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Hawking radiation as tunneling for spherically symmetric black holes: A generalized treatment

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Abstract

We present a derivation of Hawking radiation through tunneling mechanism for a general class of asymptotically flat, spherically symmetric spacetimes. The tunneling rate $\Gamma \sim \exp(\Delta S)$ arises as a consequence of the first law of thermodynamics, $T\,dS = dE + P\,dV$. Therefore, this approach demonstrates how tunneling is intimately connected with the first law of thermodynamics through the principle of conservation of energy. The analysis is also generally applicable to any reasonable theory of gravity so long as the first law of thermodynamics for horizons holds in the form, $T\,dS = dE + P\,dV$.

1. Introduction

Understanding Hawking radiation is one of the key issues in any effort to unify gravity with quantum rules. In the last few decades, we have witnessed several independent attempts to comprehend the fact that black holes radiates and therefore behaves like a perfect thermal system. The original derivation of Hawking radiation [1] involves the calculation of the Bogolyubov coefficients between asymptotic in and out states in a collapsing geometry. Another different approach uses the Euclidean quantum gravity techniques [2]. In that case, the thermal nature of horizons arises from the periodicity in Euclidean time needed to get rid of the conical singularity.

There exists another popular and may be more physically motivated approach for Hawking radiation based on quantum tunneling. Recently, it has been demonstrated that such a interpretation can really work rigorously by adopting a semiclassical method for tunneling [3,4]. The main ingredient of this work is the consideration of energy conservation in tunneling of a thin shell from the hole, motivated by some earlier work [5]. An added advantage of tunneling approach is its simplicity and as a result, it can be easily extended to different cases [6,7]. In all situations, the tunneling approach works perfectly, giving physically meaningful results. Virtually all the known solutions of the general relativity with horizons, have been investigated and the results seem to be in favor of tunneling interpretation. Despite this, *there is no general approach for the tunneling of matter from a horizon which is independent of solution* and the main motivation of this Letter is to show that *such a generalization is indeed possible*. Another important result which comes out in this context is the relationship of tunneling with the first law of thermodynamics for spacetime horizons. In fact this is observed in some earlier work [8] but again, no generalized demonstration exists. Our Letter will address this issue and explore the relationship of tunneling and first law of thermodynamics in a general spherically symmetric setting.

2. Formalism

We will consider a static, spherically symmetric horizon, in an asymptotically flat spacetime described by the metric:

$$ds^2 = -f(r)\,dt_s^2 + \frac{1}{f(r)}dr^2 + r^2\,d\Omega^2. \qquad (1)$$

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We assume this metric is the solution of field equation for a general matter described by an energy–momentum tensor T^{μ}_{ν}. We will also assume that the function $f(r)$ has a simple zero at $r = a$, and $f'(a)$ is finite, so that spacetime has a horizon at $r = a$ and periodicity in Euclidean time allows us to associate a temperature with the horizon as $T = f'(a)/4\pi$. (Even for spacetimes with multi-horizons, this prescription is locally valid for each horizon surface.) In order to generalize the tunneling formalism for such a general spherically symmetric setup, one need to tackle two key issues. The first is, how to apply energy conservation in such a general setup, and secondly, how to write the general form of the first law of thermodynamics in such a situation. The first one is tricker because the concept of energy in general relativity is difficult to define. One has to depend on some heuristic arguments to apply energy conservation for a general setup. The answer to the second issue already exists in a different approach to understand the dynamics of gravity and thermodynamics [9]. Following that methodology, it is possible to show that in a general spherically symmetric background the Einstein's equations evaluated on the horizons can be cast into a form

$$T\,dS = dE_h + P\,dV \tag{2}$$

where S is the entropy associated with the horizon at $r = a$, E_h is its energy and P is the radial pressure which is equal to $T^r_r(a)$. The expression of energy E_h is equal to $a/2$ which is the irreducible mass of the hole (E_h is also equal to the Misner–Sharp energy of the horizon) and $V = 4\pi a^3/3$ is the relevant volume for our analysis. In spite of superficial similarity, Eq. (2) is different from the conventional first law of black hole thermodynamics [10], due to the presence of $P\,dV$ term. The validity of this expression can be seen, for example, in the case of Reissner–Nordstrom black hole for which $P \neq 0$. If a chargeless particle of mass dM is dropped into a Reissner–Nordstrom blackhole, then an elementary calculation shows that the energy defined above as $E_h \equiv a/2$ changes by $dE_h = (da/2) = (1/2)[a/(a-M)]\,dM \neq dM$ while it is $dE_h + P\,dV$ which is precisely equal to dM making sure $T\,dS = dM$. So we need the $P\,dV$ term to get $T\,dS = dM$ when a *chargeless* particle is dropped into a Reissner–Nordstrom black hole. More generally, if da arises due to changes dM and dQ, it is easy to show that Eq. (2) gives $T\,dS = dM - (Q/a)\,dQ$ where the second term arises from the electrostatic contribution from the horizon surface charge. Dynamically, Eq. (2) is best interpreted as the energy balance under infinitesimal virtual displacements of the horizon normal to itself and therefore must be linked with energy conservation and thus to the tunneling process.

This result can be formally interpreted by noting that in standard thermodynamics, we consider two equilibrium states of a system differing infinitesimally in the extensive variables like entropy, energy, and volume by dS, dE_h and dV while having same values for the intensive variables like temperature (T) and pressure (P). Then, the first law of thermodynamics asserts that $T\,dS = P\,dV + dE_h$ for these states. In a similar way, Eq. (2) can be interpreted as a connection between two quasi-static equilibrium states where both of them are spherically symmetric solutions of Einstein equations with the radius of horizon differing by da while having same source $T_{\mu\nu}$ and temperature $T = f'(a)/4\pi$. This formalism does not care what causes the change of the horizon radius and therefore much more formal and generally applicable. More recently, this approach has been extended to the general class of Lovelock gravity [11], stationary Kerr like and time dependent spherically symmetric spacetimes [12] as well as for the FRW class of metrics [13]. This approach suggests that the relevant energy of the horizon which enters into the first law of thermodynamics under the change of the horizon radius is $E_h = a/2$. Since, first law of thermodynamics is basically a statement of the conservation of energy, $E_h = a/2$ should be the appropriate notion of energy of the horizon in the tunneling process. Now, to describe across-horizon phenomena like tunneling, it is necessary to choose coordinates which, unlike Schwarzschild like coordinates, are not singular at the horizon. A particularly suitable choice is obtained by introducing Painlevé type of coordinates where the metric is given by

$$ds^2 = -f\,dt^2 \pm 2\sqrt{1-f}\,dt\,dr + dr^2 + r^2\,d\Omega^2 \tag{3}$$

where the plus (minus) sign denotes the spacetime line element of the out-going (incoming) particles across the horizon. We want to consider the tunneling of matter across the horizon. By treating the matter as a sort of De-Broglie S-wave, its trajectory can be approximately determined as [4,14],

$$\dot{r} = \pm\frac{f}{2\sqrt{1-f}} \approx \pm\kappa(r-a) \tag{4}$$

where κ is the surface gravity of the horizon at $r = a$. The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from r_i to r_f can be expressed as

$$\operatorname{Im}\mathcal{S} = \operatorname{Im}\int_{r_i}^{r_f} p_r\,dr = \operatorname{Im}\int_{r_i}^{r_f}\int_0^{p_r} dp'_r\,dr. \tag{5}$$

We multiply and divide the integrand by the two sides of Hamilton's equation $\dot{r} = +\frac{d\mathcal{H}}{dp_r}\big|_r$, change variable from momentum to energy, and switch the order of integration to obtain

$$\operatorname{Im}\mathcal{S} = \operatorname{Im}\iint_{r_i}^{r_f} \frac{dr}{\dot{r}}\,d\mathcal{H}. \tag{6}$$

Now, using Eq. (4) and performing the radial integral with appropriate contour prescription, one immediately finds that, for outgoing case [3],

$$\operatorname{Im}\mathcal{S} = -\int\frac{d\mathcal{H}}{2T}, \tag{7}$$

where $T = \kappa/2\pi$, the Hawking temperature associated with the horizon. The next task is to find the correct Hamiltonian to evaluate this integral.

For this, we appeal to energy conservation to guess the form of $\mathcal{H}$.

We first note that since there is no explicit time dependence, the Hamiltonian should be equal to the total energy of the system. To find an expression for the total energy, we again note that, the horizon separates the spacetime into two parts, which can be labelled as inside and outside. Our previous arguments [9] show that the energy associated with the horizon is $E_h = a/2$. Also, for matter with T^μ_ν as the energy–momentum tensor, the contribution from the matter field in the outside region should be

$$E_m = -\int_\Sigma T^\mu_\nu \xi^\nu \, d\Sigma_\mu, \tag{8}$$

where the integration is over the 3-surface Σ which extends from the horizon to infinity. (The negative sign merely reflects the fact that in our convention $-T^t_t$ is the energy density of the matter.) Hence the total energy of the spacetime should be

$$\begin{aligned} E_T &= E_h + E_m \\ &= \frac{a}{2} - \int_\Sigma T^\mu_\nu \xi^\nu \, d\Sigma_\mu. \end{aligned} \tag{9}$$

For our particular case, $\xi^\nu = (1, 0, 0, 0)$. Then the energy expression reduces to,

$$\begin{aligned} E_T &= \frac{a}{2} - \int_{r=a}^{\infty} T^t_t 4\pi r^2 \, dr \\ &= \frac{a}{2} - \int_{r=a}^{\infty} T^r_r 4\pi r^2 \, dr. \end{aligned} \tag{10}$$

In the second step, we have used the fact that, for the metric in Eq. (3), $G^t_t = G^r_r$.

Now, in order to show that the expression of E_T is indeed the energy of the spacetime, we evaluate it for two separate cases. First, for the trivial case of Schwarzschild black hole for which $T^\mu_\nu = 0$ and $a = 2M$, which gives $E_T = M$ as desired. Next, for the Reissner–Nordstrom black hole, we have,

$$T^\mu_\nu = \frac{Q^2}{8\pi r^4}(-1, -1, 1, 1) \quad \text{and} \quad E_h = M - \frac{Q^2}{2a}. \tag{11}$$

It is possible to provide a simple explanation of the energy expression E_h. Consider any general $r > a$ and then we can write, for Reissner–Nordstrom black hole,

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 1 - \frac{2\mu(r)}{r}, \quad \text{where } \mu(r) = M - \frac{Q^2}{2r}. \tag{12}$$

Hence, the quantity $\mu(r) = M - Q^2/2r$ can be interpreted as the mass energy inside the sphere of radius r and for $r = a$, we have $\mu(a) = E_h$ validating the case for E_h as the energy of the horizon. Substituting this into Eq. (10), it is easy to show that even in this case $E_T = M$.

As a result of tunneling across the horizon, some matter either tunnels out or in across the horizon. Hence, the parameters which fix the radius of the horizon (mass, charge, etc.) change and that leads to a change in the radius of the horizon. In fact, *the only physical change occurring due to the process of tunneling is in the radius of the horizon*. We suppose, due to tunneling the radius changes from a to $a + \delta a$ (note that δa is positive for incoming shell and negative for outgoing shell). Both E_h and E_m and as a result E_T depends of a. Therefore, let the energy E_T changes to $E^i_T(a)$ to $E^f_T(a + \delta a)$ and the difference $E^f_T - E^i_T$ can be attributed to the shell. So, by energy conservation, we can immediately write,

$$\begin{aligned} d\mathcal{H} &= E^f_T(a + \delta a) - E^i_T(a) \\ &= \frac{\delta a}{2} - \left(\int_{a+\delta a}^{\infty} - \int_a^{\infty} \right) T^r_r 4\pi r^2 \, dr \\ &= \frac{\delta a}{2} + T^r_r(a) 4\pi a^2 \delta a \\ &= dE_h + P\, dV. \end{aligned} \tag{13}$$

Substituting this in Eq. (7), we ultimately get,

$$\operatorname{Im} \mathcal{S} = -\int \frac{dE_h + P\, dV}{2T} = -\int \frac{dS}{2}, \tag{14}$$

where we have used the first law of thermodynamics as in Eq. (2).

Written in this form, one can also see an immediate generalization of this approach, at least in spherical symmetry, to any theory of gravity for which a suitable generalization of $a/2$, the Misner–Sharp energy, can be motivated (note that the $P\,dV$ term generalizes in a *natural* way, i.e., with dV interpreted as the areal volume). Indeed, such an expression for energy exists for the Lanczos–Lovelock Lagrangians [11,15]. Replacing $a/2$ with that expression, one can explicitly show that we get the correct result; specifically, we obtain the correct scaling of entropy (which is no longer proportional to area) for the Lanczos–Lovelock Lagrangians.

Another interesting point which comes in this regard is that, entire analysis is totally local. In fact, all contributions from the spatial infinity cancels out in the second step of Eq. (13). Hence, in principle this approach is extendable to spacetimes having multiple horizons. Now, the semi classical tunneling rate is given by

$$\Gamma \sim e^{-2\operatorname{Im} \mathcal{S}} = e^{\int_{S_i}^{S_f} dS} = e^{\Delta S}, \tag{15}$$

where $\Delta S = S_f - S_i$. This is the well-known result obtained in [3] for a general, asymptotically flat, spherically symmetric background.

Now, we would like to discuss the case for the class of metrics in which $g_{rr} g_{t_s t_s} \neq -1$. The spacetime metric is given by,

$$ds^2 = -f(r)\, dt_s^2 + \frac{1}{g(r)} dr^2 + r^2 \, d\Omega^2. \tag{16}$$

The horizon is given by a simple zero of the function $f(r)$ at $r = a$. Temperature associated with this horizon can be shown to be $T = \sqrt{f'(a) g'(a)}/4\pi$. The fact that $r = a$ is a null surface, and requiring the regularity of Ricci scalar at $r = a$, further impose two conditions on $g(r)$ at $r = a$. (a) $g(a) = 0$ and (b) $f'(a) = g'(a)$. Using these, the temperature associated with the horizon at $r = a$ becomes $T = g'(a)/4\pi$. Using these conditions, and writing the metric in the Painlevé form, one can show that [12,17] the energy–momentum tensor, *on the horizon*, again has the form,

$$T_t^t\big|_{r=a} = T_r^r\big|_{r=a}; \qquad T_\theta^\theta\big|_{r=a} = T_\phi^\phi\big|_{r=a} \tag{17}$$

(where, as earlier, t is the Painlevé time coordinate). It can be shown that [9] the relevant energy of the horizon, E_h, is again $a/2$, so that the first equality in Eq. (10) still holds. Proceeding as before, using energy conservation, we can write,

$$\begin{aligned} d\mathcal{H} &= E_T^f(a+\delta a) - E_T^i(a) \\ &= \frac{\delta a}{2} - \left(\int_{a+\delta a}^{\infty} - \int_{a}^{\infty}\right) T_t^t 4\pi r^2\, dr \\ &= \frac{\delta a}{2} + T_t^t(a) 4\pi a^2 \delta a \\ &= \frac{\delta a}{2} + T_r^r(a) 4\pi a^2 \delta a \\ &= dE_h + P\, dV, \end{aligned} \tag{18}$$

where we have used Eq. (17) in the fourth line. The remaining steps are identical as before, and give the same tunneling rate, $\Gamma \sim \exp(\Delta S)$.

Hence, we find that, for a general spherically symmetric case, the derivation of this result requires only local physics. There was neither an appeal to Euclideanization nor any need to invoke an explicit collapse phase. The simple facts that tunneling changes the horizon radius and energy conservation, are enough to find the semi classical tunneling rate.

Another striking feature which naturally appears from this analysis is the relationship between tunneling and the first law of thermodynamics. The tunneling rate is ultimately found to be,

$$\begin{aligned} \Gamma &\sim e^{\int \frac{dE_h + P\, dV}{T}} \\ &\sim e^{\Delta S}. \end{aligned} \tag{19}$$

In order to obtain the second expression from the first, we have to apply the generalized first law $T\, dS = dE_h + P\, dV$. This fact clearly shows that tunneling interpretation of black hole radiation is intimately related with the first law of thermodynamics. This is also expected because the basic input behind both tunneling and the first law is same, namely the conservation of energy.

We would also like to point out the importance of the assumption of spherical symmetry in our calculation. The two main ingredients which depend on this assumption are:

1. The fact that $G_t^t = G_r^r$ on the horizon, which ultimately allows us to write the $P\, dV$ term. This equality can be seen as a result of the equality of g_{rr} and g_{tt}, and their first derivatives, on the horizon, for the metric Eq. (16). But, it is also possible to show that this result is valid even for non-spherically symmetric case, near the Killing horizon of any stationary but non-static (and non-extremal) black hole spacetime [16]. In that case, the near horizon structure of the Einstein tensor, and hence that of T_ν^μ, admits a block diagonal representation, in which $T_t^t = T_r^r$ (with the time coordinate interpreted appropriately). Hence, this assumption is not specific to spherical symmetry.
2. The quantities dE_h and $P\, dV$ have a natural interpretation in spherical symmetry and it is difficult to construct analogous quantities in non-spherically symmetric case. But, at least for spacetimes with a timelike killing vector, there is a notion of volume [19] and it may help to generalize our arguments for non-spherical situation.

But the fact that this generalized approach works for spherical symmetry is intriguing enough, and it may be possible that this can also be extended to non-spherically symmetric situations, by interpreting various quantities appropriately.

3. Conclusion

Our analysis generalizes the tunneling approach of Hawking radiation for a general spherically symmetric and asymptotically flat set up. The basic inputs that go into the calculation is the energy function in Eq. (10) and application of energy conservation. With these two inputs and also using Eq. (2), one immediately recovers the results of [3] in this general context. This shows the validity of tunneling process for a general spherically symmetric horizon and also its relation with the first law of thermodynamics. In fact in our analysis, the result $\Gamma \sim \exp(\Delta S)$ comes out as a natural consequence of the first law $T\, dS = dE_h + P\, dV$, and therefore suggests an intimate dependence on each other through the principle of conservation of energy.

Note added

While completing this work, we came across Ref. [18], which also reaches the same conclusion regarding the relationship of tunneling and the first law of thermodynamics through a totally different approach. In fact, unlike [18], our method is *independent of the theory of gravity and makes no assumption of entropy being proportional to area*. Another crucial difference is the presence of the $P\, dV$ term in our analysis. Hence, the approach presented here is more generally applicable as long as the first law of thermodynamics holds in the form $T\, dS = dE_h + P\, dV$.

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