

1. The principal direction of left Cauchy Green deformation tensor in a homogeneous body deforming from a stress free reference configuration and made of transversely isotropic material varies with the magnitude of the applied uniaxial load when the direction of the applied uniaxial load is not along or perpendicular to the direction of the fibers.
2. The principal direction of left Cauchy Green deformation tensor in an inhomogeneous body deforming from a stress free reference configuration and made of isotropic material does not vary with the magnitude of the applied uniaxial load.
3. The principal direction of left Cauchy Green deformation tensor in a body deforming from a stressed reference configuration and made of isotropic material varies with the magnitude of the applied uniaxial load.
4. In an inhomogeneous body made of isotropic material, the ratio of the effective stress along the x and y direction in equal biaxial experiment would differ from 1 as long as the material functions depend on the loading directions coordinates apart from the invariants of $\mathbf{C}$, right Cauchy Green deformation tensor.

Now, we provide evidence for the above statements.

3.1. Response of homogeneous body made of transversely isotropic material

The response of homogeneous cuboid made of transversely isotropic and incompressible material subjected to uniaxial load is studied. Here we note that a general constitutive relation for this class of materials (7) can be written as,

$$\boldsymbol{\sigma} = -p\mathbf{1} + \beta_1^a\mathbf{F}\mathbf{A}\mathbf{F}^t + \beta_2^a\mathbf{B}^{-1}, \tag{17}$$

where,

$$\mathbf{A} = \mathbf{1} + \frac{\beta_3^a}{\beta_1^a}\mathbf{M}\otimes\mathbf{M} + \frac{\beta_4^a}{\beta_1^a}[\mathbf{C}\mathbf{M}\otimes\mathbf{M} + \mathbf{M}\otimes\mathbf{C}\mathbf{M}]. \tag{18}$$

Towards studying the variation in the principal direction of left Cauchy–Green deformation tensor as a function of the fiber direction and magnitude of the uniaxial load, it is assumed that the uniaxial loading is along $\mathbf{e}_x$, as shown in Fig. 1. Without loss of generality, the fibers are assumed to be oriented in the plane of loading such that the fiber direction is given as:

$$\mathbf{M} = \cos(\theta)\mathbf{e}_x + \sin(\theta)\mathbf{e}_y, \tag{19}$$

where θ is a constant taking values between 0 and 90 degrees.

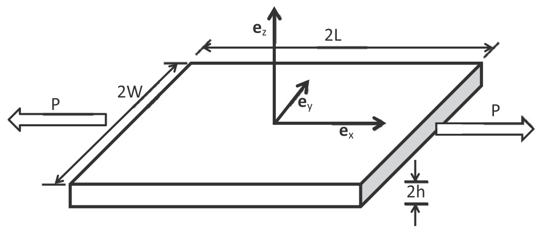


Fig. 1. Schematic of uniaxial loading of a thin sheet.

Due to this applied uniaxial load, the deformation of the homogeneous, transversely isotropic cuboid is given by

$$x = \lambda_1 X + \kappa_1 Y, \quad y = \lambda_2 Y, \quad z = \lambda_3 Z, \tag{20}$$

where (X, Y, Z) are the Cartesian coordinates of a typical material particle before the application of the load, (x, y, z) the Cartesian coordinates of the same material particle after the application of the load, κ_1 and λ_i's are constants. The deformation gradient, $\mathbf{F}$ corresponding to the deformation field (20) is computed to be

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & \kappa_1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{21}$$

The incompressibility condition would require that $\lambda_3 = 1/(\lambda_1\lambda_2)$. Substituting (21) and (19) in (18) we find the Cartesian components of $\mathbf{A}$ as,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\beta_3^a}{\beta_1^a} \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) & 0 \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$+ \frac{\beta_4^a}{\beta_1^a} \begin{pmatrix} 2\lambda_1^2\cos^2(\theta) + \lambda_1\kappa_1\sin(2\theta) & \left[\lambda_1^2 + \lambda_2^2 + \kappa_1^2\right]\sin(2\theta)/2 + \lambda_1\kappa_1 & 0 \\ \left[\lambda_1^2 + \lambda_2^2 + \kappa_1^2\right]\sin(2\theta)/2 + \lambda_1\kappa_1 & \lambda_1\kappa_1\sin(2\theta) + 2\sin^2(\theta)[\lambda_2^2 + \kappa_1^2] & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

The Cartesian components of the Cauchy stress corresponding to the applied uniaxial load along the x direction is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Substituting (22) and (21) in (17) and equating the resulting state of stress to be as given in (23), the following equations have to hold:

$$\sigma = -p + \lambda_1^2\{\beta_1^a + \beta_3^a\cos^2(\theta) + \beta_4^a[2\lambda_1^2\cos^2(\theta) + \lambda_1\kappa_1\sin(2\theta)]\}$$
$$+ \kappa_1^2\{\beta_3^a\sin(2\theta) + \beta_4^a[\lambda_1\kappa_1 + (\lambda_1^2 + \lambda_2^2 + \kappa_1^2)\sin(2\theta)]\} + \frac{\beta_2^a}{\lambda_1^2}, \tag{24}$$

$$0 = \kappa_1\lambda_2\left\{\beta_1^a + \beta_3^a\sin^2(\theta) + \beta_4^a[2(\lambda_2^2 + \kappa_1^2)\sin^2(\theta) + \lambda_1\kappa_1\sin(2\theta)]\right\}$$
$$+ \lambda_1\lambda_2\{\beta_3^a\sin(2\theta) + \beta_4^a[\lambda_1\kappa_1 + (\lambda_1^2 + \lambda_2^2 + \kappa_1^2)\sin(2\theta)]\} + \beta_2\frac{\kappa_1}{\lambda_1^2\lambda_2}, \tag{25}$$

$$0 = -p + \lambda_2^2\left\{\beta_1^a + \beta_3^a\sin^2(\theta) + \beta_4^a[2(\lambda_2^2 + \kappa_1^2)\sin^2(\theta) + \lambda_1\kappa_1\sin(2\theta)]\right\}$$
$$+ \frac{\beta_2^a}{\lambda_2^2}, \tag{26}$$

where

$$p = \frac{\beta_1^a}{(\lambda_1\lambda_2)^2} + \beta_2^a(\lambda_1\lambda_2)^2, \tag{27}$$

obtained from the requirement that the out of plane normal stress, $\sigma_{zz} = 0$ and $\beta_i^a = \hat{\beta}_i^a(J_1, J_2, J_4, J_5)$,

$$J_1 = \lambda_1^2 + \lambda_2^2 + \kappa_1^2 + \left(\frac{1}{\lambda_1\lambda_2}\right)^2, \quad J_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + (\lambda_1\lambda_2)^2 + \left(\frac{\kappa_1}{\lambda_1\lambda_2}\right)^2,$$

$$J_4 = (\lambda_1\cos(\theta))^2 + (\lambda_2\sin(\theta))^2 + \lambda_1\kappa_1\sin(2\theta) + (\kappa_1\sin(\theta))^2,$$

$$J_5 = \lambda_1^4\cos^2(\theta) + (\lambda_2^2 + \kappa_1^2)^2\sin^2(\theta) + \lambda_1\kappa_1(\lambda_1^2 + \lambda_2^2 + \kappa_1^2)\sin(2\theta) + (\lambda_1\kappa_1)^2. \tag{28}$$

For a given value of λ_1 Eqs. (25) and (26) are solved simultaneously to obtain κ_1 and λ_2 respectively. Then, the uniaxial stress required to realize the given value of λ_1 is determined using Eq. (24). In this work, *fsolve*, a built-in function in MATLAB is used to simultaneously solve the Eqs. (25) and (26). Then, using this determined value of λ_2 and κ_1 for a given value of λ_1, the principal direction of the left Cauchy–Green deformation tensor, ϕ, is computed as,

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2B_{xy}}{B_{xx} - B_{yy}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2\lambda_2\kappa_1}{\lambda_1^2 + \kappa_1^2 - \lambda_2^2}\right). \tag{29}$$

Here it is pertinent to note that in this work the principal direction of a tensor is defined as the angle made by the direction along which the maximum principal value occurs with respect to the x direction along which the components of the tensor is written.

First, let us consider the case wherein the fibers are oriented along the loading direction, that is, when $\theta = 0$ degrees. Eqs. (25) and (26) simplifies to

$$0 = \kappa_1\lambda_2\left[\beta_1^a + \frac{\beta_2^a}{\lambda_1^2\lambda_2^2}\right], \quad 0 = \left[(\lambda_1\lambda_2)^2 - \frac{1}{\lambda_2^2}\right]\left[\frac{\beta_1^a}{\lambda_1^2} - \beta_2^a\right], \tag{30}$$

when $\theta = 0$. $\kappa_1 = 0$ and $\lambda_2 = 1/\sqrt{\lambda_1}$ is a possible solution to the above Eq. (30) irrespective of the choice of the material response functions. Substituting this solution in (29), $\phi = 0$. Thus, when $\theta = 0, \phi$ does not change with the magnitude of the uniaxial load. It should be noted that, depending on the choice of the material response function, β_i, there might be other solutions to Eq. (30) than that studied here for which the principal direction of $\mathbf{B}$ vary with the magnitude of the applied uniaxial load. But this is not of interest here.

Next, we study the case wherein the fibers are oriented perpendicular to the loading direction, that is $\theta = 90$ degrees. For this case, Eqs. (24)–(26) simplifies to

$$\sigma = \left[(\lambda_1\lambda_2)^2 - \frac{1}{\lambda_1^2}\right]\left[\frac{\beta_1^a}{\lambda_2^2} - \beta_2^a\right], \tag{31}$$

$$0 = \kappa_1\lambda_2\left\{\beta_1^a + \beta_3^a + 2\beta_4^a(\lambda_2^2 + \kappa_1^2) + \frac{\beta_2^a}{\lambda_1^2\lambda_2^2}\right\}, \tag{32}$$

$$0 = \left[(\lambda_1\lambda_2)^2 - \frac{1}{\lambda_2^2}\right]\left[\frac{\beta_1^a}{\lambda_1^2} - \beta_2^a\right] + \lambda_2^2\{\beta_3^a + 2\beta_4^a(\lambda_2^2 + \kappa_1^2)\}. \tag{33}$$

In this case too, $\kappa_1 = 0$ is a possible solution to the above set of equations irrespective of the choice of the material response functions. Consequently, the principal direction of left Cauchy–Green deformation tensor does not change with the magnitude of the uniaxial load for this case too.

For values of θ other than 0 and 90 degrees, it can be seen from Eq. (25) that κ_1 cannot be zero whenever β_3^a or β_4^a is nonzero. Moreover, the value of κ_1 that satisfies (25) depends on the value of λ_1. Thus, for homogeneous bodies made of transversely isotropic material with the fiber direction not coinciding or being perpendicular to the direction of the applied uniaxial load, the principal direction of the left Cauchy–Green deformation tensor changes with the magnitude of the applied uniaxial load. Towards illustrating this, we pick the constitutive relation proposed by May-Newman and Yin (1998) and Prot et al. (2007) for porcine mitral valve. In the present notation, the May-Newman and Yin (1998) constitutive relation requires,

$$\beta_1^a = 4c_0c_1(J_1-3)\exp\left(c_1(J_1-3)^2+c_2(\sqrt{J_4}-1)^4\right), \quad \beta_2^a=0,$$

$$\beta_3^a = 2c_0c_2\frac{(\sqrt{J_4}-1)^3}{\sqrt{J_4}}\exp\left(c_1(J_1-3)^2+c_2(\sqrt{J_4}-1)^4\right), \ \beta_4^a=0, \quad (34)$$

where c_i's are constants. An alternative constitutive relation proposed by Prot et al. (2007) for the same porcine mitral valve, in the present notation requires that

$$\beta_1^a = 4\bar{c}_0\bar{c}_1(J_1-3)\exp\left(\bar{c}_1(J_1-3)^2+\bar{c}_2(J_4-1)^2\right), \quad \beta_2^a=0,$$

$$\beta_3^a = 2\bar{c}_0\bar{c}_2(J_4-1)\exp\left(\bar{c}_1(J_1-3)^2+\bar{c}_2(J_4-1)^2\right), \quad \beta_4^a=0, \quad (35)$$

where $\bar{c}_i$'s are constants.

Following May-Newman and Yin (1998) and Prot et al. (2007), we assume that the value of constants $c_i, \bar{c}_i$ to be as given in Table 1. This choice of material parameters ensures that the stresses required to realize the 3 deformation fields – equi-biaxial, strip biaxial fixed along the fiber direction, strip biaxial fixed perpendicular to the fiber direction – are the same for both the models (see Prot et al., 2007).

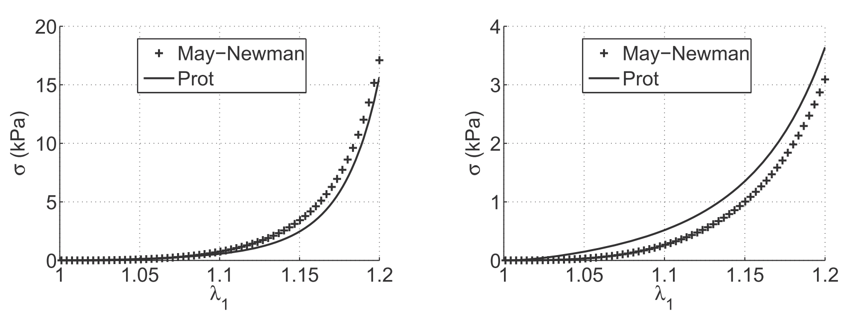
Table 1
Value of the constants in May-Newman and Yin (1998) and Prot et al. (2007) constitutive model.

Identifier	c_0 (kPa)	c_1	c_2	$\bar{c}_0$ (kPa)	$\bar{c}_1$	$\bar{c}_2$
Anterior	0.399	4.325	1446.5	0.052	4.63	22.6
Posterior	0.414	4.848	305.4	0.171	5.28	6.46

Here consistent with the observation of May-Newman and Yin (1998) the value of θ is assumed to be 7 degrees in case of anterior leaflet and 8 degrees in case of posterior leaflet.

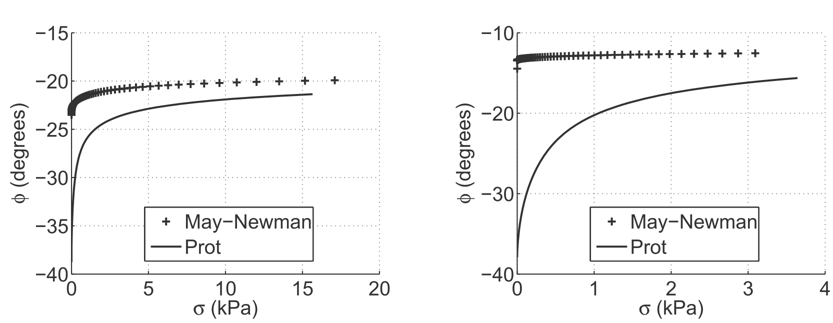
Fig. 2 plots the variation of the uniaxial stress for different values of λ_1 for the two models given by Eqs. (34) and (35). Similarly, Fig. 3 plots the variation of the principal direction of the left Cauchy–Green deformation tensor for different values of λ_1 for the same two models. It can be seen from the Fig. 2 that the uniaxial stress predicted for a given axial stretch, λ_1, by both the models are close, as in the case of strip biaxial, equal biaxial deformation states. However, when one examines the variation in the principal direction of **B** with the axial stretch it is not close. May-Newman and Yin (1998) model predicts less than 4 degree variation in the principal direction when λ_1 is varied between 1 and 1.2. But Prot et al. (2007) model predicts more than 20 degree variation in the principal direction when λ_1 varies between 1 and 1.2. Though these results are not presented here, the Prot et al. (2007) model predicts higher variation in the principal direction with increasing magnitude of the load than the May-Newman and Yin (1998) model in case of other deformation states like strip biaxial and equal biaxial deformations as well.

Thus, we conclude that for transversely isotropic bodies the principal direction of the left Cauchy–Green deformation tensor varies with the magnitude of the applied uniaxial stress when the direction of the uniaxial loading does not coincide with or is perpendicular to the fiber direction. However, for some transversely isotropic models this variation could be minimal, as small as 1 to 2 degrees. We do not look at the absolute value of the principal direction but its change with the magnitude of the applied



(a) For anterior leaflet (b) For posterior leaflet

Fig. 2. Variation of uniaxial stress, σ with λ_1 according to May-Newman and Yin (1998) and Prot et al. (2007) models.



(a) For anterior leaflet (b) For posterior leaflet

Fig. 3. Variation of principal direction of the left Cauchy–Green deformation tensor, ϕ with uniaxial stress σ according to May-Newman and Yin (1998) and Prot et al. (2007) models.

load because of the impossibility of orienting the laboratory coordinate system consistently across specimens with errors less than 1 degree, especially in cases where the deformation field is obtained by tracking some markers placed on the body.

Also, it could be seen that even if the models predict the same stresses to realize a given type of deformation, the predictions on the principal direction of the left Cauchy–Green deformation tensor could differ.

3.2. Variation in the principal direction of B in inhomogeneous body made of isotropic material

Here we study the variation in the principal direction of **B** with the magnitude of the applied uniaxial load for two classes of bodies made of isotropic material. In one case, the body is inhomogeneous and is deforming from a stress free reference configuration. In the second case, the body is deforming from a stressed reference configuration.

For both these cases we study the response to uniaxial loading of a body in the form of a cuboid as shown in Fig. 1. The body occupies the region, $\mathcal{B} = \{(X, Y, Z) | -L \leqslant X \leqslant L, -W \leqslant Y \leqslant W, -h \leqslant Z \leqslant h\}$, where L, W and h are constants. Without loss of generality we assume that the uniaxial load is applied along the x direction.

3.2.1. Case 1: stress free reference configuration

We begin by studying an inhomogeneous body deforming from a stress free reference configuration. For illustration, we shall assume that the body is a thin sheet, with its thickness being much smaller than the other two dimensions, i.e., $h \ll L$ and $h \ll W$. This requirement justifies the assumption that the state of Cauchy stress in the sheet is plane stress and can be represented in the matrix form as,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

Here we assume that $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(X, Y)$ and that σ_{xy} and σ_{yy} are not identically zero over the entire body because the body is inhomogeneous. If the body is not in the form of a thin sheet, there could be out of plane stresses as well which we shall consider subsequently.

Then, the traction boundary conditions for this problem with the uniaxial load, P, acting along the $\mathbf{e}_x$ direction are

$$\sigma_{xy}(\pm L, Y) = \sigma_{xy}(X, \pm W) = \sigma_{yy}(X, \pm W) = 0, \tag{37}$$

$$\int_{-h}^{h} \int_{-W}^{W} [det(\mathbf{F})\boldsymbol{\sigma}\mathbf{F}^{-t}\mathbf{e}_x]|_{(\pm L, Y)} dY dZ = P, \tag{38}$$

where to obtain the Eq. (38) we have appealed to the Nanson's formula (see for example Holzapfel (2000)).

When the state of stress is as given by (36), the principal direction of the Cauchy stress, θ_σ at a given location in the body is:

$$\theta_\sigma = \frac{1}{2} \tan^{-1}\left(\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}\right). \tag{39}$$

The nonzero Cartesian components of the left Cauchy–Green deformation tensor that results in the Cauchy stress being of the form (36) is

$$\mathbf{B} = \begin{pmatrix} B_{xx} & B_{xy} & 0 \\ B_{xy} & B_{yy} & 0 \\ 0 & 0 & B_{zz} \end{pmatrix}. \tag{40}$$

Next, we want to express the components of the Cauchy stress in Eq. (39) in terms of the components of the left Cauchy–Green deformation tensor given in Eq. (40). For this we use the constitutive relation for isotropic material deforming from a stress free configuration, (5) to obtain,

$$\theta_\sigma = \frac{1}{2} \tan^{-1}\left(\frac{2B_{xy}}{B_{xx} - B_{yy}}\right), \tag{41}$$

irrespective of whether the material is compressible or incompressible. Eq. (41) just says that the principal direction of the Cauchy stress and the left Cauchy–Green deformation tensor are the same at a given location. This is expected since the material is assumed to be isotropic. It then follows that if the principal direction of the Cauchy stress does not change with the magnitude of the applied uniaxial load, then the principal direction of the left Cauchy–Green deformation tensor also would not change with the magnitude of the applied load.

We next show that all the components of the stress depend linearly on P, i.e., $\sigma_{ij} = Pf_{ij}(x, y)$, where f_{ij} is a function of x and y only. It then immediately follows from Eq. (39) that the principal direction of stress is independent of the magnitude of the applied uniaxial load.

For materials that obey Hooke's law and undergo small deformations it is well known that $\sigma_{ij} = Pf_{ij}(x, y)$, since principle of superposition holds. Now, we show that this linear dependance of the components of the stress on the applied uniaxial load holds even otherwise.

In the absence of body forces and the body in static equilibrium, the equilibrium equations reduce to requiring, $div(\boldsymbol{\sigma}) = \mathbf{0}$, where $div(\cdot)$ denotes the divergence operator with respect to $\mathbf{x}$. These equilibrium equations for the assumed state of plane stress, (36) can be satisfied if the stress is obtained from Airy's potential, $\phi = \hat{\phi}(x, y)$ such that,

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \tag{42}$$

where x and y are the coordinates of a material particle in the current configuration. Now, one has to find ϕ such that the compatibility conditions are satisfied along with the boundary conditions (37) and (38). First we would like to explore the property of all ϕ which satisfies only the boundary conditions (37) and (38). Since, the boundary conditions (37) and (38) depend only on P, any ϕ satisfying the boundary condition would be a linear function of P. Hence, the Cauchy stress determined using any of these potential, ϕ which satisfies the boundary condition alone would be of the form: $\sigma_{ij} = Pf_{ij}(x, y)$. Therefore, we conclude that the principal direction of the left Cauchy–Green deformation tensor does not change with the magnitude of the applied uniaxial load in a thin sheet.

If the tested body is a thick sheet, then the state of stress need not correspond to that of a plane stress. Even when the state of stress is not plane, but the body is in static equilibrium and there are no body forces, the Cartesian components of the stress can be derived from a potential, $\phi = \hat{\phi}(x, y, z)$ as

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} & -\frac{\partial^2 \phi}{\partial x \partial y} & -\frac{\partial^2 \phi}{\partial x \partial z} \\ -\frac{\partial^2 \phi}{\partial x \partial y} & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} & -\frac{\partial^2 \phi}{\partial y \partial z} \\ -\frac{\partial^2 \phi}{\partial x \partial z} & -\frac{\partial^2 \phi}{\partial y \partial z} & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \end{pmatrix}, \tag{43}$$

so that the equilibrium equations are satisfied. The stress function, ϕ apart from having to satisfy the compatibility condition should satisfy the boundary condition:

$$\sigma_{xy}(\pm L, Y, Z) = \sigma_{xy}(X, \pm W, Z) = 0, \tag{44}$$

$$\sigma_{xz}(\pm L, Y, Z) = \sigma_{xz}(X, Y, \pm h) = 0, \tag{45}$$

$$\sigma_{yz}(X, \pm W, Z) = \sigma_{yz}(X, Y, \pm h) = 0, \tag{46}$$

$$\sigma_{yy}(X, \pm W, Z) = \sigma_{zz}(X, Y, \pm h) = 0, \tag{47}$$

$$\int_{-h}^{h}\int_{-W}^{W}\left[det(\mathbf{F})\boldsymbol{\sigma}\mathbf{F}^{-t}\mathbf{e}_x\right]|_{(\pm L,Y,Z)}dYdZ = P. \tag{48}$$

It then follows that the potential ϕ would be a linear function of P and hence the Cartesian components of the Cauchy stress also would be a such that $\sigma_{ij} = P\hat{f}_{ij}(x, y, z)$.

Now, since the state of stress is not plane, the expression for the principal direction gets complicated, but would still be a ratio of the components of the Cauchy stress tensor. Therefore, if the Cartesian components of the Cauchy stress are linear function of the applied load, the principal direction of the Cauchy stress tensor would not vary with the magnitude of the applied uniaxial load. Further, by virtue of the material being isotropic, the principal direction of the left Cauchy–Green deformation tensor and that of the Cauchy stress tensor would be the same. Hence, the principal direction of the left Cauchy–Green deformation tensor too would not vary with the magnitude of the applied uniaxial load.

3.2.2. Case 2: stressed reference configuration

Now we investigate the deformations of a residually stressed (or prestressed) thin sheet stretched by applying a uniaxial load. For illustration, it suffices to assume that the Cauchy stress field in the traction free reference configuration corresponds to that of a plane stress given by,

$$\boldsymbol{\sigma}^o = \begin{pmatrix} \sigma^o_{XX} & \sigma^o_{XY} & 0 \\ \sigma^o_{XY} & \sigma^o_{YY} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{49}$$

where

$$\sigma^o_{XY} = \sum_{p,q}\epsilon_{pq}\sin\left(\pi k_p\frac{X+L}{L}\right)\sin\left(\pi k_q\frac{Y+W}{W}\right), \tag{50}$$

$$\sigma^o_{XX} = \sum_{p,q}\epsilon_{pq}\frac{k_qL}{k_pW}\left[1-\cos\left(\pi k_p\frac{X+L}{L}\right)\right]\cos\left(\pi k_q\frac{Y+W}{W}\right), \tag{51}$$

$$\sigma^o_{YY} = \sum_{p,q}\epsilon_{pq}\frac{k_pW}{k_qL}\cos\left(\pi k_p\frac{X+L}{L}\right)\left[1-\cos\left(\pi k_q\frac{Y+W}{W}\right)\right], \tag{52}$$

k_p, k_q are integers and ϵ_{pq}'s are constants. The above representation for the Cartesian components of the Cauchy stress is obtained so that it satisfies the static equilibrium equations in the absence of body forces, $Div(\boldsymbol{\sigma}^o) = \mathbf{0}$ and the traction free boundary condition, i.e.,

$$\sigma^o_{XY}(\pm L, Y) = \sigma^o_{XY}(X, \pm W) = \sigma^o_{YY}(X, \pm W) = \sigma^o_{XX}(\pm L, Y) = 0. \tag{53}$$

Here it is pertinent to note that by virtue of the slab in the stressed reference configuration being boundary traction free and in static equilibrium, σ^o_{XY} cannot be zero identically in the interior of the body.

Substituting (49) in (10) (or equivalently in (11)), we find that

$$\mathbf{B}_o = \begin{pmatrix} B^o_{XX} & B^o_{XY} & 0 \\ B^o_{XY} & B^o_{YY} & 0 \\ 0 & 0 & B^o_{ZZ} \end{pmatrix}, \quad \mathbf{B}_o^{-1} = \begin{pmatrix} \frac{B^o_{YY}}{\Delta} & -\frac{B^o_{XY}}{\Delta} & 0 \\ -\frac{B^o_{XY}}{\Delta} & \frac{B^o_{XX}}{\Delta} & 0 \\ 0 & 0 & \frac{1}{B^o_{ZZ}} \end{pmatrix}, \tag{54}$$

where $\Delta = [B^o_{XX}B^o_{YY} - (B^o_{XY})^2]$ and B^o_{ij} are the nonzero Cartesian components of $\mathbf{B}^o$ which are related to the Cartesian components of the stress, $\boldsymbol{\sigma}^o$ through:

$$B^o_{XX} = \delta_0 + \delta_1\sigma^o_{XX} + \delta_2\left[(\sigma^o_{XX})^2 + (\sigma^o_{XY})^2\right], \tag{55}$$

$$B^o_{XY} = \sigma^o_{XY}\left[\delta_1 + (\sigma^o_{XX} + \sigma^o_{YY})\delta_2\right], \tag{56}$$

$$B^o_{YY} = \delta_0 + \delta_1\sigma^o_{YY} + \delta_2\left[(\sigma^o_{YY})^2 + (\sigma^o_{XY})^2\right], \tag{57}$$

$$B^o_{ZZ} = \delta_0. \tag{58}$$

Since, σ^o_{XY} is not identically zero in the interior of the body, B^o_{XY} is also not zero identically in the interior of the slab.

By virtue of the Cauchy stress field in the reference configuration being nonuniform, and $B^o_{XY} \neq 0$ in the interior of the slab and the slab being thin, stretching this thin slab along the x direction results in a plane state of stress as given by (36). Without making any simplifying assumptions on the form of the deformation field, the nonzero Cartesian components of the Cauchy stress is related to the Cartesian components of the deformation gradient, F_{ij} through,

$$\begin{aligned}\sigma_{xx} = {}&\alpha^r_0 + \alpha^r_1[B^o_{XX}F^2_{xX} + 2B^o_{XY}F_{xY}F_{xX} + B^o_{YY}F^2_{xY} + B^o_{ZZ}F^2_{xZ}] \\ &+ \frac{\alpha^r_2}{J_3^2\Delta}\Big[B^o_{YY}(F_{yY}F_{zZ} - F_{yZ}F_{zY})^2 + B^o_{XX}(F_{yZ}F_{zX} - F_{yX}F_{zZ})^2 \\ &-2B^o_{XY}(F_{yY}F_{zZ} - F_{yZ}F_{zY})(F_{yZ}F_{zX} - F_{yX}F_{zZ}) + \frac{\Delta}{B^o_{ZZ}}(F_{zY}F_{yX} - F_{yY}F_{zX})^2\Big],\end{aligned} \tag{59}$$

$$\begin{aligned}\sigma_{yy} = {}&\alpha^r_0 + \alpha^r_1[B^o_{XX}F^2_{yX} + 2B^o_{XY}F_{yY}F_{yX} + B^o_{YY}F^2_{yY} + B^o_{ZZ}F^2_{yZ}] \\ &+ \frac{\alpha^r_2}{J_3^2\Delta}\Big[B^o_{YY}(F_{zY}F_{xZ} - F_{zZ}F_{xY})^2 + B^o_{XX}(F_{zZ}F_{xX} - F_{xZ}F_{zX})^2 \\ &-2B^o_{XY}(F_{zY}F_{xZ} - F_{zZ}F_{xY})(F_{zZ}F_{xX} - F_{xZ}F_{zX}) + \frac{\Delta}{B^o_{ZZ}}(F_{zX}F_{xY} - F_{zY}F_{xX})^2\Big],\end{aligned} \tag{60}$$

$$\begin{aligned}\sigma_{xy} = {}&\alpha^r_1[B^o_{XX}F_{xX}F_{yX} + B^o_{XY}(F_{yY}F_{xX} + F_{yX}F_{xY}) + B^o_{YY}F_{yY}F_{xY} + B^o_{ZZ}F_{yZ}F_{xZ}] \\ &+ \frac{\alpha^r_2}{J_3^2\Delta}\Big[B^o_{YY}(F_{zY}F_{xZ} - F_{zZ}F_{xY})(F_{zZ}F_{yY} - F_{yZ}F_{zY}) \\ &+ B^o_{XX}(F_{zZ}F_{xX} - F_{xZ}F_{zX})(F_{yZ}F_{zX} - F_{yX}F_{zZ}) \\ &- B^o_{XY}[(F_{yY}F_{zZ} - F_{zY}F_{yZ})(F_{zZ}F_{xX} - F_{xZ}F_{zX}) \\ &+ (F_{zY}F_{xZ} - F_{zZ}F_{xY})(F_{yZ}F_{zX} - F_{yX}F_{zZ})] \\ &+ \frac{\Delta}{B^o_{ZZ}}(F_{zX}F_{xY} - F_{zY}F_{xX})(F_{zY}F_{yX} - F_{yY}F_{zX})\Big],\end{aligned} \tag{61}$$

obtained using (9). Recognize that in the above equation, (61) by virtue of $B^o_{XY} \neq 0$, and $\sigma_{xy} = 0$, at least on the boundary for uniaxial state of stress, F_{xY} and/or F_{yX} cannot be zero. The case here is akin to that of transversely isotropic material, except that now the anisotropic tensor, $\mathbf{A}$ in (17) is replaced by $\mathbf{B}^o$ in (9), especially when $\alpha^r_2 = 0$ and $\alpha^a_2 = 0$. Therefore, as in the case of transversely anisotropic material, the principal direction of left Cauchy–Green deformation tensor would vary with the loading, in general.

3.3. Response of inhomogeneous body made of isotropic, incompressible material subjected to equal biaxial stretch

As before, the inhomogeneous body is assumed to occupy the region, $\mathcal{B} = \{(X, Y, Z)| -L \leqslant X \leqslant L, -W \leqslant Y \leqslant W, -h \leqslant Z \leqslant h\}$, where L, W and h are constants. However, now we study the response of this inhomogeneous body made of isotropic material subjected to a biaxial loading such that,

$$\int_a \mathbf{t}_{(\mathbf{FE}_x)}(L, y, z)da = L_x\mathbf{e}_x, \quad \int_a \mathbf{t}_{(\mathbf{FE}_y)}(x, W, z)da = L_y\mathbf{e}_y, \\ \mathbf{t}_{(\mathbf{FE}_z)}(x, y, \pm h) = \mathbf{0}. \tag{62}$$

where $\mathbf{t}_{(\mathbf{n})}$ denotes the traction acting on the surface whose normal is $\mathbf{n}$ in the current configuration, $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ are coordinate basis vectors in the reference configuration, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are coordinate basis vectors in the current configuration, L_x and L_y are constants. It should be noted that in the biaxial experiment only the total applied load on the surfaces with normals initially oriented along $\mathbf{E}_x$ and $\mathbf{E}_y$ can be measured.

Assuming the thickness of the body to be small, we approximate the stresses to be uniform along the thickness, $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(X, Y)$. The assumption made here suffices to prove our thesis. Consequently, the traction boundary condition (62c) implies that the state of stress is plane, i.e.,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (63)$$

Then, without making any simplifying assumptions on the form of the deformation, the remaining two traction boundary conditions (62) requires

$$\mathcal{L}_x = \int_{-W}^{W} \left(\int_{-h}^{h} (F^{-1}_{Xx}\sigma_{xx} + F^{-1}_{Yx}\sigma_{xy}) dZ \right) dY, \quad (64)$$

$$0 = \int_{-W}^{W} \left(\int_{-h}^{h} (F^{-1}_{Xx}\sigma_{xy} + F^{-1}_{Yx}\sigma_{yy}) dZ \right) dY, \quad (65)$$

$$0 = \int_{-L}^{L} \left(\int_{-h}^{h} (F^{-1}_{Xy}\sigma_{xx} + F^{-1}_{Yy}\sigma_{xy}) dZ \right) dX, \quad (66)$$

$$\mathcal{L}_y = \int_{-L}^{L} \left(\int_{-h}^{h} (F^{-1}_{Xy}\sigma_{xy} + F^{-1}_{Yy}\sigma_{yy}) dZ \right) dX, \quad (67)$$

where F^{-1}_{Ij} are the Cartesian components of the inverse of the deformation gradient and we appealed to Nanson's formula and the fact that the $\det(\mathbf{F}) = 1$, since the material is incompressible.

Then, since the deformation is inhomogeneous, the equal stretch condition along the two directions now becomes experiment dependent. It could be based on the overall stretch or that measured over a shorter gauge length near the center region on the top surface, in each direction. Thus,

$$\frac{1}{2g_X} \int_{-g_X}^{g_X} \sqrt{F_{xX}^2 + F_{yX}^2 + F_{zX}^2} \Big|_{(X,0,h)} dX = \frac{1}{2g_Y} \int_{-g_Y}^{g_Y} \sqrt{F_{xY}^2 + F_{yY}^2 + F_{zY}^2} \Big|_{(0,Y,h)} dY, \quad (68)$$

where F_{iJ} represent Cartesian components of the deformation gradient and if $g_X = L$ and $g_Y = W$ the condition would be requiring that the overall stretch be the same, else if $g_X = g_Y = g$ it would be based on stretch measured using a shorter gauge length near the center region. The expression (68) is obtained from the formulae,

$$\Lambda_{\mathbf{E}_i} = \frac{1}{(2g_i)} \int_{-g_i}^{g_i} \sqrt{\mathbf{C}\mathbf{E}_i \cdot \mathbf{E}_i} dX_i, \quad (69)$$

where we have assumed that the stretch ratio is required for a straight line oriented along $\mathbf{E}_i$, a coordinate direction and located on the top surface of the body (i.e., $Z = h$) and passing through the point $(0, 0, h)$.

Finally, the quantity of interest is R, the ratio of the nominal stresses along the X and Y direction, where the nominal stress is defined as the applied load per unit gross undistorted cross sectional area, i.e.,

$$R = \frac{\mathcal{L}_x L}{\mathcal{L}_y W} = \frac{L}{W} \frac{\int_{-W}^{W} \left(\int_{-h}^{h} (F^{-1}_{Xx}\sigma_{xx} + F^{-1}_{Yx}\sigma_{xy}) dZ \right) dY}{\int_{-L}^{L} \left(\int_{-h}^{h} (F^{-1}_{Xy}\sigma_{xy} + F^{-1}_{Yy}\sigma_{yy}) dZ \right) dX}. \quad (70)$$

It is clear from Eq. (70) that even if the stress is obtained from constitutive relations for isotropic materials, such as (5), by virtue of the Cauchy stress and the deformation gradient being a function of (X, Y) the value of R would be different from 1 in general. Boundary conditions (65) and (66) along with the equilibrium equations $div(\boldsymbol{\sigma}) = \mathbf{0}$ and the equal stretch condition (68) are insufficient to ensure $R = 1$ for an inhomogeneous body whose constitutive function varies over X and Y.

It can also be shown easily that for a body made of isotropic material with constitutive function varying only along $Z, R = 1$, irrespective of whether the material is compressible or incompressible.

Thus, for certain classes of inhomogeneous bodies made of isotropic material, $R \neq 1$, in an equal biaxial experiment.

4. Discussion

The above theoretical study did not investigate the case wherein some subparts of the inhomogeneous bodies are made of anisotropic material and other subparts are made of isotropic material. It appears that when the inhomogeneous body is made of materials with different symmetries, in regions where the material is isotropic, the principal direction of the left Cauchy–Green deformation tensor would not vary with the magnitude of the applied uniaxial load and in regions where the material is anisotropic it would. However, this conjecture needs to be established by a more detailed study.

Next, we examine experimental issues in determining the principal direction of the left Cauchy–Green deformation tensor as a function of the magnitude of the applied uniaxial load. It looks like despite enormous amount of mechanical experiments being reported on various aspects of mechanical behavior, there has been no report on the variation of the principal direction of the left Cauchy–Green deformation tensor with the magnitude of the applied uniaxial load except by Paranjothi et al. (2011).

Now, the issue is in experimental determination of the spatial variation of the left Cauchy–Green deformation tensor. All the techniques employed to determine deformation gradient a priori assume an underlying form for the deformation field, which in many cases is a homogeneous deformation field. As shown in Saravanan (2014), in general inhomogeneous bodies do not deform homogeneously. Moreover, one cannot a priori estimate the nature of the deformation field in an inhomogeneous body subjected to uniaxial load; it depends on the form of the constitutive relation which is being sought. Therefore, there would be an error in the estimated components of the deformation gradient when the assumed form for the deformation field is not in agreement with the realized deformation field. This error is similar to that arises by approximating the tangent of a function by its secant. A study has to be undertaken to explore the influence of the error in the estimated components of the deformation gradient on the determined principal direction of the left Cauchy–Green deformation tensor to conclude on the usefulness of this proposed methodology.

Experiments are underway on various fibrous bodies to find the symmetry required in the constitutive relation by examining the variation of the principal direction of left Cauchy–Green deformation tensor with the magnitude of the applied uniaxial load. Once these results are available the appropriateness of requiring the symmetry in the constitutive relation to be same as that determined based on its internal structure established.

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