Theoretical Computer Science 412 (2011) 4629–4635

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Hardness of subgraph and supergraph problems in c-tournaments

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ARTICLE INFO

Article history:
Received 27 February 2009
Received in revised form 26 April 2011
Accepted 26 April 2011
Communicated by G. Ausiello

Keywords:
Parameterized complexity
Feedback vertex set
Dominating set
Graph modification
Tournaments

ABSTRACT

Problems like the directed feedback vertex set problem have much better algorithms in tournaments when compared to general graphs. This motivates us to study a natural generalization of tournaments, named c-tournaments, and see if the structural properties of these graphs are helpful in obtaining similar algorithms. c-tournaments are simple digraphs which have directed paths of length at most $c \geq 1$ between all pairs of vertices. We study the complexity of feedback vertex set and r-dominating set in c-tournaments. We also present hardness results on some graph editing problems involving c-tournaments.

1. Introduction

The design of fixed parameter tractable algorithms for NP hard problems is a way of coping with NP hardness. Apart from the many research articles on such algorithms, books by Downey and Fellows [1] and Niedermeier [2] form a source of information on this young but rich area. In this paper we initiate the study of c-tournaments which is a graph class that generalizes *tournaments*. For some fixed natural number c, a c-tournament is a directed graph in which for each pair of vertices u and v, there is a directed path of length at most c between the two vertices. Note that there may be a path from u to v, but no path from v to u. Clearly, a tournament is a c-tournament for $c = 1$.

Past results

The r-(out)dominating set problem asks for a vertex set such that for each vertex v in the graph there is vertex u in the set such that there is a directed path of length at most r from u to v. A famous theorem by Landau (1955) proves that in any tournament, the set consisting only of a vertex of maximum out degree is a 2-dominating set. For $r = 1$, the r-dominating set is the dominating set problem. While it is known that any tournament has a $\log_2 n$ dominating set, the status of finding a dominating set of size $r \leq \log_2 n$ in tournaments in unknown [3]. The parameterized complexity of this problem is much clearer as it is $W[2]$-complete [4].

A property is called *hereditary* if it holds for every induced subgraph of a graph which satisfies the property. Yannakakis et al. [5] proved that the problem of removing as few edges as possible from a given graph to obtain another graph which satisfies a certain fixed hereditary property is NP complete (NPC). Raman et al. [6] studied this problem from a parameterized complexity standpoint. Shamir et al. [7] studied the problem of finding a cluster subgraph, which has complete graphs as components. We note that being a cluster graph is hereditary property and so this problem is NPC.

Feedback vertex (arc) set (FVS/FAS) problems are NPC on directed graphs [8]. From a parameterized complexity standpoint, the FVS and FAS problems are fixed parameter tractable (FPT) both in the case of directed [9] and undirected graphs [10].

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2011.04.046

While the FVS problem in undirected graphs has a $O^*(\alpha^k)$ algorithm, where α is a constant, no such algorithm is known for the directed case. On the other hand there are efficient FPT algorithms [11] to solve the FVS and FAS problem in tournaments.

Our results

In Section 2.1 we study the problem of r-dominating set in c-tournaments. We prove that the c-dominating set problem in a c-tournament is $W[2]$-complete. On the other hand we show that the 2c-dominating set problem in c-tournaments is polynomial time solvable.

We note that being a c-tournament is a *hereditary* property only for $c = 1$. In Section 3.1 we study the problem of finding a graph obtained by a minimum possible number of edge deletions such that each component is a 2-*tournament*. We refer to this as the *2-tournament clustering edge set problem*. We prove that this problem is NPC, and also show that it is FPT. We then study the problem of 2-tournament completion, that is to add edges to transform a given digraph into a 2-tournament, and prove that this problem is NPC and $W[2]$ complete.

As noted earlier, the FVS problem is simpler to solve in tournaments than in general graphs owing to its structure. One natural question that arises is whether the structural properties of c-tournaments would help us solve the FVS (FAS) problem. Our study in Section 2.2 answers this question negatively. We lose most of the structural properties helpful to solve the FVS problem of tournaments when we reach 3-tournaments. Even 2-tournaments are as hard as general graphs, as we prove that given an $O^*(\alpha^k)$ algorithm, for a constant α, to solve FVS in 2-tournaments, we can solve the problem in general graphs in time $O^*(\alpha^k)$.

2. Complexity results for c-tournaments

We have used standard graph notation from the book by Bondy and Murty [12] for most part. The tth order out-degree of a vertex is the number of vertices which are at an out-distance of at most t from it. Similarly the number of vertices from which a vertex can be reached by a directed path of at most t edges is called its tth order in-degree. The sets of those vertices are called the vertex's tth order out-neighborhood and tth order in-neighborhood respectively. We say that a pair of vertices satisfy the property P_c if they are at a directed distance at most c. A digraph in which every vertex pair satisfies P_c is a c-tournament. By a dominating set in a digraph we mean the out-dominating set, unless otherwise specified. We next define the hierarchy of parameterized complexity classes.

Parameterized complexity

A parameterized problem L is a set of pairs (x, k) such that $x \in \Sigma^*$, where Σ is a finite alphabet and $k \in \mathbb{Z}_+$ is a *parameter*. A parameterized problem L is said to be *fixed parameter tractable* (FPT) if membership in L can be decided in $O(f(k) * n^{O(1)})$, where n is the input size. This class of problems can be thought of as the analogue of P. Note that in this class, for a fixed k, the running time is only polynomially dependent on the input size. When a parameterized problem cannot be shown to be FPT, then an attempt is made to identify its position in a hierarchy of parameterized complexity classes. These classes are analogous to the Π_i hierarchy in the classical complexity, and is referred to as $\mathbf{W}[t]$ hierarchy for $t \in \mathbb{Z}_+$. Completeness for a class in this hierarchy is shown via parameterized reductions. We first define the, now standard, concept of parameterized reduction or a FPT-reduction.

Definition 1. A parameterized language A FPT-many-one reduces to a parameterized language B if there are a polynomial q, functions $f, g : \mathbb{N} \to \mathbb{N}$ and a Turing machine T such that on any input (x, k), T runs for $f(k)q(|x|)$ steps and outputs $(x', g(k))$ such that $(x, k) \in A \iff (x', g(k)) \in B$.

In the natural hierarchy of classes of parameterized languages it is known that $FPT \subseteq W[1] \subseteq W[2] \ldots \subseteq W[poly]$. It has been shown, see [1,2], that Dominating Set is complete for the class $W[2]$, and Clique is complete for the class $W[1]$ under FPT-reductions. For further details and the definition of $W[t]$ we refer the reader to the texts on parameterized complexity by Downey and Fellows [1] and Niedermeier [2].

2.1. r-dominating set in c-tournaments

Following is the formal definition of the problem *r-dominating set*, for every fixed $r \in N$.
Instance Directed graph $G = (V, E)$, $k \in N$

Solution $V' \subseteq V$ such that $|V'| \leq k$ and $\forall u \in V\ \exists v \in V'$, such that, there is a directed path of length at-most r from v to u. As noted the problem of a 1-dominating set is $\mathbf{W}[2]$ hard in tournaments but the problem of an r-dominating set is in $\mathbf{P}$, for $r \geq 2$. In this section we observe that c-tournaments carry some structure of tournaments which allows us to solve the r-dominating set for $r \geq 2c$. We prove that the problem of a c-dominating set is W[2] complete in c-tournaments. We leave open the status of a $c + i$-dominating set in c-tournaments, for $1 \leq i \leq c - 1$. First we extend Landau's theorem in tournaments to c-tournaments.

Theorem 1. *Let T be a c-tournament and v a vertex with a maximum number of vertices at a distance of at most c. Then $\{v\}$ is a $(2c)$-dominating set of T.*

Proof. Let $N_c(v)$ denote the cth order out neighborhood of v. Let u be a vertex not in $N_{2c}(v)$. Since T is a c-tournament, it follows that every vertex in $N_c(v) \cup \{v\}$ is in $N_c(u)$. This is a contradiction to the fact that v is that vertex with the largest cth order neighborhood. Hence the lemma. □

Thus the problem of an r-dominating set is polynomial time solvable for $r \geq 2c$. We next show that c-dominating set in c-tournament is $W[2]$ complete. The reduction is from dominating set in tournaments [4].

Theorem 2. *The c-dominating set problem restricted to c-tournaments is W[2] complete. Here the parameter of interest is the size of the desired c-dominating set.*

Proof. We reduce the dominating set problem in tournaments to the c-dominating set problem in c-tournaments. Given a tournament $T = (V, E)$, we construct a c-tournament T_c as follows: each vertex $v \in V(T)$ is split into c copies $v_1, v_2, \ldots v_c$ which are all connected by a $c - 1$ path directed by adding edge (v_i, v_{i+1}), for all $1 \leq i \leq c - 1$. If $(u, v) \in E(T)$ we add the set $\{(u_i, v_1) | 1 \leq i \leq c\}$. We now show that T_c is a c-tournament with a c-dominating set of size at most k if and only if T has a dominating set of size at most k. This proves the $W[2]$-hardness. First, clearly T_c is a c-tournament. Secondly, if D is a dominating set of size k of $V(T)$, then $D_c = \{u_1 : u \in D\}$ is clearly a c-dominating set of T_c. This shows that if T has a dominating set of size k, then T_c has a c-dominating set of size k. Next, for any c-dominating set D' of T_c, define $D = \{v | \text{for some } i, 1 \leq i \leq c, v_i \in D'\}$. Clearly the cardinality of D is at most the cardinality of D'. Finally we observe that D is a dominating set of T: Indeed, if there is some vertex u in T which is not dominated by D, the corresponding vertex u_c is not dominated by D'. Hence the theorem follows. □

We conclude this sub-section by mentioning that a c-dominating set problem in a c-tournament is closely related to a dominating set problem in tournaments. Given a c-tournament T_c, we construct the following tournament T: $V(T) = V(T_c)$, and for each pair $\{u, v\}$, we add an edge from (u, v) in $T \iff$ there is a path of length at most c between u and v in T_c. If there is such a path in both directions, we arbitrarily pick one edge. Clearly, T is a tournament and a dominating set in T is a c-dominating set in T_c. As there is a dominating set of size $\log_2 n$ in a tournament, there is a c-dominating set of size $\log_2 n$ in a c-tournament. This gives us an algorithm to solve the minimum c-dominating set problem in c-tournaments in $O(n^{\log_2 n})$ time.

2.2. Complexity of FVS and FAS in c-tournaments

First we prove that the FVS and FAS problems in 3-tournaments are equivalent to the FVS and FAS problems in general directed graph. Given an instance G of FVS (FAS), add two new vertices, an *in-vertex* t and an *out-vertex* s. The vertex s has edges directed out of it to each vertex of G. Similarly t has edges into it from each vertex of G. Finally we add an edge from t to s. Let the resulting graph be G'. G' is a 3-tournament on $|V(G)| + 2$ vertices, and can be constructed in polynomial time. The following theorem connects the FVS (FAS) of G and G'.

Theorem 3. *G has an FVS (FAS) set of size $k \iff$ the 3-tournament G' has an FVS (FAS) set of size $k + 1$.*

Proof. We first check that G' is a 3-tournament. Consider a pair of vertices u, v in $V(G')$. If either of the vertices is s or t, then the pair of vertices is an element of $E(G')$. If neither is s or t, then there is a path from u to v formed by the three edges $(u, t), (t, s), (s, v)$. Therefore, G' is a 3-tournament.

We first prove the claim for the FVS case. In the case when G has an FVS of size k, say S, define $S' = S \cup \{s\}$. We prove, by contradiction, that $G' - S'$ is acyclic. A cycle in $G' - S'$ must contain at least one of s or t, since $G - S$ is acyclic. Since $s \in S'$, such a cycle must contain t. This is impossible since t has no out-edge in $G' - S'$. Therefore, S' is an FVS of G' and $|S'| = k + 1$. In the other direction we consider the case when G' has an FVS of size $k + 1$, say S'. By the construction of G', $S' \cap \{s, t\} \neq \phi$, since every vertex $v \in V(G)$ is in a cycle formed by the edges $(v, t), (t, s), (s, v)$. Therefore, the set $S' \cap V(G)$ is an FVS of G, and is clearly of cardinality at most k. The proof for the feedback arc set is essentially the same argument based on the observation that any feedback arc set of G' must contain the arc (t, s) and it is sufficient to add (t, s) to any feedback arc set of G to get a feedback arc set of G'. □

It might be misconstrued that since a 3-tournament can be obtained by adding just 2 vertices to any graph, they are not very restrictive. The 3-tournaments have enough structure to solve the 6-dominating set problem easily, which is $W[1]$ complete in general graphs.

We now prove that solving an FVS problem in 2-tournaments is as hard as solving them in general simple directed graphs using a more involved construction.

The reduction

Given a simple directed graph on n vertices G, we construct a 2-tournament T as following: T comprises of 3 graphs G, G', and G''. We assign distinct labels to the n vertices of G using $\lceil \log_2 n \rceil$ bits. G' is a graph on $\lceil \log_2 n \rceil$ new vertices $u_1, \ldots, u_{\lceil \log_2 n \rceil}$, and $(u_i, u_j) \in E(G)$ for each pair $1 \leq i < j \leq \lceil \log_2 n \rceil$. For each $1 \leq i \leq \lceil \log_2 n \rceil$, and $v \in V(G)$ the edge (u_i, v) is present if the i-th bit in the label of v is 0. Otherwise the edge (v, u_i) is added. G'' is a graph that has two copies, T^+ and T^-, of a transitive tournament on $\lceil \log_2 n \rceil$ vertices. To obtain T, we add edges from all vertices of G into all vertices of G'', from all vertices of G' into all vertices of T^+, from all vertices of T^+ into all vertices of T^-, and from all vertices of T^- into all vertices of G'. It is easy to observe that T is a 2-tournament.

Lemma 1. *In the above construction, G has an FVS of size k if and only if T has an FVS of size* $k + \lceil \log_2 n \rceil$

Proof. If G has an FVS of size k, say S, then $S \cup G'$ is clearly an FVS of size $k + \lceil \log_2 n \rceil$ of T. Now if S' is an FVS of T, then we claim that S' contains at least $\lceil \log_2 n \rceil$ vertices of $G'' \cup G'$. Indeed, there are at least $\lceil \log_2 n \rceil$ disjoint triangles in the subgraph of T induced by vertices of $G'' \cup G'$ (simply split $G'' \cup G'$ into triplets of vertices, each triplet consisting of exactly one vertex from each of G', T^+ and T^-). At least one vertex from each of these triangles must be present in every FVS of T. Hence the claim. Thus if T has an FVS of size $k + \lceil \log_2 n \rceil$, it must have an FVS of size at most k in G. Hence the lemma. □

The following theorem is an easy consequence of the above lemma.

Theorem 4. *An algorithm to test if a 2-tournament has an FVS of size at most k in* $O^*(2^{O(k)})$ *time can be used to test if a directed graph has an FVS of size at most k in time* $O^*(2^{O(k)})$.

3. 2-tournament subgraph and supergraph problems

The purpose of this section is to study variations of important graph editing problems involving tournaments. The variation is to study these problems with respect to 2-tournaments. We study the complexity of finding a subgraph of a digraph in which each connected component (in the underlying undirected graph) is a 2-tournament. The goal is to minimize number of edges deleted. Later, we study the problem of finding a 2-tournament which is a supergraph of a given digraph, such that the number of edges to be added is minimum.

3.1. 2-tournament clustering edge set is NPC and FPT

In the following sections, we say C is a component of digraph G, if C is a component in the underlying undirected graph of G. We show that 2-tournament clustering edge set is NPC by a reduction from the clique clustering problem [7] on undirected graphs. We subsequently show that 2-tournament clustering edge set is FPT.

Reduction

Given an undirected graph $G(V, E)$, construct the following graph $G'(V', E')$ where $V' = \{v^+, v^- : v \in V\}$ and $E' = \{(v^+, u^-), (u^+, v^-), (v^-, v^+), (u^-, u^+) : \{v, u\} \in E\}$. Throughout this section, the graph G' is used to denote the graph obtained from this reduction. Also, we refer to v^+ and v^- as complements of each other.

Lemma 2. *G is a clique* $\iff$ *G' is a 2-tournament.*

Proof. $\Rightarrow$ G is a clique. Since G is a clique, two vertices of opposite sign in G' are adjacent. Two vertices of the same sign, say u^+ and v^+ are connected by a path of length 2 involving the edges $\{u^+, v^-\}, \{v^-, v^+\}$. Therefore, G' is a 2-tournament.

$\Leftarrow$ G' is a 2-tournament. Let $\{u, v\} \in G$ be two non-adjacent vertices. Let $\{u^+, v^+\}$ be connected by a path of at most 2 edges. Since the only edge coming into v^+ is from v^-, the path of 2 edges connecting u^+ and v^+ must be $\{u^+, v^-\}, \{v^-, v^+\}$. By the construction $\{u^+, v^-\}$ is an edge if and only if $\{u, v\} \in E(G)$. This contradicts our assumption that $\{u, v\} \notin E(G)$. The same argument works if we assume that there is path from v^+ to u^+. Therefore, if G' is a 2-tournament, it follows that G is clique. □

In a directed graph G, let $\mathbf{C}$ denote a 2-tournament cluster, that is a subgraph in which each component (of the underlying undirected graph) is a 2-tournament. The edge set $E(G) \setminus E(\mathbf{C})$ is called a *clustering edge set*, and the set $V(G) \setminus V(\mathbf{C})$ is called a *clustering vertex set*.

Lemma 3. *Let C be a 2-tournament that is a subgraph of G'. There is at most one vertex $v^+ \in V(C)$ such that $v^- \notin V(C)$. Similarly, there is at most one $u^- \in V(C)$ such that $u^+ \notin V(C)$.*

Proof. Proof is by contradiction. Let $w^+, v^+ \in V(C)$ such that both $w^-, v^- \notin V(C)$. Since C is a 2-tournament, there is a path of length two involving the vertices $\{v^+, w^+\}$, and for this to happen $v^- \in V(C)$ or $w^- \in V(C)$. This is not possible since, according to our assumption, for both v^+ and w^+, v^- and w^- are not in $V(C)$. Hence the proof. □

Lemma 4. *Let M be a 2-tournament clustering edge set for G'. There is a 2-tournament clustering edge set M' such that $|M'| \le |M|$, and each pair $\{v^+, v^-\}$ occurs in one component in $G' \setminus M'$.*

Proof. If each pair $\{v^+, v^-\}$ occurs in a component in $G' \setminus M$, then M is the desired M' and the statement of the lemma is true. Let us assume that this is not the case. Let v^+ and v^- be in different components C_1 and C_2 of $C = G' \setminus M$, respectively. We first show that either moving v^+ to C_2 or moving v^- to C_1 yields a 2-tournament clustering edge set M'' such that $|M''| \le |M|$.

First, we prove that placing v^+, v^- in the same component (adding back and deleting edges appropriately) will give another 2-tournament cluster. Without loss of generality, let us place these in C_1. Let C_2 cease to be a 2-tournament. Let u and w be two (signed) vertices whose property P_2 is destroyed. Clearly u and w must form a 2-path with v^-, as the latter's removal caused the destruction of P_2 for these vertices. So both, u and w must have $'+'$ sign and one of them must be a complement of v^- i.e., one of them must be v^+, which is impossible. A similar argument shows C_1 remains a 2-tournament on addition of v^-. Hence the new graph is a 2-tournament cluster. Let c_1 be the number of vertices in $C_1 \setminus \{v^+\}$ which form an edge with v^- in G' and let c_2 be the number of vertices in $C_2 \setminus \{v^-\}$ that form an edge with v^+ in G'. In C_1 the in-degree of v^+ is zero, and in C_2 the out-degree of v^- is zero. Let d_1 and d_2 denote out-degree of v^+ in C_1 and in-degree of v^- in C_2, respectively. By applying Lemma 3, it follows that there can be at most one other vertex in C_1 whose complement is not in C_1. Therefore, it follows that $c_1 \geq d_1 - 1$. By a symmetric argument, it follows that $c_2 \geq d_2 - 1$. The clustering edge set corresponding to the transfer of v^+ to C_2 has $|M| - c_2 - 1 + d_1$ edges and this is at most $|M| + d_1 - d_2$. Similarly, moving v^- from C_2 to C_1 results in a clustering edge set of cardinality $|M| + d_2 - c_1 - 1$ which is at most $|M| + d_2 - d_1$. Therefore, it follows that one of the two moves yields a edge clustering set M'' whose cardinality is at most $|M|$, and in $G \setminus M''$ v^+ and v^- are in the same component. Performing this move for all pairs u^+ and u^- which are in different components in C, results in the desired M', and proves the lemma. □

Using Lemma 4, we now complete a reduction of finding a clique clustering edge set to finding a 2-tournament clustering edge set.

Theorem 5. *For each $k \geq 0$, an undirected graph G has a clique clustering edge set of size at most k if and only if G' has a 2-tournament clustering edge set of size at most $2k$.*

Proof. $\Rightarrow$: Let M be a clique clustering edge set of G such that $|M| = k$. Let $M' = \{(u^+, v^-), (v^+, u^-) : \{u, v\} \in M\}$. Clearly $|M'| = 2k$ and $|M'| \subseteq E(G')$, and by Lemma 2 each component of $G' \setminus M'$ is 2-tournament. Therefore, M' is a 2-tournament edge clustering set.

$\Leftarrow$ By 4, let M' be an inclusion-minimal 2-tournament clustering edge set such that $|M'| \leq 2k$ and each pair $\{v^+, v^-\}$ is in a single component of $G \setminus M'$. It now follows that if $(x^+, y^-) \in M'$, then $(y^+, x^-) \in M'$. The reason is that, otherwise we get a contradiction to the minimality of M' or a contradiction to the fact that each pair $\{v^+, v^-\}$ is in a component of $G \setminus M'$. Now, we define $M \subseteq E$ as the set $M = \{\{u, v\} : (u^+, v^-), (v^+, u^-) \in M'\}$. Since $|M'| \leq 2k$, it follows that $|M| \leq k$ and by Lemma 2 each component of $G - M$ is a clique. Therefore, M is a clique clustering edge set. Hence the lemma. □

We have reduced the clique clustering edge set problem to a 2-*tournament* clustering edge set problem. Since the former is NP hard (NPH), so is the latter, and its membership in NP is also straightforward. Therefore, the 2-tournament clustering edge set problem is NPC. Using the same reduction it follows that deciding if a directed graph has a 2-tournament of size at least k is both NPC and W[1] hard. The reduction is from the clique problem, which is NPC and W[1] hard, and the transformation is exactly the same as one outlined above. From Lemma 2 we have that there is a clique of size k in G if and only if G' has a 2-tournament of size $2k$.

2-tournament clustering edge set is FPT: We present two algorithms to find out if a given directed graph has a 2-tournament clustering edge set of size at most k. Both algorithms are based on Lemma 5 which is used to show that if a directed graph is not a cluster of 2-tournaments, then there exist two vertices, which do not have a directed 2-path between them but are at a distance of at most 3 in the underlying undirected graph.

Lemma 5. *Let G be a directed graph which is not a 2-tournament such that the underlying directed graph is connected. There exist two vertices for which the distance in the undirected graph is at most 3 but are not at directed distance 2 in G.* □

Proof. Let us assume the contrary. Let S denote the set of vertex pairs in G whose minimum distance in G, in either direction, is more than 2. Let Let $\{u_{min}, v_{min}\}$ be a pair in S such that the distance between u_{min} and v_{min} in the underlying undirected graph is the least over all pairs in S. Let this distance be d. If we assume that the statement of the lemma is wrong, then $d \geq 4$. Let the shortest undirected path be $P_{min} = \{u_{min}, v_1, v_2, v_3, \dots, v_{min}\}$. By our assumption($d \geq 4$), the pair $\{u_{min}, v_3\}$ is not an element of S. Therefore, there is a path in the directed graph between u_{min} and v_3 of length at most 2 (let w be the connecting vertex). This now gives us a path shorter than P_{min} between u_{min} and v_{min} in the underlying undirected graph (in both the cases, when the vertex w is in P_{min} or not), a contradiction to the fact that P_{min} is a path of shortest length. Therefore, our assumption is wrong. Hence the lemma. □

Algorithm 1. Lemma 5 is converted into an algorithm based on the following observation: if each component is a 2-tournament, we are done. If not, consider a vertex pair (u, v) in a component at a distance more than 2 in the directed graph, and u and v are a distance at most 3 in the underlying undirected graph. Such a pair can obviously be found in polynomial time. Let P_{uv} be any such path in the underlying undirected graph. By definition, a 2-tournament clustering edge set M ensures that u and v are in different components in $G \setminus M$. Therefore, M must contain at least one directed edge corresponding to the edges in P_{uv}. Clearly, the running time of the algorithm is $O^*(3^k)$.

Algorithm 2. We now present a $O^*(2.27^k)$ to solve the 2-tournament edge clustering set problem. We reduce this problem to the 3-hitting set problem in polynomial time which can be solved in $O^*(2.27^k)$ [13]. The reduction is again based on Lemma 5. For every pair of vertices not at directed distance 2, we find all undirected paths of length at most 3. Each such path forms a 3-set which must be *hit*. This construction can obviously be done in polynomial time. Thus from the Edge set we need to pick a 3-hitting set which hits all the 3-sets. We note that from Lemma 5 if all such 3-sets are hit we get a 2-tournaments cluster.

3.2. Digraph completion problems related to 2-tournaments

A natural question is to study the complexity of finding a set of at most k edges to be added to a digraph so that the resulting digraph becomes a 2-tournament. This is the 2-tournament completion problem, and we prove hardness results for this problem using the following problem:

Problem 1. *Single vertex satisfaction*(SVS)
Given: A simple directed graph $G = (V, E)$,a vertex $v \in V$, integer $k \geq 0$.
Question: Add at most k edges so that for all $u \in V(G)$, there exists a path of length at most 2 connecting u and v.

To prove hardness of this problem, we transform an instance of the dominating set in a bipartite graph to an instance of SVS. Dominating Set in bipartite graphs is NPC and in the parameterized context $W[2]$ hard.

Theorem 6. *SVS is NPC and $W[2]$ hard.*

Proof. Reduction: Given an instance $G = (A, B)$, an integer $k \geq 0$, of dominating set in bipartite graphs, we construct a digraph G' by adding a new vertex v to G, and then by orienting all edges from the set A to B. We now show that G has a dominating set of size at most k if and only if there is a set of at most k edges that can be added to G' so that for all $u \in V(G') \setminus \{v\}$, there exists a directed path of length at most 2 connecting u and v. If D is a dominating set of G, then $M = \{(v, u) : u \in D \cap A\} \cup \{(u, v) : u \in D \cap B\}$ is clearly a solution of size k for the SVS problem instance (G', v). In the other direction, if M is a solution of size k for the SVS problem instance (G', v), we prove that there is a dominating set of size at most k of G. First we note a simple property of dominating sets in a graph: adding m edges to a graph decreases its minimum dominating set size by at most m. Let $M' \subseteq M$ be the set of edges with both ends in G (note that G does not include v). From the above property, if d' is the minimum dominating set size of $G \cup M'$ then $d' \geq d - |M'|$, where d is the minimum dominating set size of G. Now, the edges of $M \setminus M'$ must be incident on v. We claim that $|M| - |M'| \geq d'$. Let S be the set of vertices adjacent to v. It is clear that S must be a dominating set of $G \cup M'$, since there is a directed path of length at most 2 between v and every vertex in G. Thus $d' \leq |S| = |M| - |M'|$. Thus we have $k = |M| \geq d' + |M'| \geq d$. This shows that the size of the minimum dominating set is of size at most k. This completes the reduction of the dominating set in bipartite graphs to the SVS problem. □

2-tournament completion: We now show that the problem of adding as few edges as possible to a general directed graph to make it a *2-tournament* is NPC. We reduce the problem of SVS to this problem. Given a general directed graph $G = (V, E)$ and a vertex $v \in V$, we construct the following graph $G' = (V', E')$.
Construction:

$$V' = V \cup V_1 \cup \{v_{ex}\}$$
$$V_1 = \{v_{u,w} : \forall \{u, w\} \in V - \{v\}\} \tag{1}$$
$$\begin{aligned} E' = E \ &\cup \ \{(u, v_{u,w}), (v_{u,w}, w), \forall v_{u,w} \in V_1\} \\ &\cup \ \{(u, v_{ex}), \forall u \in V\} \cup \{(v_{ex}, v_{u,w}), \forall v_{u,w} \in V_1\} \\ &\cup \ \{(u, v) \forall \{u, v\} \in V_1\}. \end{aligned} \tag{2}$$

Theorem 7. 2-*tournament completion is NPC and $W[2]$ hard.*

Proof. We present a polynomial time transformation of an instance of SVS to an instance of the 2-tournament completion. Let $(G, v, k \geq 0)$ be an instance of SVS. Let G' be the graph constructed from the above instance $(G, v, k \geq 0)$. We show that there is a solution of size at most k to the SVS if and only if there is a solution of size at most k for the 2-tournament completion on the graph G'.

Let M, a set of edges such that $|M| \leq k$, be a solution of SVS problem for the instance $I = (G, v, k)$. We prove that M is also a solution of the problem 2-tournament completion for the instance G', that is we prove that $G' \cup M$ is a 2-tournament. We prove this by showing a path of length at most 2 between every pair of vertices, $\{u, w\} \subseteq V'$. By construction, in G' there is a path of length at most 2 between all pairs except when $u \in V$ and $w = v$, and for this case since M is a solution of SVS problem of (G, v, k), there is path of length 2 connecting u and w. Therefore, it follows that $G' \cup M$ is a 2-tournament.

We now prove that a set of at most k edges, M, added to G' to make it a 2-tournament, immediately yields a solution to the SVS problem on the instance (G, v, k). We note that there are four possibilities for any edge of M,

1. It is directed from v to a vertex $v_{u,w} \in V_1$
2. It is directed from $v_{u,w} \in V_1$ to v
3. It is directed from $u \in V \setminus \{v\}$ to $v' \in V_1$
4. It is directed from $v' \in V_1$ to $u \in V \setminus \{v\}$
5. Both ends of the edge are in V

Given M, we construct M' in the following way. For each edge in M, we do the following, according to the above cases:

1. In this case, add an edge (v, w) to M'
2. Add edge (u, v) to M'
3. Add edge (u, v) to M', if not added already
4. Add edge (v, u) to M', if not added already
5. Add the edge to M'

We claim that $G' \cup M'$ is a 2-tournament and M' is also a solution of SVS instance (G, v, k). Since it is clear that $M' \leq M$, we will have our theorem. Let the former claim be false. The only way this can happen is when there exist $x = v, y \in V$ such that there is no 2-path or direct edge between them in $G' \cup M'$. They must have a 2-path in $G' \cup M$, by the definition of M. We prove that this is impossible by enumerating all cases and by showing contradictions:

1. The 2-path (in $G' \cup M$) is of the form $(v, v_{u,w}, y)$ or $(y, v_{u,w}, v)$, where $v_{u,w} \in V_1$. At least one of the edges of the 2-path must be in M, otherwise there will be a 2-path in G' and hence in $G' \cup M'$. All possibilities in this case will result a direct edge between v and y in M'. This is a contradiction to the fact that there is no direct edge between v and y in $G' \cup M'$.
2. The 2-path is of the form (v, z, y) or (y, z, y) where $z \in V$ or there is a direct edge between v and y in $G' \cup M$. We note that all these edges are, by construction, present in M', which is again a contradiction.

Hence, $G' \cup M'$ is a 2-tournament. We also note that all paths in $G' \cup M'$ required to satisfy the property P_2 of v, y are contained in V and hence M' is a solution of the SVS instance (G, v, k), as $|M'| \leq |M| \leq k$. Hence the claim and the theorem. □

4. Conclusion

A natural generalization of tournaments, r-tournaments, is the focus of current article. Hardness of some problems which are easy to solve in tournaments but *'hard'* to solve in general digraphs is studied in r-tournaments. These graphs seem to carry on some structure of tournaments, when we look for an r-dominating set. The structure, however, does not help us solve the FVS problem: given efficient algorithm to solve 2-tournament FVS, we can obtain efficient algorithms for general digraph FVS. We also studied a generalization of a cluster graph editing problem and obtained efficient FPT algorithms. Finally we consider an interesting graph augmentation problem, 2-tournament completion and prove that this problem is NP hard and $W[2]$ hard. As an intermediate step to proving the NP hardness of this problem, we prove the hardness of another graph augmenting problem named Single Vertex Satisfaction . It would be very interesting to see if other NPC problems, like the longest path problem, have efficient algorithms in r-tournaments. While it is known that tournaments always have a Hamiltonian path (in fact more), 2-tournaments need not have one (it is easy to construct a counter example). An interesting question would be finding out if the longest path problem is polynomial time solvable or proving that it is NPC. Other interesting questions would be finding out the status of an r-dominating set in c-tournaments, where $r \in [c + 1, 2c - 1]$. In conclusion, r-tournaments seems to be an interesting class of graphs. The study of hard problems in r-tournaments could help in obtaining better algorithms in general graphs, as these graphs form a natural bridge between two extremes: tournaments and general connected digraphs.

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